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PRIMARY POWERS OF A PRIME IDEAL

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In this paper we show that the powers of a prime ideal $\mathfrak{p}$ are primary iff the direct summands of the graded ring associated with $\mathfrak{p}$ are torsion-free. We prove some consequences of this fact especially in connection with geometric situations.

Let k be a field, Y, X closed subschemes of $\boldsymbol{P}_k^n$; suppose that Y is irreducible, reduced and contained in X and let $\mathfrak{p}$ be the prime ideal corresponding to Y in the homogeneous coordinate ring of X (which is the unique ring $B = k[x_0, \cdots, x_n]$ such that $X = \operatorname{Proj}(B)$ and $(x_0, \cdots, x_n)$ does not belong to $\operatorname{Ass}(B)$). Is it true that $\mathfrak{p}^n$ is primary for every n?

The general answer is of course in the negative (see for instance Corollary 3.2). On the other hand it is well known that the answer is in the affirmative if $k = \boldsymbol{C}$, $X = \boldsymbol{P}_k^n$ and Y is a complete intersection; this fact has been improved by Bonardi in [1] and recently by Hochster, who more generally proved that $\mathfrak{p}^n$ is primary for all n if $\mathfrak{p}$ is a prime ideal generated by a regular sequence in a domain (see [3]).

The main purpose of this paper is the study of the case where Y is not a complete intersection in X, and we get two essentially different situations when $\dim(Y) = 0$ and $\dim(Y) > 0$. More precisely, if Y is a closed rational point, we get the following complete answer: $\mathfrak{p}^n$ is primary for all n if and only if x is a "cone" having Y in its vertex (for precise statement see Theorem 3.1). Instead, when $\dim(Y) > 0$, if Y and X are complete intersections in $\boldsymbol{P}_k^n$, such that Y is nonsigular and X is nonsingular in the points of Y, then $\mathfrak{p}^2$ is primary and $\mathfrak{p}^n$ is primary for all n if we add the condition $\dim(Y) \geqq \operatorname{codim}(X)$ (see Theorem 3.3), in particular if X is a hypersurface (Corollary 3.4).

Suitable example at the end of § 3 justify the hypotheses we need in the above mentioned theorems.

As to the proofs, first we develop criteria for $\mathfrak{p}^n$ to be primary, showing that this property is connected with the fact that certain modules are torsion-free (Proposition 1.1 and Corollaries). Then, essentially using homological methods, we can prove "algebraic" theorems (Theorems 2.2 and 2.3), from which the "geometric" ones easily follow.

In this paper all rings are supposed to be commutative, noetherian and with identity.

1. Let B be a ring, $\mathfrak{p}$ a prime ideal and $A = B/\mathfrak{p}$; denote by G_n the A-module $\mathfrak{p}^n/\mathfrak{p}^{n+1}$ and by $G(\mathfrak{p})$ the graded A-algebra $\bigoplus_{n=0}^{\infty} G_n$ (for more details see [6]).

PROPOSITION 1.1. *If N is a positive integer, the following conditions are equivalent:*

(a) *$\mathfrak{p}^n$ is primary for $n \leqq N$.*

(b) *G_n is torsion-free for $n \leqq N-1$.*

Proof. It is clear that $\mathfrak{p}^n$ primary implies G_{n-1} torsion-free. Let $n \leqq N$, $x \notin \mathfrak{p}$, $y \notin \mathfrak{p}^n$. If $y \in \mathfrak{p}^r$, $y \notin \mathfrak{p}^{r+1}$ then $0 \leqq r < n$; we have $0 \neq \bar{x} \in A$, $0 \neq \bar{y} \in G_r$, therefore $\bar{x}\bar{y} \neq 0$ or, which is the same, $xy \notin \mathfrak{p}^{r+1}$; but we have $\mathfrak{p}^{r+1} \supseteqq \mathfrak{p}^n$ and so $xy \notin \mathfrak{p}^n$.

REMARK 1. As a consequence of Proposition 1.2 we get that $\mathfrak{p}^2$ is primary iff G_1 is torsion-free. Nevertheless the following example shows that if $n > 2 \mathfrak{p}^n$ need not to be primary even if G_{n-1} is torsion-free. Let $B = k[x, y, z] \cong k[X, Y, Z]/(Y^2, YZ, XY - Z^3)$ and $\mathfrak{p} = (y, z)$. We get in this case $G(\mathfrak{p}) \cong k[X][T_1, T_2]/(XT_1, T_1T_2, T_1^2, T_2^4)$, hence G_2 is torsion-free, but $\mathfrak{p}^3$ is not primary.

REMARK 2. In the above example $\mathfrak{p}^4$ is primary and this shows that G_N torsion-free does not imply G_n torsion-free for $n < N$.

COROLLARY 1.2. *The following conditions are equivalent:*

(a) *$\mathfrak{p}^n$ is primary for every n.*

(b) *$G(\mathfrak{p})$ is torsion-free.*

(c) *The canonical homomorphism $G(\mathfrak{p}) \to G(\mathfrak{p}B_{\mathfrak{p}})$ is injective.*

In particular $G(\mathfrak{p})$ is a domain iff $\mathfrak{p}^n$ is primary for every n and $G(\mathfrak{p}B_{\mathfrak{p}})$ is a domain.

Proof. It follows from Proposition 1.1 that (a) and (b) are equivalent. Denoting with K the quotient field of A, the equivalence of (b) and (c) easily follows after remarking that $G(\mathfrak{p}B_{\mathfrak{p}}) \cong G(\mathfrak{p}) \otimes_A K$.

COROLLARY 1.3. *If $\mathfrak{p}$ is locally generated by a regular sequence, $\mathfrak{p}^n$ is primary for all n. In particular if $V(\mathfrak{p})$ is regular in* Spec (B) *and* Spec (A) *is regular, $\mathfrak{p}^n$ is primary for every n.*

Proof. For an ideal it is clear that to be primary is a local property, hence we may assume that $\mathfrak{p}$ is generated by a regular sequence and the conclusion follows since $G(\mathfrak{p})$ is a polynomial ring over A (see [5] Theorem 2.1).

2. Let B be a ring, $\mathfrak{p}$ a prime ideal and $A = B/\mathfrak{p}$.

LEMMA 2.1. *The following conditions are equivalent*:
(a) $\mathfrak{p}$ *is locally generated by a regular sequence.*
(b) $\mathfrak{p}/\mathfrak{p}^2$ *is a projective A-module and* $B_\mathfrak{p}$ *is regular.*
(c) $\mathfrak{p}^n/\mathfrak{p}^{n+1}$ *is a projective A-module for all n and* $B_\mathfrak{p}$ *is regular.*

Proof. We may assume that B is local. If $\mathfrak{p}$ is generated by a regular B-sequence, $B_\mathfrak{p}$ is regular and since $G(\mathfrak{p})$ is a polynomial ring over A, $\mathfrak{p}^n/\mathfrak{p}^{n+1}$ is a free A-module for all n. Let us now assume that (b) holds, and $\bar{a}_1, \cdots, \bar{a}_r \in \mathfrak{p}/\mathfrak{p}^2$ be a free basis over A. Using Nakayama we get $\mathfrak{p} = (a_1, \cdots, a_r)$, hence $\mathfrak{p}B_\mathfrak{p} = (a_1 \cdots, a_r)B_\mathfrak{p}$; we claim that $(a_1, \cdots, a_r)$ is a minimal basis for $\mathfrak{p}B_\mathfrak{p}$. On the contrary, let $\mathfrak{p}B_\mathfrak{p}$ be generated by a proper subset of $\{a_1, \cdots, a_r\}$ say $\{a_2, \cdots, a_r\}$; then there exists an element t not in $\mathfrak{p}$ such that $ta_1 \in (a_2, \cdots, a_r)B$, which contradicts the hypothesis that $\bar{a}_1, \cdots, \bar{a}_r$ are linearly independent over A. Combining with $B_\mathfrak{p}$ regular, it follows that $a_1, \cdots, a_r$ is a regular $B_\mathfrak{p}$-sequence. Let us now consider the graded homomorphism $\varphi: A[T_1, \cdots, T_r] \to G(\mathfrak{p})$ defined by $\varphi(T_i) = \bar{a}_i \in \mathfrak{p}/\mathfrak{p}^2$. It is clear that φ is onto, hence $G(\mathfrak{p}) \cong A[T_1, \cdots, T_r]/I$. On the other hand $G(\mathfrak{p}B_\mathfrak{p}) \cong G(\mathfrak{p}) \otimes_A K \cong K[T_1, \cdots, T_r]$ ($K =$ quotient field of A) because $a_1, \cdots, a_r$ is a regular $B_\mathfrak{p}$-sequence generating $\mathfrak{p}B_\mathfrak{p}$. It follows that $I \otimes_A K = 0$, hence $I = 0$ as I is obviously a torsion-free A-module. Applying Rees criterion (see [5] Theorem 2.2) we get that $a_1, \cdots, a_r$ is a regular B-sequence.

THEOREM 2.2. *If* $B_\mathfrak{p}$ *and* A *are regular and* $\dim(A) = 1$ *the following conditions are equivalent*:
(a) $\mathfrak{p}^2$ *is primary.*
(b) $\mathfrak{p}^n$ *is primary for every n.*
(c) $V(\mathfrak{p})$ *is regular in* Spec(B).
(d) $\mathfrak{p}$ *is locally generated by a regular sequence.*
(e) $\mathfrak{p}/\mathfrak{p}^2$ *is a projective A-module.*
(f) $\mathfrak{p}^n/\mathfrak{p}^{n+1}$ *is a projective A-module for every n.*

Proof. It is obvious that (b) implies (a). By Lemma 2.1 (d), (e), (f) are equivalent and (d) implies (b) by Corollary 1.3. If now $\mathfrak{p}^2$ is primary, by Proposition 1.1 $\mathfrak{p}/\mathfrak{p}^2$ is a finitely generated torsion-free A-module, hence projective, because A is a Dedekind domain and so (a) implies (e). The equivalence between (c) and (d) is well known since A is regular.

LEMMA 2.3. *Let A be a domain, M a finitely generated A-module, and* $\mathfrak{a}$ *an ideal of A such that* $\text{h.d.}_A M < \text{gr}(\mathfrak{a})$ *and* $M_\mathfrak{p}$ *is torsion-*

free for every prime $\mathfrak{p}$ *such that* $\mathfrak{a} \not\subset \mathfrak{p}$. *Then* M *is torsion-free.*

Proof. If not, we can choose a prime $\mathfrak{p} \neq (0)$ such that $\mathfrak{p} \in \mathrm{Ass}(M)$, and then $\mathfrak{p}A_{\mathfrak{p}} \in \mathrm{Ass}(M_{\mathfrak{p}})$, hence depth $(M_{\mathfrak{p}}) = 0$. Therefore $\mathrm{h.d.}_{A_{\mathfrak{p}}} M_{\mathfrak{p}} = \mathrm{gr}(\mathfrak{p}A_{\mathfrak{p}})$. But $\mathrm{h.d.}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leqq \mathrm{h.d.}_A M$ and so $\mathrm{gr}(\mathfrak{p}A_{\mathfrak{p}}) \leqq \mathrm{h.d.}_A M < \mathrm{gr}(\mathfrak{a})$; this implies $\mathfrak{a} \not\subset \mathfrak{p}$, then $M_{\mathfrak{p}}$ is torsion-free over $A_{\mathfrak{p}}$, a contradiction.

THEOREM 2.4. *Let* R *be a ring,* $\mathfrak{a}$, $\mathfrak{P}$, $\mathfrak{n}$ *ideals such that* $\mathfrak{a}$, $\mathfrak{P}$ *are locally generated by regular sequence,* $\mathfrak{P}$ *is prime and* $\mathfrak{a} \subseteq \mathfrak{P} \subseteq \mathfrak{n}$. *If* $V(\mathfrak{P}/\mathfrak{a}) - V(\mathfrak{n}/\mathfrak{a})$ *is regular in* $\mathrm{Spec}(R/\mathfrak{a})$, $\mathrm{Spec}(R/\mathfrak{P}) - V(\mathfrak{n}/\mathfrak{P})$ *is regular and* $\mathrm{gr}(\mathfrak{n}/\mathfrak{P}) = d \geqq 2$, *we have*:

(a) $(\mathfrak{P}/\mathfrak{a})^2$ *is primary*

(b) *If* $d > \mathrm{gr}(\mathfrak{a}R_{\mathfrak{m}})$ *for every maximal ideal* $\mathfrak{m} \supseteq \mathfrak{n}$ (for istance if $d > \dim R$), *then* $(\mathfrak{P}/\mathfrak{a})^n$ *is primary for every* n.

Proof. Using Corollary 1.3 and the local character of the property of being primary, we can restrict our attention to the maximal ideals containing $\mathfrak{n}$. Hence we may assume that R is local.

We shall denote by $a_1, \cdots, a_r$ the elements of the regular R-sequence generating $\mathfrak{a}$, by $\mathfrak{a}_i$ the ideal $(a_1, \cdots, a_i)$ $(\mathfrak{a}_0 = 0)$, by B the ring $R/\mathfrak{a}$, by B_i the ring $R/\mathfrak{a}_i$, by $\mathfrak{p}$ the ideal $\mathfrak{P}/\mathfrak{a}$ in B and by A the ring $R/\mathfrak{P} \cong B/\mathfrak{p}$.

We shall give the proof in several steps.

1. $V(\mathfrak{P}/\mathfrak{a}_i) - V(\mathfrak{n}/\mathfrak{a}_i)$ is regular in $\mathrm{Spec}(B_i)$ for $i = 1, \cdots, r$.
It follows from the property that a local ring is regular if its quotient by a regular sequence is regular.

2. $a_i \notin \mathfrak{P}^2 + (a_1, \cdot, \hat{a}_i, \cdot, a_r)$ for $i = 1, \cdots, r$.
If we denote by $\mathfrak{a}_{\hat{i}}$ the ideal $(a_1, \cdot, \hat{a}_i, \cdot, a_r)$, the ring

$$B_{\mathfrak{p}} \cong R_{\mathfrak{P}}/\mathfrak{a}R_{\mathfrak{P}} \cong (R_{\mathfrak{P}}/\mathfrak{a}_{\hat{i}}R_{\mathfrak{P}})/(\bar{a}_i)$$

is regular, hence $a_i \notin \mathfrak{P}^2R_{\mathfrak{P}} + \mathfrak{a}_{\hat{i}}R_{\mathfrak{P}}$.

3. Let s, N be integers $0 \leqq s \leqq r$, $0 < N$; if $(\mathfrak{P}/\mathfrak{a}_i)^t$ is primary for every nonnegative integer $i \leqq s - 1$ and for every $t = 1, \cdots, N$, then $\mathfrak{a}_s \cap \mathfrak{P}^t = \mathfrak{a}_s\mathfrak{P}^{t-1}$ for every $t = 1, \cdots, N$.

The proof is by induction on s; the case $s = 0$ is trivial. Hence we may assume $s \geqq 1$ and $\mathfrak{a}_{s-1} \cap \mathfrak{P}^t = \mathfrak{a}_{s-1}\mathfrak{P}^{t-1}$ for every $t = 1, \cdots, N$. Let now $\sum_{i=1}^{s} a_i x_i \in \mathfrak{P}^t$, then $a_s x_s \in \mathfrak{P}^t + \mathfrak{a}_{s-1}$; by step 2 $a_s \notin \mathfrak{P}^2 + \mathfrak{a}_{s-1}$ so $0 \neq \bar{a}_s \in (\mathfrak{P}/\mathfrak{a}_{s-1})/(\mathfrak{P}/\mathfrak{a}_{s-1})^2$. Let $x_s \in \mathfrak{P}^m + \mathfrak{a}_{s-1}$, $x_s \notin \mathfrak{P}^{m+1} + \mathfrak{a}_{s-1}$, thus $0 \neq \bar{x}_s \in (\mathfrak{P}/\mathfrak{a}_{s-1})^m/(\mathfrak{P}/\mathfrak{a}_{s-1})^{m+1}$. On the other hand, by Proposition 1.1, $\bigoplus_{j=1}^{N} (\mathfrak{P}/\mathfrak{a}_{s-1})^{j-1}/(\mathfrak{P}/\mathfrak{a}_{s-1})^j$ can be imbedded as a graded module in $G(\mathfrak{P}/\mathfrak{a}_{s-1}) \bigotimes_A K$ (where K is the quotient field of A), which is an integral domain by Corollary 1.3 and step 1. Therefore, if $m < t-1$, $0 \neq \bar{a}_s\bar{x}_s \in (\mathfrak{P}/\mathfrak{a}_{s-1})^{m+1}/(\mathfrak{P}/\mathfrak{a}_{s-1})^{m+2}$ i.e. $a_s x_s \notin \mathfrak{P}^{m+2} + \mathfrak{a}_{s-1}$ hence $a_s x_s \notin \mathfrak{P}^t + \mathfrak{a}_{s-1}$, a contradiction. In conclusion $m \geqq t-1$ and so $x_s \in \mathfrak{P}^{t-1} + \mathfrak{a}_{s-1}$. We get $\sum_{i=1}^{s} a_i x_i = \sum_{i=1}^{s-1} a_i y_i + a_s y_s$ with $y_s \in \mathfrak{P}^{t-1}$; hence $\sum_{i=1}^{s-1} a_i y_i \in \mathfrak{a}_{s-1} \cap \mathfrak{P}^t = \mathfrak{a}_{s-1}\mathfrak{P}^{t-1}$ by induction and the conclusion immediately follows.

4. The following sequence of A-modules is exact for all t and i.

$$(1)\quad 0 \longrightarrow \mathfrak{a}_i \cap \mathfrak{P}^t/\mathfrak{a}_i \cap \mathfrak{P}^{t+1} \longrightarrow \mathfrak{P}^t/\mathfrak{P}^{t+1} \longrightarrow \mathfrak{a}_i + \mathfrak{P}^t/\mathfrak{a}_i + \mathfrak{P}^{t+1} \longrightarrow 0$$

The proof is standard.

5. $(\mathfrak{P}/\mathfrak{a}_s)^2$ is primary for $s \leqq r$. If $s = 0$ it follows from Corollary 1.3. Therefore we may assume that $(\mathfrak{P}/\mathfrak{a}_i)^2$ is primary for $i \leqq s-1$. Using step 3 we get $\mathfrak{a}_s/\mathfrak{a}_s \cap \mathfrak{P}^2 \cong \mathfrak{a}_s/\mathfrak{a}_s\mathfrak{P}$. This is a free A-module generated by $\bar{a}_1, \cdots, \bar{a}_s$; indeed if $a_1x_1 + \cdots + a_sx_s \in \mathfrak{a}_s\mathfrak{P}$, we get $\sum_{i=1}^{s} a_i x_i = \sum_{i=1}^{s} a_i y_i$ with $y_i \in \mathfrak{P}$, hence $x_i - y_i \in \mathfrak{a}_s \subseteqq \mathfrak{P}$. Using the exact sequence (1) with $t = 1$, $i = s$ we get $\text{h.d.}_A\,(\mathfrak{P}/\mathfrak{a}_s)/(\mathfrak{P}/\mathfrak{a}_s)^2 \leqq 1$.

On the other hand by step 1 $V(\mathfrak{P}/\mathfrak{a}_s) - V(\mathfrak{n}/\mathfrak{a}_s)$ is regular in $\operatorname{Spec}(B_s)$ and by hypothesis $\operatorname{Spec}(R/\mathfrak{P}) - V(\mathfrak{n}/\mathfrak{P})$ is regular; applying Corollary 1.3 we get that $(\mathfrak{P}/\mathfrak{a}_s)^2$ is primary at every point of $\operatorname{Spec}(B_s) - V(\mathfrak{n}/\mathfrak{a}_s)$, hence by Proposition 1.1 $(\mathfrak{P}/\mathfrak{a}_s)/(\mathfrak{P}/\mathfrak{a}_s)^2$ is torsion-free at the same points. Applying Lemma 2.3, we are through.

6. Let s be an integer, $1 \leqq s \leqq r$. If $(\mathfrak{P}/\mathfrak{a}_i)^t$ is primary for $i = 0, \cdots, s-1$ and for every t, the following exact sequence of A-modules holds for all t:

$$(2)\quad \begin{aligned} 0 &\longrightarrow \mathfrak{a}_{s-1} \cap \mathfrak{P}^t/\mathfrak{a}_{s-1} \cap \mathfrak{P}^{t+1} \longrightarrow \mathfrak{a}_s \cap \mathfrak{P}^t/\mathfrak{a}_s \cap \mathfrak{P}^{t+1} \\ &\xrightarrow{\varphi} \mathfrak{a}_{s-1} + \mathfrak{P}^{t-1}/\mathfrak{a}_{s-1} + \mathfrak{P}^t \longrightarrow 0\,. \end{aligned}$$

The first homomorphism is the canonical one. Let $\bar{a}$ be an element of $\mathfrak{a}_s \cap \mathfrak{P}^t/\mathfrak{a}_s \cap \mathfrak{P}^{t+1}$; by step 3 $a = \sum_{i=1}^{s} a_i x_i$, $x_i \in \mathfrak{P}^{t-1}$. We define $\varphi(\bar{a}) = \bar{x}_s$ and the exactness easily follows.

7. If $0 \leqq s < d$, $(\mathfrak{P}/\mathfrak{a}_s)^t$ is primary for every t. We shall prove by induction on s that $\text{h.d.}_A\,(\mathfrak{a}_s \cap \mathfrak{P}^t/\mathfrak{a}_s \cap \mathfrak{P}^{t+1}) \leqq s-1$ for all t, $\text{h.d.}_A\,(\mathfrak{a}_s + \mathfrak{P}^t/\mathfrak{a}_s + \mathfrak{P}^{t+1}) \leqq s$ for all t and $(\mathfrak{P}/\mathfrak{a}_s)^t$ is primary for all t (we use the convention that $\text{h.d.}_A M = -1$ if M is the A-module with unique element 0).

The case $s = 0$ is clear. Letus now suppose that

$$\text{h.d.}_A(\mathfrak{a}_{s-1} \cap \mathfrak{P}^t/\mathfrak{a}_{s-1} \cap \mathfrak{P}^{t+1}) \leqq s-2$$

for all t, $\text{h.d.}_A(\mathfrak{a}_{s-1} + \mathfrak{P}^t/\mathfrak{a}_{s-1} + \mathfrak{P}^{t+1}) \leqq s - 1$ for all t and $(\mathfrak{P}/\mathfrak{a}_i)^t$ is primary for $i \leqq s - 1$ and for every t. By step 4 and 6, the exact sequences (1) and (2) hold for any t and for $i = s$. From (2) and Theorem B ([4] pg. 124) we get $\text{h.d.}_A(\mathfrak{a}_s \cap \mathfrak{P}^t/\mathfrak{a}_s \cap \mathfrak{P}^{t+1}) \leqq \max(\text{h.d.}_A(\mathfrak{a}_{s-1} \cap \mathfrak{P}^t/\mathfrak{a}_{s-1} \cap \mathfrak{P}^{t+1}), \text{h.d.}_A(\mathfrak{a}_{s-1} + \mathfrak{P}^{t-1}/\mathfrak{a}_{s-1} + \mathfrak{P}^t)) = s - 1$ for all t, hence we deduce from (1) $\text{h.d.}_A(\mathfrak{a}_s + \mathfrak{P}^t/\mathfrak{a}_s + \mathfrak{P}^{t+1}) \leqq s < d$ for all t. Using Lemma 2.3 and the same kind of argument of step 5 we get that $\mathfrak{a}_s + \mathfrak{P}^t/\mathfrak{a}_s + \mathfrak{P}^{t+1}$ is torsion-free for all t. Hence, by Proposition 1.1, $(\mathfrak{P}/\mathfrak{a}_s)^t$ is primary for all t.

8. (Conclusion.) Applying step 5 with $s = r$ we get (a); applying step 7 with $s = r$ we get (b).

3. In this section k will denote a field and P_k^n the n-dimensional projective space over k; if $\mathfrak{a}$ is a homogeneous ideal of $k[X_0, \cdots, X_n]$, we shall denote by $V = V(\mathfrak{a}) = \text{Proj}(k[X_0, \cdots, X_n]/\mathfrak{a})$ the associated projective scheme. If P is a closed rational point on V and $\mathfrak{p}$ the homogeneous prime ideal of $k[x_0, \cdots, x_n] \cong k[X_0, \cdots, X_n]/\mathfrak{a}$ corresponding to P, we may assume in the following that $P = (1, 0, \cdots, 0)$, hence $\mathfrak{p} = (x_1, \cdots, x_n)$.

THEOREM 3.1. *With the same assumptions, the following conditions are equivalent*:

(a) $\mathfrak{p}^n$ *is primary for every* n.

(b) $\mathfrak{a}$ *is generated by forms in* $k[X_1, \cdots, X_n]$.

Proof. Let $\mathfrak{a} = (F_1, \cdots, F_r)$, $F_i \in k[X_1, \cdots, X_n]$ and $R = k[X_1, \cdots, X_n]/\mathfrak{a}^*$ where $\mathfrak{a}^* = (F_1, \cdots, F_r)k[X_1, \cdots, X_n]$; we get $k[X_0, \cdots, X_n]/\mathfrak{a} \cong R[X_0]$ and so $\mathfrak{p}$ is the extension to $R[X_0]$ of a maximal ideal of R. Hence (b) implies (a). Now we prove that (a) implies (b). Let $\mathfrak{a} = (F_1, \cdots, F_r)$; we may write $F_1 = X_0^{m-1}G_1(X_1, \cdots, X_n) + X_0^{m-2}G_2(X_1, \cdots, X_n) + \cdots + G_m(X_1, \cdots, X_n)$ where $m = \partial F_1$ and $i = \partial G_i$. Reducing modulo $\mathfrak{a}$ we get $x_0^{m-1}g_1 \in \mathfrak{p}^2$; but $x_0^{m-1} \notin \mathfrak{p}$ and $\mathfrak{p}^2$ is primary, hence $g_1 \in \mathfrak{p}^2$ i.e. $G_1 \in (X_1, \cdots, X_n)^2 + \mathfrak{a}$ which implies $G_1 \in \mathfrak{a}$. By repeating this argument we get $G_1, \cdots, G_m \in \mathfrak{a}$, therefore $\mathfrak{a} = (G_1, \cdots, G_m, F_2, \cdots, F_r)$; the same for $F_2, \cdots, F_r$ and we are done.

COROLLARY 3.2. *With the same assumptions as above, if* P *is regular the following conditions are equivalent*:

(a) $\mathfrak{p}^2$ *is primary.*

(b) $\mathfrak{p}^n$ *is primary for every* n.

(c) V *is a linear space* (i.e. $\mathfrak{a}$ is generated by linear forms).

Proof. It follows from Theorem 3.1 that (c) and (b) are equivalent after remarking that a cone is a linear space if a point of its vertex is nonsingular for the cone. The equivalence of (a) and (b) follows from Theorem 2.2.

THEOREM 3.3. *Let X, Y be closed subschemes of $\boldsymbol{P}_k^n$, which are complete intersections in $\boldsymbol{P}_k^n$. Suppose that Y is an irreducible, reduced, positive dimensional, normal subscheme of X and* $\mathrm{Sing}(X) \cap Y \subseteqq \mathrm{Sing}(Y)$ (where "Sing" stands for "singular locus of"). *If $\mathfrak{p}$ denotes the prime ideal corresponding to Y in the projective coordinate ring of X, then*:

(a) *$\mathfrak{p}^2$ is primary.*

(b) *If* $\dim Y \geqq \operatorname{codim} X$, *then $\mathfrak{p}^n$ is primary for every n.*

Proof. If we denote by $\mathfrak{n}$ the ideal associated with $\mathrm{Sing}(Y)$, we get the proof as a strightforward consequence of Theorem 2.3.

COROLLARY 3.4. *With the same hypotheses of Theorem* 3.3, *if X is a hypersurface, $\mathfrak{p}^n$ is primary for all n.*

Now we shall try to justify the hypotheses of the previous theorems with same examples.

EXAMPLE 1. In Theorem 3.1 and Corollary 3.2 the condition "P rational" is essential. Let

$$B = \boldsymbol{R}[x_0, x_1, x_2]/(X_0^2 + 2X_1^2 - 2X_1X_2 + X_2^2) ,$$

$\mathfrak{p} = (x_1 - x_2)$; we have $B/\mathfrak{p} \cong \boldsymbol{R}[X_0, X_1]/(X_0^2 + X_1^2)$, hence P is a non-rational closed point on V. By strightforward computation V is a regular conic and $\mathfrak{p}^n$ is primary for all n since $\mathfrak{p}$ is generated by a regular element of B, but V is obviously not a linear space.

EXAMPLE 2. In Corollary 3.2 the condition "P regular" is essential. Let $B = k[x_0, x_1, x_2] \cong k[X_0, X_1, X_2]/(X_0X_1^2 - X_2^3)$, $\mathfrak{p} = (x_1, x_2)$; it is clear that $\mathfrak{p}^2$ is primary.

REMARK. In Theorem 3.3 the condition "$\dim Y > 0$" is essential because if $\dim Y = 0$ we have the counterexamples given by Theorem 3.1.

EXAMPLE 3. Let

$$B = k[x_0, \cdots, x_4]/(X_0X_2 - X_3X_4,\ X_1X_2 - X_3^2,\ X_0X_3 - X_1X_4) ,$$

$\mathfrak{p} = (x_2, x_3, x_4)$. In this case the hypotheses of Theorem 3.3 are fullfilled, save "X complete intersection", and $\mathfrak{p}^2$ is not primary.

EXAMPLE 4. (see [2]). Let $X = \boldsymbol{P}^5_k$, $\mathfrak{p}$ the prime ideal defining the Veronese surface i.e. the prime ideal generated by the 2 by 2 minors of the matrix $M = \begin{pmatrix} X_0 & X_1 & X_2 \\ X_1 & X_3 & X_4 \\ X_2 & X_4 & X_5 \end{pmatrix}$. In this case the hypotheses of Theorem 3.3 are fullfilled, save "Y complete intersection", and $\mathfrak{p}^2$ is not primary. In fact if $d = \det M$, $d \notin \mathfrak{p}_2$ (for $i = 0, \cdots, 5$).

EXAMPLE 5. Let $B = k[x_0, x_1, x_2, x_3] \cong k[X_0, X_1, X_2, X_3[/(X_1X_2 - X_3^2)$, $\mathfrak{p} = (x_2, x_3)$. In this case the hypotheses of Theorem 3.3 are fullfilled, save "$\operatorname{Sing}(X) \cap Y \subseteq \operatorname{Sing}(Y)$", and $\mathfrak{p}^2$ is not primary.

EXAMPLE 6. Let

$$B = k[x_0, \cdots, x_7]/(X_0X_2 + X_1X_3 + X_4^2, X_0X_5 + X_1X_6 + X_7^2) ,$$

$\mathfrak{p} = (x_2, \cdots, x_7)$. In this case the hypotheses of Theorem 3.3 are fullfilled hence $\mathfrak{p}^2$ is primary, but $1 = \dim Y < \operatorname{codim} X = 2$ and we are going to prove that $\mathfrak{p}^3$ is not primary. In fact if we call $f_1 = X_0X_2 + X_1X_3 + X_4^2$, $f_2 = X_0X_5 + X_1X_6 + X_7^2$, from the identity

$$X_5f_1 - X_2f_2 = X_1(X_3X_5 - X_2X_6) + X_5X_4^2 - X_2X_7^2$$

we get $x_1(x_3x_5 - x_2x_6) = x_2x_7^2 = x_5x_4^2 \in \mathfrak{p}^3$ with $x_3x_5 - x_2x_6 \in \mathfrak{p}^3$ and $x_1 \notin \mathfrak{p}$.

REFERENCES

1. M. Teresa Bonardi, *Alcune estensioni di un teorema di G. Z. Giambelli e R. Torelli ed applicazioni*, Pubblicazioni dell' Istituto Matematico dell' Università di Genova, 1967.
2. ———, *Sopra le potenze di un ideale primo di un anello di serie formali o convergenti*, Le Matematiche, vol. XXIII, 1968.
3. Melvin Hochster, *Criteria for equality of ordinary and symbolic powers of primes*, Math. Zei., **133** (1973).
4. Irving Kaplansky, *Commutative Rings*, Allyn and Bacon, Boston, 1970.
5. David Rees, *The grade of an ideal of module*, Proc. Cambridge Phil. Soc., **53** (1957).
6. Oscar Zariaki and Pierre Samuel, *Commutative Algebra*, Vol. II, D. Van Nostrand, 1959.

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