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SETS OF UNIQUENESS AND MULTIPLICITY FOR L^p

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In the first part of this paper, it is proved that: if $1 < q < p \leqq 2$ and G is a nondiscrete, locally compact abelian (LCA) group with character group Γ, there exists a subset of positive measure $E \subset G$ which is a set of uniqueness for $L^q(\Gamma)$ and, at the same time, a set of multiplicity for $L^p(\Gamma)$.

This is followed by some results of the same type concerning the spaces $L^{p,\alpha}(\Gamma)$, $\alpha \neq 0$, when G is the Cantor group.

I. I. Hirschman, Jr., and Y. Katznelson prove all the results of this paper in the case when $G = \boldsymbol{T}$ (the circle group) and $\Gamma = \boldsymbol{Z}$ (the integer) (see [5]). They also prove results concerning the spaces $L^{p,\alpha}(\boldsymbol{Z})$. The core of the present work consists in proving our theorem for the case in which G is a compact group the elements of which have bounded order. To obtain the theorem for a general LCA group, we use the latter result, the theorem of Hirschman and Katznelson, and the structure theory for LCA groups. The existence of sets of uniqueness for $L^p(\Gamma)$, $1 \leqq p < 2$, which are of positive measure (and therefore are sets of multiplicity for $L^2(\Gamma)$) was proved by Y. Katznelson ([6]; [7], p. 101), for the case $G = \boldsymbol{T}$, and by A. Figà-Talamanca and G. I. Gaudry for the general case (see [2]). The results contained in this paper appeared in part in our thesis for the "laurea" in Mathematics at the University of Genova. This thesis was prepared under the guidance of Prof. A. Figà-Talamanca, to whom we are grateful for advice and assistance.

1. Preliminaries. Let G be a nondiscrete LCA group and Γ its character group. If f is an integrable function on G,

$$\hat{f}(\gamma) = \int_G f(x)\cdot\gamma(-x)dx \qquad \text{for any } \gamma \in \Gamma$$

denotes the Fourier transform of f.

Let us define now a uniqueness set and a multiplicity set for $L^p(\Gamma)$ as follows:

DEFINITION. Let E be a measurable subset of G; then E is called a *set of uniqueness* for $L^p(\Gamma)$ (or set of p-uniqueness), if no nonzero integrable function f, carried by E, satisfies the condition $\hat{f} \in L^p(\Gamma)$.

A subset which is not a set of uniqueness is called a *set of multiplicity* for $L^p(\Gamma)$ (or set of p-multiplicity).

Throughout this paper we indicate the Haar measure of a set S, by $m(S)$.

2. **The main result.** We can now state the first result of this paper.

THEOREM 1. *For every real numbers p, q such that $1 < q < p \leqq 2$ there exists a subset $E \subset G$, of positive measure, which is a set of q-uniqueness and p-multiplicity.*

Proof. From a theorem of structure [8, p. 40] we know that every LCA group has an open subgroup which is the direct sum of a compact group K and a Euclidean space $R^n (n \geqq 0)$. Thus it is sufficient to prove the theorem for groups of the form

$$G = K \oplus R^n \qquad \text{for } n \geqq 0\,.$$

We must distinguish several cases:

Case (1). G is a compact group with Haar measure normalized ($n = 0$). In this case the proof of the theorem can be obtained from the following two lemmas:

LEMMA 1. *Given a real numbers $\varepsilon, 0 < \varepsilon < 1$, and $p, q \in (1, 2]$ such that $q < p$, we can find a measurable set $E \subset G$ and two functions F and ϕ, defined on G, satisfying the following properties:*

(a) $m(E) \geqq 1 - \varepsilon$

(b) $F(x) = 1$ *for any* $x \in E$

(c) $\|\hat{F}\|_{q'} \leqq \varepsilon$ *if* $1/q + 1/q' = 1$

(d) $\phi(x) \geqq 0$ *for any* $x \in G$

(e) $m\{x \in G: \phi(x) = 1\} \geqq 1 - 2\varepsilon$

(*f*) $\log \|\hat{\phi}\|_p \leqq \varepsilon$

(g) $\{x \in G: \phi(x) \neq 0\} \subset E$.

Proof of the Lemma 1. We consider three cases according to the order of the elements of Γ.

Case 1 (a). Γ is a torsion group of bounded order. Let Δ be an independent set of generators for Γ. Let $\{\Gamma_n\}_{n=1}^{\infty}$ be a sequence of subgroups of Γ, all having the same finite order r and which are generated by elements of Δ in such a way that, if $(\Gamma_1, \Gamma_2, \cdots)$ denotes the group generated by $\Gamma_1, \Gamma_2, \cdots$, then

$$\Gamma_{n+1} \cap (\Gamma_1, \Gamma_2, \cdots, \Gamma_n) = \{0\} \quad \text{for} \quad n = 1, 2, \cdots\,.$$

Now, for any (large) positive integer N, we can define—on the group

G—two sets of N functions, $F_1, F_2, \cdots, F_N$ and $\phi_1, \phi_2, \cdots, \phi_N$ in the following way:

$$F_i(x) = 1 - \sum_{\gamma \in \Gamma_i} \gamma(x) \qquad \text{for } i = 1, \cdots, N ;$$

$$\phi_i(x) = 1 - \frac{1}{r} \sum_{\gamma \in \Gamma_i} \gamma(x) \qquad \text{for } i = 1, \cdots, N .$$

Since F_i and ϕ_i $(1 \leqq i \leqq N)$ are trigonometric polynomials, we can trivially deduce that:

(I) $\|\hat{F}_i\|_{q'} \leqq r^{1/q'}$ for any $i = 1, \cdots, N$.

(II) $\log \|\hat{\phi}_i\|_p \leqq r^{1-p}$ for any $i = 1, \cdots, N$.

Moreover:

(III) $m\{x \in G: F_i(x) = 1\} = 1 - 1/r$ for $i = 1, \cdots, N$.

(IV) $m\{x \in G: \phi_i(x) = 1\} = 1 - 1/r$ for $i = 1, \cdots, N$.

Now, let us define, for every integer N:

$$E = \bigcap_{i=1}^{N} \{x \in G: F_i(x) = 1\} ,$$

$$F(x) = \frac{1}{N} \sum_{i=1}^{N} F_i(x) \qquad \text{for } x \in G$$

$$\phi(x) = \prod_{i=1}^{N} \phi_i(x) \qquad \text{for } x \in G .$$

It follows that:

(i) $m(E) = m\{x \in G: F(x) = 1\} \geqq 1 - N/r$.

Because the F_i are supported by disjoint subgroups and $\hat{F}_i(0) = 0$ $(1 \leqq i \leqq N)$, then (I) yields:

(ii) $\|\hat{F}\|_{q'} = (1/N)(\sum_{i=1}^{N} \|\hat{F}_i\|_{q'}^{q'})^{1/q'} \leqq N^{-1+1/q'} \cdot r^{1/q'}$.

Since $\|\hat{\phi}_i\|_p = \|\hat{\phi}_j\|_p$ for any $i, j \in \{1, 2, \cdots, N\}$, and by the definition of Γ_n there exists at most one way to write $\gamma \in \Gamma$ as the sum of elements $\gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2, \cdots, \gamma_N \in \Gamma_N$, then

(iii) $\log \|\hat{\phi}\|_p = \log (\prod_{i=1}^{N} \|\hat{\phi}_i\|_p) \leqq N \cdot r^{1-p}$.

Since $p > q$, it is possible to choose a positive integer N and a large positive integer r such that the conditions

$$\begin{cases} \|\hat{F}\|_{q'} \leqq N^{-1/q} \cdot r^{1/q'} \leqq \varepsilon \\ \log \|\hat{\phi}\|_p \leqq N \cdot r^{1-p} \leqq \varepsilon , \end{cases}$$

are simultaneously satisfied.

Since Γ is of bounded order (and therefore it has infinitely many elements of the same order) we can choose r arbitrarily large so that it is the common order of infinitely many disjoint subgroups $\Gamma_1, \Gamma_2, \cdots$.

To complete the proof of the lemma in Case 1 (a), we note that:

(i) $\phi(x) = \prod_{i=1}^{N} (1 - \chi_{M_i})(x) \geqq 0$ for $x \in G$

where—for every $i = 1, 2, \cdots, N$ — $M_i = \{x \in G: \gamma(x) = 1, \gamma \in \Gamma_i\}$ and

χ_{M_i} is the characteristic function of M_i.

(ii) $m(E) \geqq 1 - N/r \geqq 1 - N \cdot r^{1-p} \geqq 1 - \varepsilon$
bceause $r > 1$ and $(p - 1) \leqq 1$.

(iii) $m\{x \in G: \phi(x) = 1\} = m(E) \geqq 1 - \varepsilon$.

Case 1 (b). Γ contains an element of infinite order. Let γ' be an element of Γ such that $n \cdot \gamma' \neq 0$, for every $n \in N$. We call Γ' the subgroup of Γ generated by γ'. Thus Γ' is isomorphic to Z and its character group $\hat{\Gamma}'$ is a compact group isomorphic to G/M, where

$$M = \{x \in G: \gamma'(x) = 1\}$$

is the annihilator of Γ' [8, p. 35]. But $\hat{\Gamma}'$ is also isomorphic to $\boldsymbol{T}$. Thus we can consider—given a positive real number ε—the construction of the functions which satisfy the Lemma 1 in the case $G = \boldsymbol{T}$, given by Hirschman and Katznelson in [5, p. 226]. We call these functions f and φ. Since f and φ belong to $L^2(\boldsymbol{T})$, we can write:

$$f(\vartheta) = \sum_{k \in \boldsymbol{Z}} a_k \cdot e^{ik\vartheta} \qquad \text{for any} \quad \vartheta \in (0, 2\pi]$$

$$\varphi(\vartheta) = \sum_{k \in \boldsymbol{Z}} b_k \cdot e^{ik\theta} \qquad \text{for any} \quad \vartheta \in (0, 2\pi] \, .$$

Now we define:

$$F(x) = \sum_{k \in \boldsymbol{Z}} a_k \cdot \gamma'(x)^k \qquad \text{for} \quad x \in G$$

and

$$\phi(x) = \sum_{k \in \boldsymbol{Z}} b_k \cdot \gamma'(x)^k \qquad \text{for} \quad x \in G \, .$$

From the definition of M it follows that such functions are constant on the sets $x + M$, where $x \in G$, and therefore we can consider F and ϕ as defined on G/M. Hence for every $x \in G/M$ we have

$$F(x) = \sum_{\gamma \in \Gamma'} \hat{F}(\gamma) \cdot x(\gamma) = \sum_{k \in \boldsymbol{Z}} \hat{f}(k) e^{ik\vartheta} = f(\vartheta) \, ,$$

where $\vartheta \in (0, 2\pi]$ is such that $e^{i\vartheta} \in \boldsymbol{T}$ is the image of $x \in G/M$ under the isomorphism mapping G/M onto $\boldsymbol{T}$.

Similarly:

$$\phi(x) = \varphi(\vartheta) \, .$$

It follows that [4, p. 91, vol. II]:

$$m\{x \in G: F(x) = 1\} = m\{\vartheta \in (0, 2\pi]: f(\vartheta) = 1\}$$

and

$$m\{x \in G: \phi(x) = 1\} = m\{\vartheta \in (0, 2\pi]: \varphi(\vartheta) = 1\} .$$

Thus, by observing that:

$$\|\hat{F}\|_{q'} = \|\hat{f}\|_{q'} ; \qquad \|\hat{\phi}\|_p = \|\hat{\varphi}\|_p ;$$

and recalling [5], the proof of the lemma is easily completed in this case.

Case 1 (c). Γ is a torsion group of infinite order. Suppose that $\gamma' \in \Gamma$ has order n. We call Γ' the cyclic subgroup of Γ generated by γ'. The discrete group Γ' is isomorphic to $\boldsymbol{Z}(n)$. Its character group $\hat{\Gamma}'$ is a cyclic compact subgroup of $\boldsymbol{T}$ which is also isomorphic to G/M, if M is the annihilator of Γ'.

Similarly to the Case 1 (b), if

$$f(\vartheta) = \sum_{k \in \boldsymbol{Z}} A_k \cdot e^{ik\vartheta} \qquad \text{for} \quad \vartheta \in (0, 2\pi]$$

and

$$\varphi(\vartheta) = \sum_{k \in \boldsymbol{Z}} B_k \cdot e^{ik\vartheta} \qquad \text{for } \vartheta \in (0, 2\pi]$$

are the functions satisfying the Lemma 1 when $G = \boldsymbol{T}$ and $\varepsilon/2$ replaces ε, we pose:

$$F(x) = \sum_{k=-[(n-1)/2]}^{[n/2]} \left(\sum_{i=-\infty}^{+\infty} A_{k+ni} \right) \cdot \gamma'(x)^k \qquad \text{for} \quad x \in G ,$$

$$\phi(x) = \sum_{k=-[(n-1)/2]}^{[n/2]} \left(\sum_{i=-\infty}^{+\infty} B_{k+ni} \right) \cdot \gamma'(x)^k \qquad \text{for} \quad x \in G .$$

Since these functions are constant on $x + M \subset G\,(x \in G)$, we can consider them as defined on G/M. Thus, for every $x \in G/M$, if $e^{i\vartheta}$ is the image of x under the isomorphism mapping G/M onto $\boldsymbol{T}$,

$$F(x) = \sum_{\gamma \in \Gamma'} \hat{F}(\gamma) x(\gamma) = \sum_{m \in \boldsymbol{Z}} \hat{f}(m) e^{im\vartheta} = f(\vartheta) .$$

Using the same reasoning we get, for any $x \in G/M$ and for any ϑ defined as above,

$$\phi(x) = \varphi(\vartheta) .$$

In order that the functions F and ϕ satisfy the Lemma 1, they must verify these non-obvious conditions:

(1) $\|\hat{F}\|_{q'} \leqq \varepsilon$,

(2) $\log \|\hat{\phi}\|_p \leqq \varepsilon$,

(3) $m\{x \in G: F(x) = 1\} \geqq 1 - \varepsilon$,

(4) $m\{x \in G: \phi(x) = 1\} \geqq 1 - 2\varepsilon$.

We will show that these conditions are satisfied if the order of the element γ' is large enough. Indeed, given $\hat{f} \in L^1(Z)$ and $\varepsilon > 0$ there exists n' so large that for $n \geqq n'$ we have:

$$\sum_{|k|>[(n-1)/2]} |A_k| \leqq \varepsilon/2 .$$

Consequently, by Minkowski's inequality and recalling that if $1 \leqq s$ then $L^1(Z) \subseteqq L^s(Z)$ we have:

$$\begin{aligned} ||\hat{F}||_{q'} &= \Big(\sum_k \Big|A_k + \sum_{i\neq 0} A_{k+ni}\Big|^{q'}\Big)^{1/q'} \\ &\leqq \Big(\sum_k |A_k|^{q'}\Big)^{1/q'} + \Big(\sum_k \Big|\sum_{i\neq 0} A_{k+ni}\Big|^{q'}\Big)^{1/q'} \\ &\leqq ||\hat{f}||_{q'} + \sum_k \Big|\sum_{i\neq 0} A_{k+ni}\Big| \\ &\leqq ||\hat{f}||_{q'} + \sum_{\substack{k,i \\ i\neq 0}} A_{k+ni} \\ &\leqq ||\hat{f}||_{q'} + \varepsilon/2 \leqq \varepsilon . \end{aligned}$$

Similarly, given $\hat{\varphi} \in L^1(Z)$ and $\varepsilon > 0$, there exists n'' so large that for $n \geqq n''$ we have

$$\sum_{|k|>[(n-1)/2]} |B_k| \leqq ||\hat{\varphi}||_p \cdot \varepsilon/2 .$$

Thus if γ' has order $n \geqq n''$,

$$\begin{aligned} \lg ||\hat{\phi}||_p &= \lg \Big(\sum_k \Big|B_k + \sum_{i\neq 0} B_{k+ni}\Big|^p\Big)^{1/p} \\ &\leqq \lg \Big(\Big(\sum_k |B_k|^p\Big)^{1/p} + \Big(\sum_k \Big|\sum_{i\neq 0} B_{k+ni}\Big|^p\Big)^{1/p}\Big) \\ &\leqq \lg \Big(||\hat{\varphi}||_p + \sum_{\substack{k,i \\ i\neq 0}} |B_{k+ni}|\Big) \\ &\leqq \lg (||\hat{\varphi}||_p \cdot (1 + \varepsilon/2)) \leqq \varepsilon . \end{aligned}$$

Finally, if η is the measure of the subset of $(0, 2\pi]$ where $f(\vartheta) \neq 1$ and the order n of γ' satisfies

$$n > (\eta/\varepsilon)^{1/2}$$

we have:

$$\begin{aligned} m\{x \in G: F(x) = 1\} &\geqq 1 - \varepsilon , \\ m\{x \in G: \phi(x) = 1\} &\geqq 1 - 2\varepsilon . \end{aligned}$$

Therefore, to fulfill conditions (1) to (4) and to prove the lemma it is enough to choose $\gamma' \in \Gamma$ with order n satisfying

$$n > \max (n', n'', (\eta/\varepsilon)^{1/2}) .$$

LEMMA 2. *Given a sequence of positive real numbers* $\{\varepsilon_k\}_{k=1}^{\infty}$ *such that* $2\sum_k \varepsilon_k < 1$, *it is possible—for every* $k \in N$ *to find* $E^{(k)}, F^{(k)}, \phi^{(k)}$ *satisfying Lemma* 1 *with* $\varepsilon = \varepsilon_k$ *such that for all* $N \in N$

$$\left\|\left(\prod_{k=1}^{N} \phi^{(k)}\right)^{\wedge}\right\|_p \leqq \exp\left[\sum_{k=1}^{N-1} \varepsilon_k\right] \cdot \prod_{k=1}^{N} \|\hat{\phi}^{(k)}\|_p \leqq e \, . \tag{1}$$

Proof. Inequality (1) can be proved easily enough by induction. Suppose that (1) holds for $N = 1, 2, \cdots, R-1$. We want to show that:

$$\left\|\left(\prod_{k=1}^{R} \phi^{(k)}\right)^{\wedge}\right\|_p \leqq \exp(\varepsilon_{R-1})\left\|\left(\prod_{k=1}^{R-1} \phi^{(k)}\right)^{\wedge}\right\|_p \cdot \|\hat{\phi}^{(R)}\|_p \, . \tag{2}$$

We must distinguish two cases:
(1°) G is as in the Case 1(a) of Lemma 1. Then we must choose $\phi^{(R)}$ such that

$$\operatorname{supp} \hat{\phi}^{(R)} \cap (\operatorname{supp} \hat{\phi}^{(1)}, \cdots, \operatorname{supp} \hat{\phi}^{(R-1)}) = \{0\} \, .$$

(2°) G is as the Cases 1 (b) and 1 (c) of Lemma 1. Then we define $\phi^{(R)}$ starting from the function φ_R which satisfy the analogous inequality in [5]. Using condition (f) of Lemma 1, we see that (1) is true in both cases.

Conclusion of the proof of Theorem 1 *in Case* (1). Choose a sequence of real numbers $\varepsilon_k > 0$, $k = 1, 2, \cdots$, such that $2\sum_k \varepsilon_k < 1$. We define

$$E = \bigcap_{k=1}^{\infty} E^{(k)} \, .$$

Since we assume that $2\sum_k \varepsilon_k < 1$, then clearly:

$$m(E) = 1 - \sum_{k=1}^{\infty} \varepsilon_k > 0 \, .$$

We assert now that E is a set of uniqueness for $L^q(\Gamma)$. Let g be an integrable function with support included in E for which $\|\hat{g}\|_q < +\infty$. Since $F^{(k)}(x) = 1$ for $x \in E, k \in N$ we have, for γ fixed:

$$\begin{aligned} \hat{g}(\gamma) &= \int_G g(x)\gamma(-x)dx = \int_G F^{(k)}(x)g(x)\gamma(-x)dx \\ &= F^{(k)} * \hat{g}(\gamma) \, , \end{aligned}$$

and, from Hölder's inequality:

$$|\hat{g}(\gamma)| \leqq \|\hat{g}\|_q \cdot \|\hat{F}^{(k)}\|_{q'} \leqq \|\hat{g}\|_q \cdot \varepsilon_k \, .$$

Letting $k \to \infty$ we have that $\hat{g}(\gamma) = 0$ for all $\gamma \in \Gamma$ and thus that $g \equiv 0$.

We also assert that E is a set of multiplicity for $L^p(\Gamma)$. Let $\phi(N, x) = \prod_{k=1}^{N} \phi^{(k)}(x)$ for $x \in G$. Since $L^p(\Gamma)$ is weakly boundedly compact for $p \in (1, 2]$, it follows from (1) that there exists $\lambda(\gamma) \in L^p(\Gamma)$ which is a weak limit point of $\hat{\phi}(N, \gamma)$, $N = 1, 2, \cdots$. By (e) of Lemma 1,

$$m\{x \in G: \phi(N, x) = 1\} \geqq 1 - 2 \sum_k \varepsilon_k > 0 ,$$

and since, for every $x \in G$, $\phi(N, x) \geqq 0$, by (d) of Lemma 1, we see that:

$$\hat{\phi}(N, 0) \geqq 1 - 2 \sum \varepsilon_k , \qquad N = 1, 2, \cdots ,$$

and hence

$$\lambda(0) = 1 - 2 \sum \varepsilon_k > 0 .$$

Therefore $\lambda(\gamma) \not\equiv 0$.

Let h be a function defined on G such that $\hat{h}(\gamma) = \lambda(\gamma)$. Moreover, $h \in L^1(G)$ since it has support of finite measure and, by the Hausdorff-Young theorem, $h \in L^{p'}(G)$. Then h is a nonzero function supported in the intersection of the supports of $\phi(N, x)$, $x \in G$ and $N \in \boldsymbol{N}$, and therefore $(\operatorname{supp} h) \subset E$.

Case (2). G is a Euclidean space $\boldsymbol{R}^n$ for some $n > 0$. We can identify $\boldsymbol{T}$ with the real interval $[-1/2, 1/2)$ and therefore we can identify $\boldsymbol{T}^n$ to be the vector product $[-1/2, 1/2)^n$. Choose now a sequence of real numbers $\{\varepsilon_k\}_{k=1}^{\infty}$ such that $4 \sum_k \varepsilon_k < 1$.

For every ε_k we consider the corresponding function f_k which satisfies Lemmas 1 and 2 in the case of the torus. Thus we can extend periodically the function f_k to all the real line and we call f_k also this (periodic) extension. Now we extend the domain of definition of f_k from $\boldsymbol{R}$ to $\boldsymbol{R}^n$ as follows:

$$f_k^{\dagger}(x_1, \cdots, x_n) = f_k(x_1)$$

and we choose a positive, continuous function P defined on $\boldsymbol{R}^n$ which satisfies:

(1) $0 \leqq P(x_1, \cdots, x_n) \leqq 1$ for any $(x_1, \cdots, x_n) \in \boldsymbol{R}^n$
(2) $P(x_1, \cdots, x_n) = 1$ on $[-1/2 + 1/8, 1/2 - 1/8]^n$
(3) $P(x_1, \cdots, x_n) = 0$ out of $[-1/2, 1/2]^n$
(4) $\hat{P} \in C^{\infty}(\hat{\boldsymbol{R}}^n)$.

By defining F_k as

$$F_k(x_1, \cdots, x_n) = f_k^{\dagger}(x_1, \cdots, x_n) \cdot P(x_1, \cdots, x_n)$$

we have a function which is obtained from $f_k^{\dagger}$ by dropping to zero its value outside of $[-1/2, 1/2]^n$ and which equals $f_k^{\dagger}$ on a smaller cube.

Now we show that the range of the operator T,

$$T: \hat{g} \longrightarrow (g^{\curlyvee} P)^{\wedge}$$

where we define $g^{\curlyvee}(x_1, \cdots, x_n) = g(x_1)$ is contained in $L^s(\hat{\boldsymbol{R}}^n)$, $1 \leqq s \leqq \infty$, for all continuous g on $\boldsymbol{T}$ with $\hat{g} \in L^1(\boldsymbol{Z})$, that is for all $g \in A(\boldsymbol{T})$. Moreover, we show that T is continuous from $L^1(\boldsymbol{Z})$-equipped with $L^s(\boldsymbol{Z})$-norm into $L^s(\hat{\boldsymbol{R}}^n)$, $1 \leqq s \leqq \infty$.

To keep the notation simple we only prove the case when $n = 1$. We also make use of the following formula for $\alpha \in \hat{\boldsymbol{R}}$:

$$(gP)^{\wedge}(\alpha) = \sum_{n=-\infty}^{+\infty} \hat{g}(n)\hat{P}(\alpha - n) .$$

By the Riesz-Thorin theorem, it is sufficient to prove that T is continuous for $p = 1$ and $p = \infty$.
If $p = 1$,

$$\begin{aligned} \|Tg\|_1 = \int_{\boldsymbol{R}} |(gP)^{\wedge}(\alpha)|\, d\alpha &\leqq \sum_{n=-\infty}^{+\infty} |\hat{g}(n)| \left(\sup_{\alpha} \int_{\boldsymbol{R}} |\hat{P}(\alpha - n)|\, d\alpha \right) \\ &\leqq \|\hat{g}\|_1 \cdot \|\hat{P}\|_{L^1(\hat{\boldsymbol{R}})} . \end{aligned}$$

If $p = \infty$,

$$\begin{aligned} \|Tg\|_\infty = \sup_{\alpha} |(gP)^{\wedge}(\alpha)| &\leqq \sup \sum_{n=-\infty}^{+\infty} |\hat{g}(n)| \cdot |\hat{P}(\alpha - n)| \\ &\leqq \|\hat{g}\|_\infty \cdot \sup_{0 \leqq \alpha < 1} \sum_n |\hat{P}(\alpha - n)| ; \end{aligned}$$

the sum $\sum_n |\hat{P}(\alpha - n)|$ is bounded independently of α and therefore

$$\|Tg\|_\infty \leqq C_1 \cdot \|\hat{g}\|_\infty ,$$

where C_1 depends only on P.
Therefore,

$$\|\hat{F}^{(k)}\|_{L^{q'}(\hat{\boldsymbol{R}})} \leqq C_2 \cdot \|\hat{f}_k\|_{1^{q'}}$$

where C_2 depends only on P.
We claim that

$$E = \bigcap_{k=1}^{\infty} \{x \in \boldsymbol{R}^n : F_k(x) = 1\}$$

is a set of positive measure, of q-uniqueness and p-multiplicity.

It is easy to check that

$$m(E) \geqq (1 - \textstyle\sum_k \varepsilon_k - 1/4)(3/4)^{n-1} > 0 .$$

The proof for E that is a set of q-uniqueness is the same as for Theorem 1, Case (1).

To prove the p-multiplicity of E, let $h^{\curlyvee}$ denote the periodic extension to $\boldsymbol{R}^n$—defined in the same way as for $f_k^\vee$ in Case (2)—of the function h found in Case (1), where $G = \boldsymbol{T}$.

Let P be defined on $\boldsymbol{R}^n$ as above. It remains only to show that the function

$$H(x_1, \cdots, x_n) = h^\vee(x_1, \cdots, x_n)P(x_1, \cdots, x_n)$$

is a nonzero function supported by E (obvious), and satisfying $H \in L^p(\hat{\boldsymbol{R}}^n)$.

This last fact follows from the boundness of the operator $T: \hat{g} \to (g^\vee P)^\wedge$ from $L^1(\boldsymbol{Z})$ to $L^p(\hat{\boldsymbol{R}}^n)$, which we have proved above.

Thus the theorem is completed also in Case (2).

Case (3). G is a direct sum of a compact group K and a Euclidean space $\boldsymbol{R}^n$ for $K \neq \{0\}$ and $n > 0$. We can consider the set $E \subset \boldsymbol{R}^n$ found in Case (2) and, by choosing the Haar measure on K such that $m(K) = 1$, it is easy to check that

$$E \oplus K$$

is a set which resolves our theorem. We simply recall that the dual group of a direct sum is a sum of the corresponding dual groups [8, p. 36].

3. A variant.

We consider the case when G is the Cantor group.

DEFINITION. We call Cantor group D, the complete direct sum of $Z(2)_n$ for $n \in N$ where each $Z(2)_n$ is the group of order two. Thus D is the group of all sequences $\{\xi_n\}$, $\xi_n = 0$ or $\xi_n = 1$ with coordinatewise addition modulo 2, and with the topology that makes the mapping

$$\{\xi_n\} \rightsquigarrow 2\sum_{n=1}^{\infty} \xi_n \cdot 3^{-n}$$

a homeomorphism of the group onto the classical Cantor set of the real line. It will be convenient to identify D with the interval [0, 1]. This is done by considering the mapping

$$t: D \longrightarrow [0, 1]$$

$$\{\xi_n\}_{n=1}^{\infty} \rightsquigarrow \sum_{n=1}^{\infty} \xi_n \cdot 2^{-n}$$

which is continuous, invertible a.e. and Haar measure preserving. Because the dual of a group of order two is again the group of order two, the dual of the group D is the direct sum of a sequence of groups of order two. In other words, the generators of Γ (the dual of D), are the functions

$$r_n(x) = (-1)^{\xi_n},$$

and every element of Γ has the form

$$w(x) = \prod_{i=1}^{k} r_{n_i}(x) = (-1)^{\xi_{n_1}+\cdots+\xi_{n_k}},$$

If $t(x)$ is not a dyadic rational number, we can identify x with $t(x)$ and then we can identify the r_n functions with the classical Rademacher functions on $[0, 1]$. It is clear that the harmonic analysis on $L^p(D)$ is the same as on $L^p([0, 1])$, if we identify D with the real interval $[0, 1]$, its Haar measure with the Lebesgue measure, and its characters with the Walsh functions (finite products of Rademacher functions).

We recall now the Paley order for the Walsh functions and hence for the characters of D:

$$\begin{aligned} w_0 &= r_0 \equiv 1 , \\ w_{2^n-1} &= r_n , && \text{for} \quad n \geqq 1 , \\ w_n &= r_{n_1}\cdot \cdots \cdot r_{n_k} && \text{where} \quad n = 2^{n_1-1} + \cdots + 2^{n_k-1} . \end{aligned}$$

Let us set, for f defined on D:

$$\|\hat{f}\|_{p,\alpha} = \Big(\sum_{n\in N} |\hat{f}(w_n)|^p(1+n)^{p\alpha}\Big)^{1/p}$$

where $1 < p < \infty$ and $0 < \alpha < 1 - 1/p$.

If we make the obvious generalization of a set of uniqueness and a set of multiplicity for $L^p(\Gamma)$—given in §1—to the case $L^{p,\alpha}(\Gamma)$, $\alpha \neq 0$, we can prove:

THEOREM 2. *For every real numbers p, q such that $1 < p, q < \infty$, if $p'\alpha < \beta q'$, where $\alpha, \beta \in \boldsymbol{R}$, $0 < \alpha < 1/p'$ and $0 < \beta < 1/q'$, there exists a subset $E \subset D$ of positive measure which is a set of uniqueness for $L^{q,\beta}(\Gamma)$ and a set of multiplicity for $L^{p,\alpha}(\Gamma)$.*

Proof. Given a real number $\varepsilon > 0$ we must find a function F such that
(1) $\|\hat{F}\|_{q',-\beta} \leqq \varepsilon$ where $1/q + 1/q' = 1$,
(2) $m\{x \in D: F(x) = 1\} \geqq 1 - \varepsilon$,
and a function ϕ such that
(3) $\phi(x) \geqq 0$, for any $x \in D$,
(4) $m\{x \in D: \phi(x) = 1\} \geqq 1 - \varepsilon$,
(5) $\log \|\hat{\phi}\|_{p,\alpha} \leqq \varepsilon$.
To this end we define a sequence $\{F_m\}_{m=0}^{\infty}$ of functions:

$$F_m(x) = 1 - \sum_n w_n \qquad \text{for} \quad m = 0, 1, 2, \cdots ,$$

where the sum is taken over the finite set

$$\{0, 1\cdot 2^{km}, 2\cdot 2^{km}, 3\cdot 2^{km}, \cdots, (2^k - 1)2^{km}\}$$

and k is a positive integer to be specified later.

We can give the following equivalent definition for F_0:

$$F_0(x) = \begin{cases} 1 - 2^k & x \in (0, 1/2^k) \\ 1 & \text{elsewhere} . \end{cases}$$

If we extend F_0 to a periodic function F_0^* of period one, defined on all of $\boldsymbol{R}$ and equal to F_0 on $[0, 1]$ we can also define:

$$F_m(x) = F_0^*(2^{km}x) \qquad \text{for} \quad 0 \leqq x \leqq 1 .$$

It is clear that:

$$\int F_m(x)dx = 0 \quad \text{and} \quad m\{x \in D: F_m(x) = 1\} = 1 - 1/2^k .$$

Moreover, from the following relation:

$$\hat{F}_m(w_n) = \begin{cases} 1 & \text{if } n = 0, 1\cdot 2^{km}, 2\cdot 2^{km}, 3\cdot 2^{km}, \cdots, (2^k - 1)2^{km} , \\ 0 & \text{otherwise} , \end{cases}$$

we deduce:

$$\|\hat{F}_m\|_{q',-\beta} = \left(\sum_n (1 + n)^{-\beta q'}\right)^{1/q'} \leqq A 2^{k(-\beta+1/q')}\cdot 2^{km(-\beta)} .$$

Analogously we define a sequence $\{\phi_m\}_{m=0}^{\infty}$ of functions such that

$$\phi_0(x) = 1 ,$$

$$\phi_m(x) = 1 - \frac{1}{2^k} \sum_n w_n(x) , \qquad \text{for} \quad m = 1, 2, \cdots ,$$

where the sum is taken over the set

$$\{0, 1\cdot 2^{k(m-1)}, 2\cdot 2^{k(m-1)}, \cdots, (2^k - 1)2^{k(m-1)}\} .$$

Since the ϕ_m are trigonometric polynomials, it is trivial that:

$$\hat{\phi}_m(w_n) = \begin{cases} 1 - 1/2^k & \text{if} \quad n = 0 \\ 1/2^k & \text{if} \quad n = 2^{k(m-1)}, \cdots, (2^k - 1)2^{k(m-1)} \\ 0 & \text{otherwise} , \end{cases}$$

and so, an easy calculation yields:

$$\log \|\hat{\phi}_m\|_{p,\alpha} \leqq B\cdot 2^{k(1-p)}\cdot 2^{km(p\alpha)} .$$

Moreover, we have

$$m\{x: \phi_m(x) = 1\} \geqq 1 - 1/2^k .$$

Indeed we can give the following equivalent definition for ϕ_1:

$$\phi_1(x) = \begin{cases} 0 & \text{for } x \in (0, 1/2^k) \\ 1 & \text{elsewhere} \end{cases}$$

and notice that, if ϕ_1^* is a periodic extension of ϕ_1 on all the real line, the following relation holds:

$$\phi_m(x) = \phi_1^*(2^{k(m-1)}x) \qquad \text{for any} \quad x, 0 \leqq x \leqq 1 .$$

Now, if we choose k and m such that are solutions of the system:

$$(2) \qquad \begin{cases} A2^{k(1/q'-\beta)} & 2^{km(-\beta)} \leqq \varepsilon \\ B2^{k(1-p)} & 2^{km(p\alpha)} \leqq \varepsilon \end{cases}$$

(we observe that A and B are constants depending on p, q, α, β), F_m and ϕ_m satisfy properties (1), (2), (3), (4), and (5).

These solutions exist because $p'\alpha < \beta q'$, by hypothesis.
Let us define:

$$E = \{x \in D: F_m(x) = 1\} .$$

Choose a sequence of real numbers $\varepsilon_i > 0$, $i = 1, 2, \cdots$, such that $2\sum_i \varepsilon_i < 1$, and repeat the preceding construction for every $\varepsilon = \varepsilon_i$. If we also choose the solutions k and m of (2) in such a way that:

$$k_i > k_{i-1}m_{i-1} ,$$

we can show, by induction, that, for every $N \in \boldsymbol{N}$:

$$\left\|\left(\prod_{i=1}^{N} \phi_i\right)^{\wedge}\right\|_{p,\alpha} \leqq \prod_{i=1}^{N} \|\hat{\phi}_i\|_{p,\alpha} .$$

We assert now that our set of (p, α)-multiplicity and (q, β)-uniqueness is

$$E = \bigcap_{i=1}^{\infty} E_i .$$

The proof of the theorem is now the same as that for Theorem 1, Case (1).

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