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ON THE ASSOCIATIVITY AND COMMUTATIVITY OF ALGEBRAS OVER COMMUTATIVE RINGS

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Let A be an algebra (not necessarily associative) over a commutative ring R. A is left scalar associative if for each $a, b, c \in A$ there exists $\alpha \in R$ depending on a, b, c such that $(ab)c = \alpha a(bc)$. A right scalar associativity is defined similarly. A is scalar commutative if for each a, b in A, there exists $\alpha \in R$ depending on a, b such that $\alpha ab = ba$. In this paper, it is shown that if A is right and left scalar associative and scalar commutative then $(ab)c - a(bc)$ and $ab - ba$ are nilpotent for every a, b and c in A. If $1 \in A$, then $[(ab)c - a(bc)]^2 = 0$. If R is a principal ideal domain then A is associative and commutative.

Introduction. Coughlin and Rich [1] and Coughlin, Kleinfeld and Rich [2] have studied algebras A over a field F with the property that for any $x, y, z \in A$ there exists $\alpha \in F$ depending on x, y, z such that $(xy)z = \alpha x(yz)$. They show that if A has a nonzero idempotent then this condition implies associativity. Rich [4] has shown that if for each $x, y \in A$ there exists $\alpha \in F$ depending on x, y such that $xy = \alpha yx$, then A is either commutative or anti-commutative. In this paper we study related conditions for an algebra A over a comutative ring R. If either A has no zero divisors or if A has identity element and R is a principal ideal domain, then we can still prove associativity and commutativity under the respective conditions. In general with some additional minor constraints we are able to prove the nilpotency of associators and commutators in A.

1. Preliminaries. Throughout this paper N will denote the set of natural numbers and Z^+ will denote the set of positive integers. R will denote a commutative, associative ring which may or may not have an identity element. A will denote a not necessarily associative ring which may or may not have an identity element. We assume that A is an algebra over R in the sense that for all $a, b \in A$ and $\alpha, \beta \in R$, $(\alpha + \beta)a = \alpha a + \beta a$, $\alpha(a + b) = \alpha a + \alpha b$, $(\alpha\beta)a = \alpha(\beta a)$, and $\alpha(ab) = (\alpha a)b = a(\alpha b)$. As usual if $a, b, c \in A$, the *associator* $(a, b, c) = (ab)c - a(bc)$ and the *commutator* $[a, b] = ab - ba$. We will be concerned with the following generalizations of concepts introduced in [1], [4].

DEFINITION 1. A is left scalar associative if for each $a, b, c \in A$ there exists $\alpha \in R$ depending on a, b, c such that $(ab)c = \alpha a(bc)$.

2. A is right scalar associative if for each $a, b, c \in A$ there exists $\alpha \in R$ depending on a, b, c such that $a(bc) = \alpha(ab)c$.

3. A is scalar associative if it is both right and left scalar associative.

4. A is scalar commutative if for each $a, b \in A$ there exists $\alpha \in R$ depending on a, b such that $ab = \alpha ba$.

LEMMA 1.1. (i) *If A is scalar associative, then for all $a, b, c \in A$, $\alpha \in R$, $\alpha a(bc) = 0$ if and only if $\alpha(ab)c = 0$. Also, $(ab)c = 0$ if and only if $a(bc) = 0$.*

(ii) *If A is scalar commutative, then for all $a, b \in A$, $\alpha \in R$, $\alpha ab = 0$ if and only if $\alpha ba = 0$. Also, $ab = 0$ if and only if $ba = 0$.*

LEMMA 1.2. (i) *Suppose A is scalar associative. Let $x, y, z, u \in A$, $\alpha, \beta \in R$ such that $(x, y, u) = 0$, $(xy)z = \alpha x(yz)$ and $x(y(z + u)) = \beta(xy)(z + u)$. Then $xyu - \alpha xyu - \beta xyu + \alpha\beta xyu = 0$.*

(ii) *Suppose A is scalar commutative. Let $x, y, u \in A$, $\alpha, \beta \in R$ such that $yu = uy$, $yx = \alpha xy$ and $(x + u)y = \beta y(x + u)$. Then $yu - \alpha yu - \beta yu + \alpha\beta yu = 0$.*

Proof. (i) We have $x(y(z + u - \alpha\beta z - \beta u)) = 0$ and hence by Lemma 1.1, $(xy)(z + u - \alpha\beta z - \beta u) = 0$. Also $\alpha\beta(xy)(z + u) = (xy)z + \alpha xyu$. Combining these we have the result.

(ii) We have $(x + u - \alpha\beta x - \beta u)y = 0$. Hence by Lemma 1.1, $y(x + u - \alpha\beta x - \beta u) = 0$. Also $\alpha\beta y(x + u) = yx + \alpha yu$. Combining these we have the result.

2. Associators and commutators.

The main purpose of this section is to study nilpotency of associators and commutators.

THEOREM 2.1. *Let A be scalar commutative. If either A has an identity element or is scalar associative, then the square of every commutator in A is zero.*

Proof. Let $x, y \in A$. There exists $\alpha \in R$ such that $yx = \alpha xy$. First assume A has an identity element 1. Then there exists $\beta \in R$ such that $(x + 1)y = \beta y(x + 1)$. By Lemma 1.2 we set $y - \alpha y - \beta y + \alpha\beta y = 0$. Hence $y(x + 1) - \alpha y(x + 1) = \beta y(x + 1) - \alpha\beta y(x + 1) = (x + 1)y - \alpha(x + 1)y$. This implies that $(xy - yx)^2 = (xy - \alpha xy)^2 = 0$.

Next assume A is scalar associative. There exists $\gamma \in R$ such that $(x + y)y = \gamma y(x + y)$. By Lemma 1.2, $y^2 - \alpha y^2 = \gamma y^2 - \alpha\gamma y^2$. Multiplying the first equation by α and then subtracting from itself, we get $xy - \alpha xy = \gamma yx - \alpha\gamma yx$. Now there exists $\delta \in R$ such that $y(yx) = \delta y^2 x$. So $y(xy - \alpha xy) = \gamma y(yx) - \alpha\gamma y(yx) = \delta(\gamma y^2 - \alpha\gamma y^2)x = \delta(y^2 - \alpha y^2)x = y(yx) + \alpha y(yx) = \alpha y(\gamma y) - \alpha^2 y(xy)$. Hence $y(xy - 2\alpha xy + \alpha^2 xy) = 0$. By Lemma 1.1, $(xy - yx)^2 = (xy - \alpha xy)^2 = 0$.

THEOREM 2.2. *Suppose A has an identity element 1 and is scalar associative. Then the square of every associator in A is zero.*

Proof. Let $x, y, z \in A$. There exist $\alpha, \beta \in R$ such that $(xy)z = \alpha x(yz)$ and $x(y(z+1)) = \beta(xy)(z+1)$. By Lemma 1.2, $xy - \alpha xy = \beta xy - \alpha\beta y$. So

$$\begin{aligned}(xy)(z+1) - \alpha(xy)(z+1) &= \beta(xy)(z+1) - \alpha\beta(xy)(z+1)\\ &= x(y(z+1)) - \alpha x(y(z+1)) .\end{aligned}$$

Thus $(x, y, z)^2 = 0$.

THEOREM 2.3. *Suppose A has no zero divisors.*

(i) *If A is scalar commutative then A is commutative.*

(ii) *If A is scalar associative then A is associative.*

Proof. (i) Let $x, y \in A$. There exist $\alpha, \beta \in R$ such that $yx = \alpha xy$ and $(x+y)y = \beta y(x+y)$. By Lemma 1.2, $(y - \alpha y)(y - \beta y) = 0$. So $y = \alpha y$ or $y = \beta y$. In either case $xy = yx$.

(ii) Let $u, v, w \in A$ be nonzero. Let $\alpha \in R$ be such that $uv = \alpha u(vw)$. Let $w' \in A$. We make the following claim.

$$\text{(1)} \qquad \text{If} \quad [w, w'] \neq 0 , \quad \text{then} \quad (uv)w' = \alpha u(vw') .$$

There exists $\gamma \in R$ such that $(uv)w' = \gamma u(vw')$. Suppose (1) is not true. Then $\alpha u \neq \gamma u$. There exists $\beta \in R$ such that $(uv)(w + w') = \beta u(v(w + w'))$. So $\alpha u(vw) + \gamma u(vw') = \beta u(vw) + \beta u(vw')$. Since A has no zero divisors we get $\alpha w + \gamma w' = \beta w + \beta w'$. Hence $(\alpha - \beta)w = (\beta - \gamma)w'$. Thus w' commutes with $(\alpha - \beta)w$ and w commutes with $(\beta - \gamma)w'$. So,

$$\begin{aligned}(\alpha - \beta)[w, w'] &= [(\alpha - \beta)w, w'] = 0 ;\\ (\beta - \gamma)[w', w] &= [(\beta - \gamma)w', w] = 0 .\end{aligned}$$

Hence

$$(\alpha - \beta)u[w, w'] = 0 = (\beta - \gamma)u[w, w'] .$$

Since $[w, w'] \neq 0$ we get $(\alpha - \beta)u = 0 = (\beta - \gamma)u$. Hence $\alpha u = \gamma u$, a contradiction. So (1) is true. Similarly, it can be seen that if $u', v' \in A$, then

$$(2) \qquad \begin{aligned} &[u, u'] \neq 0 \text{ implies } (u'v)w = \alpha u'(vw)\,; \\ &[v, v'] \neq 0 \text{ implies } (uv')w = \alpha u(v'w)\,. \end{aligned}$$

Next we show that for all $x \in A$, $x \cdot x^2 = x^2 \cdot x$. Suppose not. Then there exists $x \in A$ such that $[x, x^2] \neq 0$. So $x \neq 0$ and there exists $\delta \in R$ such that $(x \cdot x)x = \delta x(x \cdot x)$. By (1), (2) we get $(x^2 \cdot x)x = \delta x^2(x \cdot x)$, $(x \cdot x^2)x = \delta x(x^2 \cdot x)$ and $(x \cdot x)x^2 = \delta x(x \cdot x^2)$. So $(x^2 \cdot x)x = \delta x^2(x \cdot x) = \delta(x \cdot x)x^2 = \delta^2 x(x \cdot x^2) = \delta x(\delta x \cdot x^2) = \delta x(x^2 \cdot x) = (x \cdot x^2)x$. This contradiction shows that $x^2 \cdot x = x \cdot x^2$ for all $x \in A$. Now let $a, b, c \in A$, $a, b, c \neq 0$ such that $(a, b, c) = 0$. Let $c' \in A$. There exist $\mu, \nu \in R$ such that $(ab)c' = \mu a(bc')$, $a(b(c' + c)) = \nu(ab)(c' + c)$. By Lemma 1.2 and the fact that A is a domain we get $(a - \mu a)(a - \nu a) = 0$. So $a = \mu a$ or $a = \nu a$. In either case $(a, b, c') = 0$. By duality we can therefore conclude that for $a, b, c \neq 0$, $(a, b, c) = 0$ implies that $(a', b, c) = (a, b', c) = (a, b, c') = 0$ for all $a', b', c' \in A$. Starting with the fact that $x \cdot x^2 = x^2 \cdot x$ for all $x \in A$, we can use the above repeatedly to conclude that A is associative.

If I is an ideal of A which is also a subalgebra, then A/I is also an algebra over R. If A is scalar associative (resp. scalar commutative) then so is A/I. If A is scalar associative and $x \in A$, then by Lemma 1.1, the nth power of x is zero in one association if and only if it is zero in every association. Hence it makes sense to talk about nilpotent elements in A.

THEOREM 2.4. *Let A be scalar associative and scalar commutative. Then every associator and every commutator in A is nilpotent.*

Proof. The hypothesis implies that the nilpotents of A form an ideal and a subalgebra. So without loss of generality, let A have no nilpotent elements. By Theorem 2.1, A is commutative. Call an ideal I of A prime if $ab \in I$ implies $a \in I$ or $b \in I$. We now follow some well known ideas (cf. [3; Chapter 4]). Let $x \in A$, $x \neq 0$. Let T be the groupoid generated by x. Then $0 \notin T$. By Zorn's lemma there exists a maximal ideal P of A not intersecting T. We claim that P is prime. Suppose there exist $a, b \in A$ such that $a, b \notin P$ but $ab \in P$. Let $P_1 = \{na + ra + u \mid n \in N, r \in A, u \in P\}$. Then by commutativity and scalar associativity, P_1 is an ideal of A containing P and a. Hence $P \cap T \neq \emptyset$. So there exists $d_1 = n_1 a + r_1 a \in T$ for some $n_1 \in N$, $r_1 \in A$. Similarly there exists $d_2 = n_2 b + r_2 b \in T$ for some $n_2 \in N$, $r_2 \in A$. So $d_1 d_2 \in T$. By commutativity, scalar associativity and the fact that $ab \in P$, we get $d_1 d_1 \in P$. This contradiction shows that P is a prime ideal. Hence the intersection of all prime ideals is zero. We claim that each prime ideal P is a subalgebra. For let $x \in P$. Then for any $\alpha \in R$, $(\alpha x)(\alpha x) = (\alpha^2 x) \cdot x \in P$. Hence $\alpha x \in P$.

Thus A is a subdirect sum of algebras without zero divisors, each of which scalar associative. By Theorem 2.3, A is associative.

3. Algebras with identity elements. Throughout this section we will assume that R has an identity element 1 satisfying $1 \cdot a = a$ for all $a \in A$ and that A itself has an identity element which will also be denoted by 1.

LEMMA 3.1. *A is right scalar associative if and only if A is left scalar associative if and only if A is scalar associative.*

Proof. By the dual nature of the conditions we may assume A is left scalar associative and prove that A is right scalar associative. Since A is left scalar associative we have that for any $x, y, z \in A$, $x(yz) = 0$ implies $(xy)z = 0$. Now let $a, b, c \in A$. There exist $\alpha, \beta \in R$ such that $(ab)c = \alpha a(bc)$, and $((a+1)b)c = \beta(a+1)(bc)$. Hence $\alpha a(bc) + bc = \beta a(bc) + \beta bc$. Thus $(\alpha a + (1-\beta)\cdot 1 - \beta a)(bc) = 0$. So $((\alpha a + (1-\beta)\cdot 1 - \beta a)b)c = 0$. It follows that $(\alpha - \beta)(ab)c = (\beta - 1)bc = (\alpha - \beta)a(bc)$. So $(\beta - 1)\alpha(bc) = (\alpha - \beta)\alpha a(bc) = (\alpha - \beta)(ab)c = (\beta - 1)bc$. Hence $(\beta - 1)(a+1)(bc) = a[(\beta - 1)bc] + (\beta - 1)bc = a[(\beta - 1)\alpha bc] + (\beta - 1)bc = (\beta - 1)\alpha a(bc) + (\beta - 1)bc = (\beta - 1)(ab)c + (\beta - 1)bc = (\beta - 1)((a+1)b)c$. Consequently, $((a+1)b)c = \beta(a+1)(bc) = (\beta - 1)\cdot(a+1)(bc) + (a+1)(bc) = (\beta - 1)((a+1)b)c + (a+1)(bc)$. Thus $(a+1)(bc) = (2-\beta)((a+1)b)c$. Since a, b, c are arbitrary we obtain that A is right scalar associative.

Suppose R is a P.I.D. (Principal Ideal Domain) and $a \in A$. Then we define order of a, $o(a)$, to be the generator of the ideal $I = \{\alpha \mid \alpha \in R, \alpha a = 0\}$ of R. Thus $o(a)$ is unique up to associates in R. $o(a) = 1$ if and only if $a = 0$.

LEMMA 3.2. *Suppose R is a P.I.D.*

(i) *If A is scalar associative, $a, b \in R$ and $o(ab) = 0$ then $(a, b, c) = 0$ for all $c \in A$.*

(ii) *If A is scalar commutative, $b \in R$ and $o(b) = 0$ then b is in the center of A.*

Proof. (i) Let $c \in A$. There exist $\alpha, \beta \in R$ such that $(ab)c = \alpha a(bc)$ and $a(b(c+1)) = \beta(ab)(c+1)$. By Lemma 1.2, $(1-\alpha)(1-\beta)ab = 0$. So $\alpha = 1$ or $\beta = 1$ and thus $(a, b, c) = 0$.

(ii) Let $b \in A$. There exist $\alpha, \beta \in R$ such that $ba = \alpha ab$ and $(a+1)b = \beta b(a+1)$. So by Lemma 1.2, $(1-\alpha)(1-\beta)b = 0$. Hence $\alpha = 1$ or $b = 1$ and thus $ab = ba$.

LEMMA 3.3. *Suppose R is a P.I.D. and A is scalar associative.*

Assume further that there exists a prime $p \in R$, $m \in Z^+$ *such that* $p^m A = 0$. *Then* A *is associative.*

Proof. Let $\mathscr{A}_1 = \{(x, y) \mid x, y \in A$ and $(x, y, z) = 0$ for all $z \in A\}$, $\mathscr{A}_2 = \{(x, y) \mid x, y \in A$ and $(u, x, y) = 0$ for all $u \in A\}$ and $\mathscr{A} = \mathscr{A}_1 \cap \mathscr{A}_2$. Let $x, y \in A$. Let $o(xy) = p^k$, $k \in N$. We prove by induction on k that $(x, y) \in \mathscr{A}$. If $k = 0$ then $xy = 0$ and by Lemma 1.1, $(x, y) \in \mathscr{A}$. So we assume $k > 0$ and that the statement is true for $l < k$. We first show that for any $z \in A$, $(x, y, z) \neq 0$ implies $(x, yz, w) = 0$ for all $w \in A$. So let $(x, y, z) \neq 0$. There exist $\alpha, \beta \in R$ such that $(xy)z = \alpha x(yz)$ and $((x + 1)y)z = \beta(x + 1)(yz)$. So $\alpha x(yz) + yz = \beta x(yz) + \beta yz$. Hence $(\alpha - \beta)x(yz) = (\beta - 1)yz$. If $(\alpha - \beta)x(yz) = 0$, then $\beta yz = yz$ whence $(x, y, z) = 0$, a contradiction. So $(\alpha - \beta)x(yz) \neq 0$. In particular $\alpha - \beta \neq 0$ and $\alpha - \beta = \delta p^t$ for some $t \in N$, $\delta \in R$, $(\delta, p) = 1$. If $t \geqq k$ then since $o(xy) = p^k$ we would get $(\alpha - \beta)xy = 0$. But then $(\alpha - \beta)(xy)z = 0$ which by Lemma 1.1 implies $(\alpha - \beta)x(yz) = 0$, a contradiction. Hence $t < k$. Now since $p^k(xy)z = 0$ we get $p^k x(yz) = 0$. So $p^{k-t}(\beta - 1)yz = p^k \delta x(yz) = 0$. Let $o(yz) = p^i$. We claim that $i \geqq k$. For suppose $i < k$. Then by induction hypothesis $(y, z) \in \mathscr{A} \subseteqq \mathscr{A}_2$. In particular $(x, y, z) = 0$, a contradiction. So $i \geqq k$. Hence $p^k \mid p^i \mid p^{k-t}(\beta - 1)$. Thus $p^t \mid \beta - 1$ and $\beta - 1 = p^t \gamma$ for some $\gamma \in R$. So $p^t[\delta x(yz) - \gamma yz] = 0$. Therefore $p^t[(\delta x - \gamma \cdot 1)(yz)] = 0$. By induction hypothesis $(\delta x - \gamma \cdot 1, yz) \in \mathscr{A} \subseteqq \mathscr{A}_1$. Hence for all $w \in A$, $(\delta x - \gamma \cdot 1, yz, w) = 0$. Since $(\gamma \cdot 1, yz, w) = 0$ we get $\delta(x, yz, w) = (\delta x, yz, w) = 0$. Since $(\delta, p) = 1$, there exist $\mu, \nu \in R$ such that $\mu p^m + \nu \delta = 1$. So $(x, yz, w) = \nu \delta(x, yz, w) = 0$. Thus we have shown that

$$(3) \qquad (x, y, z) \neq 0 \text{ implies } (x, yz, w) = 0 \text{ for all } w \in A .$$

Now we proceed to show that $(x, y) \in \mathscr{A}_1$. For suppose not. Then $(x, y, u) \neq 0$ for some $u \in A$. Then we also have $(x, y, u + 1) \neq 0$. So by (3) we get $(x, yu, w) = 0 = (x, yu + y, w)$ for all $w \in A$. This implies $(x, y, w) = 0$ for all $w \in A$. In particular $(x, y, u) = 0$, a contradiction. This contradiction shows that $(x, y) \in \mathscr{A}_1$. A right-left dual argument can be given to show that $(x, y) \in \mathscr{A}_2$. Hence $(x, y) \in \mathscr{A}$. This completes the induction step.

LEMMA 3.4. *Suppose* R *is a P.I.D. and* A *is scalar commutative. Assume fuyther that there exists a prime* $p \in R$, $m \in Z^+$ *such that* $p^m A = 0$. *Then* A *is commutative.*

Proof. Let C be the center of A. Let $x \in A$ and $o(x) = p^k$, $k \in N$. We prove by induction on k that $x \in C$. If $k = 0$, then $x = 0$ and there is nothing to prove. So let $k > 0$ and assume the statement for $l < k$. We first show that for any $y \in A$, $[x, y] \neq 0$ implies $[yx, y] = 0$. So let $[x, y] \neq 0$. There exist $\alpha, \beta \in R$ such that $xy = \alpha yx$ and $(x+1)y = \beta y(x+1)$. So $\alpha yx + y = \beta yx + \beta y$. Hence $(\alpha - \beta)yx = (\beta - 1)y$. If $(\alpha - \beta)yx = 0$, then $\beta y = y$ and $[x, y] = 0$, a contradiction. So $(\alpha - \beta)yx \neq 0$. In particular $\alpha - \beta \neq 0$ and $\alpha - \beta = p^t\delta$ for some $t \in N$, $\delta \in R$ with $(\delta, p) = 1$. If $t \geqq k$, then $(\alpha - \beta)yx = 0$, a contradiction. So $t < k$. Then $p^{k-t}(\beta - 1)y = p^{k-t}(\alpha - \beta)yx = p^k\delta yx = 0$. Let $o(y) = p^i$, $i \in N$. If $i < k$ then by induction hypothesis $y \in C$, a contradiction. So $i \geqq k$. Hence $p^k | p^i | p^{k-t} \times (\beta - 1)$. So $p^t | \beta - 1$ and $\beta - 1 = p^t\gamma$ for some $\gamma \in R$. So $p^t(\delta yx - \gamma y) = 0$ and by induction hypothesis $\delta yx - \gamma y \in C$. In particular $[\delta yx - \gamma y, y] = 0$. Since $[\gamma y, y] = 0$ we get $\delta[yx, y] = [\delta yx, y] = 0$. Since $(\delta, p) = 1$ there exist $\mu, \nu \in R$ such that $\mu\delta + \nu p^m = 1$. So we get $[yx, y] = \mu\delta[yx, y] = 0$. Thus we have shown that for all $y \in A$,

$$(4) \qquad [x, y] \neq 0 \quad \text{implies} \quad [yx, y] = 0 .$$

Now we proceed to show that $x \in C$. Suppose not. Then there exists $u \in A$ such that $[x, u] \neq 0$. So $[x, u+1] \neq 0$. By (4) we get $[ux, u] = [ux + x, u + 1] = 0$. Hence $[x, u] = 0$, a contradiction. This proves $x \in C$ completing the induction step.

THEOREM 3.5. *Suppose R is a P.I.D.*
(i) *If A is scalar associative, then A is associative.*
(ii) *If A is scalar commutative, then A is commutative.*

Proof. (i) Suppose A is not associative. We will get a contradiction. There exist $x, y, z \in A$ such that $(x, y, z) \neq 0$. So $(x+1, y, z) \neq 0$. By Lemma 3.2, $o(xy) \neq 0$, $o(xy + y) \neq 0$. Thus $o(y) \neq 0$. Since $(x, y+1, z) \neq 0$, the above argument shows that $o(y+1) \neq 0$. Hence $o(1) \neq 0$. Let $o(1) = d \neq 0$. Then d is not a unit and hence $d = p_1^{i_1} \cdots p_t^{i_t}$ for some primes $p_1, \cdots, p_t \in A$ and some positive integers $i_1, \cdots, i_t$. Let $A_t = \{a \mid a \in A, p_i^{\alpha_i}a = 0\}$. Then each A_i is a nonzero subalgebra of A and $A = A_1 \oplus \cdots \oplus A_t$. Being subalgebras of A, the A_i's are scalar associative. Being homomorphic images of A, all the A_i's have identity elements. By Lemma 3.3, each A_i and hence A is associative, a contradiction.

(ii) Suppose A is not commutative. We will obtain a contradiction. There exists $x \in A$ such that $x \notin C$, the center of A. So $x + 1 \notin C$. By Lemma 3.2, $o(x) \neq 0$ and $o(x+1) \neq 0$. Hence $o(1) \neq 0$. By using Lemma 3.4, we obtain as in (i), that A is commutative, a contradiction.

EXAMPLE 3.6. If R is a field and A is scalar commutative then even if A does not have an identity element, Rich [4] shows that

A must be either commutative or anti-commutative. This is not true for P.I.D.'s. To see this let A_1 be the non-commutative, anti-commutative algebra over Z_5 given in [4]. As a Z-algebra A_1 satisfies $xy = 4yx$. Now Z_3 as a Z-algebra satisfies the same identity. So the Z-algebra $A = A_1 \times Z_3$ satisfies $xy = 4yx$. However A is not commutative (since A_1 is not) and A is not anti-commutative (since Z_3 is not).

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