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GENERA IN NORMAL EXTENSIONS

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Let K/F be a finite normal extension of algebraic number fields and let C_K be the ideal class group of K. There are two fundamentally different ways to define the principal genus of C_K with respect to F. Classically the principal genus is described by norm residue symbols. By the modern definition it is the class group of the maximal unramified extension of K which is the composite of K with an abelian extension of F. It is shown here that the two definitions are equivalent.

Let F be a finite algebraic number field and K a finite normal extension of F with $G = \mathrm{Gal}(K/F)$. Let $\bar{K}$ be the Hilbert class field of K and let C_K be the ideal class group of K. By class field theory the fields lying between K and $\bar{K}$ are in one-one correspondence with the subgroups of C_K. (See [3] or [4] for the class field theory involved.) Let L be the genus field for K/F. As defined by Fröhlich ([1]), L is the composite of K with the maximal abelian extension of F in $\bar{K}$. Calling this maximal abelian extension E, we have $\bar{K} \supseteqq L = KE \supseteqq K$, $E \supseteqq F$ and $K \cap E$ is the maximal abelian extension of F in K. The subgroup of C_K corresponding to L is the principal genus of C_K. Gauss's definition of the principal genus is based on arithmetic characters. In [2] we showed that when G is abelian the two definitions are equivalent. Here we will show that in fact they are equivalent for any G.

Let C_F be the ideal class group of F and let $N_{K/F}: C_K \rightarrow C_F$ be the norm map on ideal class groups. Let $\bar{F}$ be the Hilbert class field of F and ${}_NC_K$ the kernel of the norm map. Then the subgroup ${}_NC_K$ of C_K corresponds to the extension $K\bar{F}$ of K. Clearly $L \supseteqq K\bar{F}$ and, letting H denote the principal genus of C_K, we see that ${}_NC_K \supseteqq H$.

We now proceed to describe the characters in Gauss's definition. Let $P_1, \cdots, P_t$ be the primes of K, finite or infinite, ramified in K/F. For each i choose a prime $\bar{P}_i$ in $\bar{K}$ such that $\bar{P}_i \cap K = P_i$. This allows a consistent choice of primes in each subfield k by $P_{k,i} = \bar{P}_i \cap k$. And we will denote the completed localization of k at $P_{k,i}$ by k_i. In particular we have the chain $\bar{K}_i \supseteqq L_i \supseteqq K_i$, $E_i \supseteqq F_i$ of local fields. For an ideal $\mathfrak{A}$ of a field k let $[\mathfrak{A}]$ denote the ideal class of $\mathfrak{A}$. Now let $\mathfrak{A}$ be an ideal of K such that $[\mathfrak{A}] \in {}_NC_K$. Thus $N_{K/F}(\mathfrak{A})$ is a principal ideal of F, say $N_{K/F}(\mathfrak{A}) = (a)$, $a \in F$. For each i we have a norm residue symbol $((K_i/F_i)/a)$ which we will also write $((a, K/F)/P_i)$ or most simply $\chi_i(a)$. This symbol is an element of the local group $\mathrm{Gal}(K_i/F_i)$ modulo its commutator. We may identify $\mathrm{Gal}(K_i/F_i)$ with the decomposition group Z_i of P_i in K/F. Thus we have a homomorphism $\chi_i: F^\times \to Z_i/[Z_i, Z_i] = Z_i^{ab}$. Let $\chi: F^\times \to \prod_{i=1}^t Z_i^{ab}$ by $\chi(a) = (\chi_1(a), \cdots, \chi_t(a))$. Let U_F denote the units of F, P_F the principal ideals of F, and $S = \chi(U_F)$. Then χ induces a homomorphism which we'll also denote by $\chi: P_F = F^\times/U_F \to \prod_{i=1}^t Z_t^{ab}/S$. Let $(a) \in P_F$ and $(a) = (N_{K/F}(b)) = N_{K/F}((b))$. Then $a = \varepsilon \cdot N_{K/F}(b)$ for some $\varepsilon \in U_F$ and $\chi_i(a) = \chi_i(\varepsilon)\chi_i(N_{K/F}(b)) = \chi_i(\varepsilon)$ for $i = 1, \cdots, t$ since $\chi_i(N_{K/F}(b)) = ((K_i/F_i)/N_{K/F}(b))$ and every global norm is a local norm everywhere. It follows that $\chi: P_F \to \prod Z_i^{ab}/S$ vanishes on $N_{K/F}(P_K) \subseteq P_F$. Note that $N_{K/F}: {}_N C_K \to P_F/N(P_K)$ since if $[\mathfrak{A}] = [\mathfrak{B}] \in {}_N C_K$ then $\mathfrak{A} = (\alpha)\mathfrak{B}$ and $N(\mathfrak{A}) = N((\alpha))N(\mathfrak{B}) \in P_F$. Now we can define $f = \chi \circ N_{K/F}: {}_N C_K \to P_F/N(P_K) \to \prod Z_i^{ab}/S$. The formal statement of the equivalence of the two definitions of principal genus is given by

THEOREM. *Let K/F be a finite normal extension of number fields and let H be the principal genus in the sense of Fröhlich. Let $f: {}_N C_K \to \prod_{i=1}^t Z_i^{ab}/S$ be the modified product of local norm residue symbols described above. Then* $H = \mathrm{Ker}(f)$.

Proof. First we show that $\mathrm{Ker}(f) \subseteq H$. Let P be a prime ideal of K, $P \neq P_i$, $i = 1, \cdots, t$; $[P] \in \mathrm{Ker}(f)$, and P of absolute degree 1. Since $[P] \in \mathrm{Ker}(f)$ and P is of degree 1, $N_{K/F}(P) = \mathfrak{p} = (\rho)$ where $\mathfrak{p} = P \cap F$ and $\rho \in F$. Moreover ρ may be chosen so that $\chi_i(\rho) = 1$, $i = 1, \cdots t$, since ρ times any unit of F generates $\mathfrak{p}$ and $[P] \in \mathrm{Ker}(f)$ implies $\chi(\rho) = \chi(\varepsilon)$ for some $\varepsilon \in U_F$.

Let M_i be the maximal abelian extension of F_i in L_i. So $K_i \cap M_i$ is the maximal abelian extension of F_i in K_i. Then

$$\left(\frac{M_i/F_i}{\rho}\right)\Big|_{M_i \cap K_i} = \left(\frac{M_i \cap K_i|F_i}{\rho}\right) = \left(\frac{K_i|F_i}{\rho}\right) = \chi_i(\rho) = 1 .$$

The second equality here follows from the fact that $N_{K_i/F_i}(K_i) = N_{K_i \cap M_i/F_i}(K_i \cap M_i)$. Therefore

$$\left(\frac{M_i/F_i}{\rho}\right) \in \mathrm{Gal}(M_i/M_i \cap K_i) \subseteq \mathrm{Gal}(M_i/F_i) .$$

Since $P \neq P_i$, any i, ρ is a P_i-unit for each i. Thus $((M_i/F_i)/\rho) \in T(M_i/F_i) \subseteq \mathrm{Gal}(M_i/F_i)$ where T is the inertia group of the local extension. So we have

$$\left(\frac{M/_iF_i}{\rho}\right) \in T(M_i/F_i) \cap \mathrm{Gal}(M_i/M_i \cap K_i) = T(M/_iM_i \cap K_i)$$

LEMMA. *L_i/F_i be a normal extension of local fields and M_i the* *maximal abelian extension of F_i in L_i. If $L_i \supseteqq K_i \supseteqq F_i$ and L_i/K_i is unramified, then $M_i/M_i \cap K_i$ is unramified.*

The lemma, to be proved below, implies that $T(M_i/M_i \cap K_i) = \{1\}$ and therefore $((M_i/F_i)/\rho)=1$. Since $M_i \supseteqq E_i$ it follows that $((E_i/F_i)/\rho)= 1 = ((\rho, E/F)/P_i)$ for all i. So we have E/F abelian, $\rho \in F$, and $((\rho, E/F)/\mathfrak{p}_i) = 1$ for $\mathfrak{p}_i = F \cap P_i$, $i = 1, \cdots, t$. Since $\bar{K} \supseteqq E$, the $\{\mathfrak{p}_i\}$ includes all primes of F ramified in E/F. For every unramified prime of F at which ρ is a unit the norm residue symbol is 1. The only undetermined symbol is $((\rho, E/F)/\mathfrak{p})$. By the product formula for norm residue symbols, the product of all symbols is 1. Hence we must have $((\rho, E/F)/\mathfrak{p}) = 1$. Recall that $(\rho) = \mathfrak{p}$, i.e. ρ is a prime element at $\mathfrak{p}$, and $\mathfrak{p}$ is unramified in E/F. Hence $((\rho, E/F)/\mathfrak{p})$ generates the decomposition group of $\mathfrak{p}$ in E/F. We conclude that $\mathfrak{p}$ is completely decomposed in E/F. It follows by standard arguments that P is completely decomposed in L/K since $L = KE$. The subgroup of C_K corresponding to a subfield k of $\bar{K}$ can be characterized as the classes of all prime ideals of K which are completely decomposed in k/K. Thus $[P] \in H$ since H corresponds to L.

Now we show that $\mathrm{Ker}(f) \supseteqq H$. Let P be a prime of K of absolute degree 1 with $[P] \in H$. Let $N_{K/F}(P) = \mathfrak{p} = (\rho)$, $\rho \in F$ and as above let P_i, $i = 1, \cdots, t$ be the primes of K ramified in K/F. We may assume also $P \neq P_i$ for any i. Since $[P] \in H$, P is completely decomposed in L/K. Say, $P = Q_1 \cdots Q_g$ so that $N_{L/F}(Q_1) = (\rho)$. Let $\mathfrak{m}$ be a divisor of F divisible by high powers of all P_i and prime to P. Since E is the maximal abelian extension of F in L and in $\bar{K}$ the norm limitation theorem implies that

$$(*) \quad N_{E/F}(I_\mathfrak{m}(E))\cdot S_\mathfrak{m}(F) = N_{L/F}(I_\mathfrak{m}(L))\cdot S_\mathfrak{m}(F) = N_{\bar{K}/F}(I_\mathfrak{m}(\bar{K}))\cdot S_\mathfrak{m}(F)$$

where $I_\mathfrak{m}(k)$ is the group of ideals of k relatively prime to $\mathfrak{m}$ and $S_\mathfrak{m}(k)$ is the ideal ray (Strahl) $\mathrm{mod}\, \mathfrak{m}$.

We have noted that $(\rho) = N_{L/F}(Q)$ with $Q_i \in I_\mathfrak{m}(L)$. It follows from (*) that we can write $(\rho) = N_{\bar{K}/F}(\mathfrak{A})\cdot(\alpha)$ where $\mathfrak{A} \in I_\mathfrak{m}(\bar{K})$ and $(\alpha) \in S_\mathfrak{m}(F)$. The norm from $\bar{K}$ to K of any ideal of $\bar{K}$ is a principal ideal of K. Let $N_{\bar{K}/K}(\mathfrak{A}) = (a)$, $a \in K$. So $(\rho) = (\alpha)N_{\bar{K}/F}(\mathfrak{A}) = (\alpha)N_{K/K}(N_{E/K}(\mathfrak{A})) = (\alpha)(N_{K/F}(a))$ or $\varepsilon\rho = \alpha \cdot N_{K/F}(a)$ for some unit $\varepsilon \in U_F$. Therefore

$$\left(\frac{\varepsilon\rho,\, K/F}{P_i}\right) = \left(\frac{\alpha,\, K/F}{P_i}\right)\cdot\left(\frac{N_{K/F}(a),\, K/F}{P_i}\right).$$

Since a global norm is certainly a local norm $((N_{K/F}(a), K/F)/P_i) = 1$. Also since $\alpha \in F$, $\alpha \equiv 1(\mathfrak{m})$ and $\mathfrak{m}$ is divisible by high powers of the

P_i we see that $((M_i \cap K_i/F_i)/\alpha) = 1$. And therefore

$$\left(\frac{K_i/F_i}{\alpha}\right) = \left(\frac{\alpha,\ K/F}{P_i}\right) = 1\ .$$

Thus $((\varepsilon\rho,\ K/F)/P_i) = 1$ for all i. In other words $\chi(\rho) = \chi(\varepsilon^{-1})$, which gives $[P] \in \operatorname{Ker}(f)$.

Proof of the lemma. Let $T(L/F)$ be the inertia subgroup of $\operatorname{Gal}(L/F)$. The quotient $\operatorname{Gal}(L/F)/T(L/F)$ is a cyclic group, hence $T(L/F)$ contains the commutator subgroup of $\operatorname{Gal}(L/F)$, which is $\operatorname{Gal}(L/M)$. Thus L/M is totally ramified. Letting e denote the ramification index, we have $e(L/K \cap M) \geqq [L\colon M] \geqq [K\colon K \cap M]$. This last inequality follows from the fact that $L \supseteqq KM$ and, since $M/K \cap M$ is galois, $[KM\colon M] = [K\colon K \cap M]$. Since L/K is unramified, $e(L/K \cap M) \leqq [K\colon K \cap M]$. Therefore $e(L/K \cap M) = [K\colon K \cap M] = [L\colon M] = e(L/M)$ and so $e(M/K \cap M) = 1$.

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