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THE SPLITTING OF EXTENSIONS OF SL(3, 3) BY THE VECTOR SPACE F_3^3

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We give two proofs that $H^2(\mathrm{SL}(3,3), F_3^3) = 0$. This result has appeared in a paper by Sah, [6], but our methods are relatively elementary, i.e., we require only elementary homological algebra and do a group-theoretic analysis of an extension of SL (3, 3) by F_3^3 to show that the extension splits. The starting point is to notice that the vector space is a free module for $F_3(\langle x\rangle)$, where x has Jordan canonical form $\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}$. We then can exploit the vanishing of $H^i(\langle x\rangle, F_3^3)$ $i=1, 2$.

For elementary linear algebra, we refer to [2] and for cohomology of groups, we refer to [1], [4], [5] or [6]. Group theoretic notation is standard and follows [3]. Let V be a 3-dimensional F_3-vector space and let SL(3, 3) be the associated special linear group. Let v_1, v_2, v_3 be a basis for V. Define, for $i, j \in \{1, 2, 3\}$, $i \neq j$, and $t \in F_3$, $x_{ij}(t) \in \mathrm{SL}(3,3)$ by

$$x_{ij}(t)\colon v_k \longmapsto \begin{cases} v_k & k \neq i \\ v_i + tv_j, & k = i\ . \end{cases}$$

Inspection of the Jordan canonical form shows that all $x_{ij}(t)$, $t \neq 0$, are conjugate in $\mathrm{GL}(3,3) = \{\pm I\} \times \mathrm{SL}(3,3)$, hence in SL(3, 3).

Set $G = \mathrm{SL}(3,3)$. We let

$$(*) \qquad 1 \longrightarrow V \longrightarrow G^* \xrightarrow{\ \pi\ } G \longrightarrow 1$$

be an arbitrary extension of G by V with the above action. We will show (*) is split. We use the convention that $u^* \in G^*$ is a representative (arbitrary, unless otherwise specified) for $u \in G$.

The alternate proof of splitting (given later) is much neater than the first version. The methods are quite different, however, and it seems worthwhile to give two proofs.

LEMMA 1. *Let $x = x_{12}(1)x_{23}(1)x_{13}(-1)$. Then $C_G(x) = \langle x, x_{13}(1)\rangle$. If $t \in G$ is an involution which inverts x, then t centralizes $x_{13}(1)$.*

Proof. The first statement is elementary linear algebra. Namely, x has a cyclic vector in V, so that any transformation which commutes with x is a polynomial in x. Since x has minimal polynomial of degree 3, its full commuting algebra is all matrices of the shape

$$\begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}, \quad a, b, c \in \boldsymbol{F}_3 .$$

The first statement is now clear. As for the second, it suffices to display an element t with the required properties, e.g.

$$t = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The lemma is proven.

LEMMA 2. *$x_{12}(1)x_{23}(1)x_{13}(-1)$ and all its conjugates are represented in G^* by elements of order 3. Any two such representatives are conjugate by an element of V.*

Proof. The Jordan canonical form for $x = x_{12}(1)x_{23}(1)x_{13}(-1)$ indicates that V is a free $\boldsymbol{F}_3\langle x\rangle$-module. So $H^i(\langle x\rangle, V) = 0$ for $i \geqq 1$. Both statements follow.

LEMMA 3. *Each $x_{ij}(t)$ is represented in G^* by an element of order 3, which commutes with an involution of G^*.*

Proof. We may assume $i = 1, j = 3, t = 1$. Let

$$x = x_{12}(1)x_{23}(1)x_{13}(-1) ,$$

and let $x^* \in G^*$ represent x, $|x^*| = 3$. Again by Lemma 2, a Frattini—like argument shows that $N_G(\langle x\rangle)^* = V \cdot N_{G^*}(\langle x^*\rangle)$. Choose $y \in N_{G^*}(\langle x^*\rangle)$ with $y^\pi = x_{13}(1)$. Then $C_{G^*}(x^*) = \langle x^*, y, v_3\rangle$ is abelian. Let $t \in N_{G^*}(\langle x^*\rangle)$ be an involution inverting x^*. Then by Lemma 1, t^π centralizes $x_{13}(1)$ and inverts v_3. By Fittings theorem.

$$C_{G^*}(x^*) = \langle y_1\rangle \times \langle x, v_3\rangle$$

where $\langle y_1\rangle = C_{G^*}(\langle x^*, t\rangle)$. Clearly $|y_1| = 3$ and $1 \neq y_1^\pi \in \langle x_{13}(1)\rangle$. This proves the lemma.

LEMMA 4. *If t is an involution of G^*, $C_{G^*}(t)$ has a Sylow 3-subgroup isomorphic to $\boldsymbol{Z}_3 \times \boldsymbol{Z}_3$.*

Proof. Since G has one class of involutions, so does G^*. So, we apply Lemma 3 to see that t centralizes an element of order 3 outside V. Since $|C_V(t)| = 3$ and $C_G(t^\pi) \cong \mathrm{GL}(2, 3)$, we are done by the Frattini argument namely, $\langle t\rangle \in \mathrm{Syl}_2(V\langle t\rangle)$ and $V\langle t\rangle \lhd H$, where H is the preimage in G^* of $C_G(t^\pi)$.

In what follows, let $R = N_{G^*}(\langle v_3\rangle)$ and $Q = O_3(R)$. Then

$$R^\pi = \left\{\begin{pmatrix} A & & a \\ & & b \\ 0 & 0 & c\end{pmatrix} \middle| A \in \mathrm{GL}(2, 3),\ a, b \in F_3,\ c = (\det A)^{-1}\right\}$$

$$Q^\pi = \left\{\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1\end{pmatrix} \middle| a, b \in F_3\right\}.$$

Let

$$h = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1\end{pmatrix}$$

and let—denote images under $R \to R/\langle v_3\rangle$. Let $h^* \in R$ be an involution representing h.

Lemma 5. *$\bar{Q}$ is inverted by h^*. Also, $\bar{Q}$ is elementary abelian and Q is extra special of order 3^5, exponent 3, with center $\langle v_3\rangle$.*

Proof. The first statement is clear since h inverts Q^π and $\langle v_1, v_2\rangle$. Therefore, $\bar{Q}$ is abelian. From Lemma 3, we get that $\bar{Q}$ is elementary and the action of members of Q^π on V implies that Q is extra special. Since Q^π is generated by elements of order 3, by Lemma 3 again, Q has exponent 3.

We now require a technical result for studying automorphisms of Q. Since automorphisms commute with commutation, we have a homorphism (which is actually onto) $\mathrm{Aut}(Q) \to \mathrm{Sp}_0(4, 3)$, the group of similitudes of a nondegenerate alternating bilinear form from F_3^4 to F_3 (a similitude preserves the form up to a scalar multiple; we have $|\mathrm{Sp}_0(4, 3)\colon \mathrm{Sp}(4, 3) = |F_3^\times| = 2$, where $\mathrm{Sp}(4, 3)$ is the symplectic group, i.e. the group preserving the form).

Lemma 6. *Let M be a 4-dimensional F_3-vector space supporting a nondegenerate alternating form $(\,,)$ and let* $\mathrm{Sp}_0(4, 3)$, $\mathrm{Sp}(4, 3)$ *be the associated group of similitudes, resp. symplectic group. Let I be a maximal totally isotropic subspace and let K be its* (global) *stabilizer in* $\mathrm{Sp}_0(4, 3)$. *Then*

(i) $\dim I = 2$

(ii) *If J is a maximal totally isotropic subspace complementing*

I in M we may choose a basis a_1, b_1 for I, a_2, b_2 for J so that $(a_i, b_j) = \delta_{ij}$ and $(a_i, a_j) = (b_i, b_j) = 0$. With respect to the basis $\{a_1, a_2, b_1, b_2\}$ for V, elements of K have the shape

$$\begin{pmatrix} A & B \\ 0 & c\,{}^tA^{-1} \end{pmatrix},$$

$A \in \mathrm{GL}(2, 3)$, B a symmetric 2×2 matrix, $c \in \boldsymbol{F}_3^\times$; $c = 1$ if and only if the matrix lies in Sp(4, 3). *In this notation, $0_3(K)$ consists of those matrices with $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and if L is the set of matrices with $B = 0$, L complements $0_3(K)$ in K.*

(iii) *If $Y \in \mathrm{Syl}_3(L)$, $0_3(K)$ is a free $\boldsymbol{F}_3 Y -$ module.*

(iv) *Any subgroup of K meeting $0_3(K)$ trivially stabilizes a maximal totally isotropic subspace which complements I, and is in fact conjugate to a subgroup of L.*

Proof. Statements (i) and (ii) are straightforward. To prove (iii), we may assume $Y = \langle y \rangle$,

$$y = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

Take

$$k(\alpha, \beta, \gamma) = \begin{pmatrix} 1 & 0 & \alpha & \beta \\ 0 & 1 & \beta & \gamma \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

a typical element of $0_3(K)$. A matrix calculation show that $y^{-1}k(\alpha, \beta, \gamma)y = k(\alpha - 2\beta + \gamma, \beta - \gamma, \gamma)$. To show $0_3(K)$ is a free Y-module, it suffices, since $0_3(K) \cong \boldsymbol{Z}_3 \times \boldsymbol{Z}_3 \times \boldsymbol{Z}_3$, to find a triple (α, β, γ) such that the three elements $y^{-i}k(a, \beta, \gamma)y^i$, $i = 0, 1, 2$ are linearly independent. Any (α, β, γ) with $\beta \neq 0$ does the trick.

We now prove statement (iv). First (iii) implies that $H^i(Y, 0_3(K)) = 0$ for $i \geqq 1$. Secondly, if $X \leqq K$, $X \cap 0_3(K) = 1$, then a Sylow 3-subgroup X_3 of X is conjugate to a subgroup of Y, and so $H^i(X_3, 0_3(K)) = 0$ for $i \geqq 1$. Finally, we quote the injectiveness of the restriction $H^i(X, 0_3(K)) \to H^i(X_3, 0_3(K))$. A consequence is that X is conjugate in $0_3(K)X$ to $L \cap 0_3(K)X$, whence X stabilizes a maximal totally isotropic subspace complementing I.

THEOREM. *The extension* (*) *is split. Consequently,* $H^2(\mathrm{SL}(3, 3), F_3^3) = 0$.

Proof. Let $S \cong \mathrm{GL}(2, 3)$ complement $\langle v_3 \rangle$ in $C_{G^*}(h^*)$ (use Lemma 4 and Gaschütz' theorem). Easily, we see that S is faithful on Q and the map $\mathrm{Aut}\,(Q) \to \mathrm{Sp}_0(4, 3)$ embeds S as a sugroup S_0 of K, where, in the notation of Lemma 6, $M = \bar{Q}$, $I = \bar{V}$. Also, $0_3(S) = 1$ implies $0_3(K) \cap S_0 = 1$. Hence, by Lemma 6 (iv), S_0 stabilizes a complement $\bar{J}$ to $\bar{V}$ in $\bar{Q}$, where $\bar{J}$ is totally singular. Letting J be the preimage of $\bar{J}$ in Q, J is elementary abelian. Since h^* inverts $\bar{J}$ and centralizes v_3, we $J = \langle v_3 \rangle \times [J, h^*]$. Then $[J, h^*]S$ complements V in R. Since $(|G\colon R^\pi|, 3) = 1$, Gaschütz theorem implies that G^* splits over V, as required.

An alternate proof was suggested by V. Landazuri in a conversation. We sketch the argument. Using Lemmas 2 and 3, we get

(i) every element of order 3 in G is represented in G^* by an element of order 3.

Let $y \in G^*$ represent $x_{ij}(1)$, $|y| = 3$. Since $[V, y, y] = 1$, a simple calculation shows

(ii) every element of the coset $Vx_{ij}(1)^* = Vy$ has order 3.

Now take $a, b \in U^*$, $a^\pi = x_{12}(1)$, $b^\pi = x_{12}(-1)x_{23}(1)$, $|b| = 3$ (using (i)). By (ii), $|a| = |ab| = |ba| = 3$. An elementary argument shows that, if ξ_1, ξ_2 are elements in any group such that $|\xi_1| = |\xi_2| = |\xi_1\xi_2| = 3$, then $\langle \xi_1\xi_2^{-1}, \xi_2^{-1}\xi_1 \rangle$ is a normal abelian subgroup of index 3 in $\langle \xi_1, \xi_2 \rangle$. Applying this to $\xi_1 = ab$, $\xi_2 = ba$ we see that $\langle a, b \rangle$ has a normal abelian subgroup $H = \langle [a^{-1}, b^{-1}], [a, b] \rangle$ of index 3. By (ii), H is elementary abelian. Therefore, $|\langle a, b \rangle| = 3^3$. It is easily seen that $\langle a^\pi, b^\pi \rangle = U$, and this means $\langle a, b \rangle \cap V = 1$. Our theorem now follows from Gaschütz' theorem.

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