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FORMULAS FOR THE NEXT PRIME

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In 1971, J. M. Gandhi showed that if the first n primes, $p_1, p_2, \cdots, p_n$ are known, then the next prime, p_{n+1}, is given "explicitly" by the formula:

$$(1) \qquad 1 < b^t\left(\sum_{d|P_n} \frac{\mu(d)}{b^d - 1} - \frac{1}{b}\right) < b\,,$$

where b is any positive integer $\geqq 2$, where $P_n = p_1 p_2 \cdots p_n$, where $\mu(d)$ is the Möbius function, and where the unique integer value of t which satisfies the indicated inequalities is in fact p_{n+1}.

In this paper, we obtain of the following formulas for p_{n+1}:

$$(2) \qquad p_{n+1} = \lim_{s\to\infty} \{P_n(s)\zeta(s) - 1\}^{-1/s}$$

$$(3) \qquad p_{n+1} = \lim_{s\to\infty} \{P_n(s) - \zeta^{-1}(s)\}^{-1/s}$$

$$(4) \qquad p_{n+1} = \lim_{s\to\infty} \{\zeta(s) - Q_n(s)\}^{-1/s}$$

and

$$(5) \qquad p_{n+1} = \lim_{s\to\infty} \{1 - \zeta^{-1}(s)Q_n(s)\}^{-1/s}\,.$$

Here $\zeta(s) = \sum_{n=1}^\infty n^{-s}$ for (real) $s > 1$ is the Riemann Zeta Function, with $\zeta^{-1}(s) = \sum_{n=1}^\infty \mu(n)/n^s$; $P_n(s) = \prod_{p_i|P_n}(1 - p_i^{-s})$, and $Q_n(s) = \{P_n(s)\}^{-1} = \sum_{n=1}^{\prime\infty} n^{-s}$, where the *prime* indicates that summation is extended over those values of n having no prime factors exceeding p_n.

The approach to be followed here involves the derivation of a more general formula, based on the notion of probability distributions on the positive integers, from which both the Gandhi formula and the new formulas listed above follow as special cases.

2. Probability formulas for the integers. Let $\alpha(n)$ be a *probability function* on the positive integers. That is, $\alpha(n) \geqq 0$ for all $n = 1, 2, 3, \cdots$, and $\sum_{n=1}^\infty \alpha(n) = 1$.

Let $\beta(m) = \sum_{n=1}^\infty \alpha(mn)$. In the probability distribution D determined by $\{\alpha(n)\}$, $\beta(m)$ is the *probability that a randomly chosen integer is a multiple of m*. Next, let $\gamma(k) = \sum_{d|k} \mu(d)\beta(d)$. Then $\gamma(k)$ is the probability (in D) that a randomly chosen integer is relatively prime to k, because

$$\gamma(k) = 1 - \sum_{p_i|k} \beta(p_i) + \sum_{p_i p_j|k} \beta(p_i p_j) - + \cdots\,.$$

Let $P_n = p_1 p_2 \cdots p_n$ be the product of the first n primes. Then

$\gamma(P_n) = \sum_{d|P_n} \mu(d)\beta(d)$ by the definition of $\gamma(k)$; but also

$$(6) \qquad \gamma(P_n) = \alpha(1) + \alpha(p_{n+1}) + \cdots = \sum_{j=1}^{\infty}{}'' \alpha(j) ,$$

where $\sum''$ indicates summation over all positive integers divisible by none of the first n primes.

THEOREM 1. *Suppose* $1 \leqq n_1 < n_2 < n_3 < \cdots$ *is any subsequence of the positive integers, and there exists an operator* T *such that*

$$T\left(\sum_{i=1}^{\infty} \alpha(n_i)\right) = n_1$$

for all such subsequences. Then

$$(7) \qquad T(\gamma(P_n) - \alpha(1)) = p_{n+1}$$

is a "formula" for the next prime, p_{n+1}.

Proof. Since $\gamma(P_n) - \alpha(1) = \sum''_{j>1} \alpha(j) = \alpha(p_{n+1}) + \cdots$, we have $T(\gamma(P_n) - \alpha(1)) = p_{n+1}$ by the hypothesis concerning the operator T.

Another general result is given by:

THEOREM 2.

$$(8) \qquad \gamma(0) = \lim_{n\to\infty} \gamma(P_n) = \sum_{d=1}^{\infty} \mu(d)\beta(d) = \alpha(1) .$$

Proof. This follows directly from

$$\gamma(P_n) = \sum_{d|P_n} \mu(d)\beta(d) = \sum_{j=1}^{\infty}{}'' \alpha(j) .$$

As we shall see, Theorem 2 is a generalization of Euler's product formula for the Zeta Function.

3. Some special cases. Suppose $\alpha(n) = (b-1)b^{-n}$ for $n = 1, 2, 3, \cdots$ where $b > 1$ is a positive integer. This is a geometric distribution on the positive integers. Then

$$\beta(m) = \sum_{n=1}^{\infty} \alpha(mn) = \frac{b-1}{b^m - 1} ,$$

and

$$\gamma(k) = \sum_{d|k} \mu(d)\beta(d) = (b-1)\sum_{d|k} \frac{\mu(d)}{b^d - 1} .$$

In particular, $\gamma(P_n) = (b-1)\sum_{d|P_n}\mu(d)/(b^d-1)$, and

$$\gamma(P_n) - \alpha(1) = (b-1)\Big(\sum_{d|P_n}\frac{\mu(d)}{b^d-1} - \frac{1}{b}\Big) = (b-1)\Big\{\frac{1}{b^{p_{n+1}}} + \cdots\Big\} ,$$

and to recover p_{n+1} it suffices to divide by $b-1$, and then multiply by the smallest power b^t of b, t an integer, such that

$$b^t\Big(\sum_{d|P_n}\frac{\mu(d)}{b^d-1} - \frac{1}{b}\Big) > 1 .$$

This is Gandhi's Formula (1). (For other derivations, see [1] and [2].)

Alternatively, let $\alpha(n) = n^{-s}/\zeta(s)$ for a fixed real value of s, $s > 1$. (Note that $\sum_{n=1}^{\infty}\alpha(n) = (\sum_{n=1}^{\infty} n^{-s})/\zeta(s) = 1$.) Then $\beta(m) = \sum_{n=1}^{\infty}\alpha(mn) = m^{-s}$, and $\gamma(k) = \sum_{d|k}\mu(d)d^{-s} = \sum_{p|k}(1-p^{-s})$. Specifically, $\gamma(P_n) = \sum_{d|P_n}(\mu(d))/d^s = \prod_{i=1}^{n}(1-p_i^{-s}) = P_n(s)$. Note also that $\alpha(1) = 1/\zeta(s)$. Thus $\gamma(P_n) - \alpha(1) = (p_{n+1})^{-s} + \cdots$, and an appropriate operator T to recover the term p_{n+1}, in the sense of Theorem 1, is $T = \lim_{s\to\infty}(\quad)^{-1/s}$. Thus

$$P_{n+1} = \lim_{s\to\infty}\{P_n(s) - \zeta^{-1}(s)\}^{-1/s} . \tag{3}$$

Each of the formulas (2), (3), (4), (5) can be given a direct interpretation. Thus

$$P_n(s)\zeta(s) - 1 = \sum\nolimits_1 a^{-s} \tag{9}$$

$$P_n(s) - \zeta^{-1}(s) = -\sum\nolimits_2 \mu(a)a^{-s} \tag{10}$$

$$\zeta(s) - Q_n(s) = \sum\nolimits_2 a^{-s} \tag{11}$$

$$1 - \zeta^{-1}(s)Q_n(s) = -\sum\nolimits_1 \mu(a)a^{-s} \tag{12}$$

where $\sum_1$ indicates summation over those integers $a > 1$ *all* of whose prime factors exceed p_n; where $\sum_2$ indicates summation over those integers $a > 1$ having at least one prime factor exceeding p_n; and where $\mu(a)$ is the Möbius function. In all four of these expressions, the first surviving term in p_{n+1}^{-s}, which is recovered by the inversion operator T to yield the formulas (2), (3), (4), and (5).

For the case $\alpha(n) = n^{-s}/\zeta(s)$, Theorem 2 yields the identity

$$\gamma(0) = \prod_{i=1}^{\infty}(1-p_i^{-s}) = \sum_{d=1}^{\infty}\mu(d)\cdot d^{-s} = 1/\sum_{n=1}^{\infty} n^{-s} \tag{13}$$

which includes the Euler Product Formula for the Zeta Function (cf. [3]).

The reader is invited to find other distributions on the positive integers for which Theorems 1 and 2 yield interesting formulas. A simpler proof of Gandhi's formula was given by the present author in 1974.

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