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NIELSEN NUMBERS AS A HOMOTOPY TYPE INVARIANT

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Let $f: X \to X$ denote a self map of a compact ANR and let $N(f)$ denote the Nielsen number of f which measures the number of essential fixed points of f. Then it is well-known that $f \sim g: X \to X$ implies $N(f) = N(g)$. Suppose Y is another ANR and $g: Y \to Y$ is a map such that for a homotopy equivalence $h: X \to Y$, we have $hf \sim gh$. Then Jiang (1964) proved that in these more general circumstances, $Nf = N(g)$, in the special case when $\pi_1(X)$ is finite. This paper contains a proof of the result without this restriction and applies it to give a technique for extending results in the Nielsen theory of fiber-preserving maps from locally trivial fiber bundles in the category of polyhedra to Hurewicz fibrations in the ANR category.

1. Intoduction. As usual in dealing with the Nielsen number $N(f)$ of a map $f: X \to X$, we restrict our spaces to the category of ANR's (compact metric)(see [1]). Since it is well-known that $f \sim g: X \to X$ (f homotopic to g) implies that $N(f) = N(g)$ a word of explanation of the title is in order. Jiang Bo-Ju in [6] considered the following situation. Given the homotopy commutative diagram

$$(1) \qquad \begin{array}{ccc} X & \xrightarrow{f} & X \\ h\downarrow & & \downarrow h \\ Y & \xrightarrow{g} & Y \end{array}$$

where h is a homotopy equivalence, Jiang called f and g maps of the same *homotopy type*. The main theorem of the last section of his paper stated that if f and g have the same homotopy type and if $\pi_1(X)$ *is finite*, then $N(f) = N(g)$. The proof consisted of employing the approach to Nielsen theory using lifts in the universal cover. Then, h was used to establish a correspondence between the fixed point classes of f to these of g. However, in order to establish that *essential* classes corresponded to *essential* classes it was necessary to compute local indices. This was done using Lefschetz numbers in the universal cover, which is compact when the space has finite fundamental group. This technique has its limitations and about as far as one can go with it is to relax the finiteness condition on $\pi_1(X)$ to a finiteness condition on the kernel of $f_{\#}: \pi_1(X) \to \pi_1(X)$.

Fortunately, there is a very simple proof of the *general result* which is quite useful in the study of Nielsen numbers in fiber spaces.

2. Homotopy type invariance.

THEOREM 2.1. *Given the homotopy commutative diagram maps*

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ h\downarrow & & \downarrow h \\ Y & \xrightarrow{g} & Y \end{array} \tag{2}$$

where h is a homotopy equivalence. Then, $N(f) = N(g)$.

The basic idea is to use mapping cylinder $C(h)$ of the homotopy equivalence h and the following basic properties of the local index [1]:

(a) (Homotopy Invariance) $f \sim g: X \to X$ implies $N(f) = N(g)$

(b) (Commutativity) Given $\varphi: X \to Y$, $\psi: Y \to X$, then

$$i(X, \psi\varphi, U) = i(Y, \varphi\psi, \psi^{-1}(U)) , \quad (i = \text{index}) \tag{3}$$

provided $\psi\varphi$ is fixed point free on the boundary of U.

We use (b) to prove the following lemma.

LEMMA 2.2. *Let X denote a retract of Z and let $i: X \to Z$, $r: Z \to X$ denote inclusion and retraction, respectively. Given $f: X \to X$, set $\hat{f} = ifr: Z \to Z$, the natural extension of f. Then $N(\hat{f}) = N(f)$.*

Proof. The fixed point set $\Phi(f)$ of f is the fixed point set of $\hat{f}$. We next show that the Nielsen classes of f and $\hat{f}$ are identical. Let x and y denote two fixed points. If x and y are Nielsen equivalent *wrt* $\hat{f}$, then there is a path α in Z joining x and y such that $\alpha \sim \hat{f}\alpha$ (rel endpoints). This forces $r\alpha \sim r\hat{f}\alpha$ in X. But $r\hat{f}\alpha = rifr\alpha = fr\alpha$, so that $r\alpha \sim fr\alpha$, where $r\alpha$ is a path in X joining x and y. Thus, x and y are Nielsen equivalent *wrt* f. See equivalence *wrt* f implies directly equivalence *wrt* $\hat{f}$, we see that the Nielsen classes of f and $\hat{f}$ are identical. We are left with checking local indices to insure that the number of essential classes is the same for f and $\hat{f}$. Let $\{U_i\}$ be a finite open cover of $\Phi(f)$ with mutually disjoint open sets, each covering a Nielsen class E_i. We apply the commutativity paoperty to the maps $\varphi =$ if: $X \to Z$, $\psi = r: Z \to X$. Then,

$$i(X, f, U_i) = i(X, \psi\varphi, U_i) = i(Z, \varphi\psi, \psi^{-1}(U_i)) = i(Z, \hat{f}, \psi^{-1}(U_i)) \tag{4}$$

and the local indices of each E_i *wrt* f and $\hat{f}$ are the same. Thus, $N(\hat{f}) = N(f)$.

Proof of Theorem 2.1. Starting with the diagram (2) let $C(h)$ denote the mapping cylinder of the homotopy equivalence h. Then, we have inclusion maps $i: X \to C(h)$, $j: Y \to C(h)$ and retractions $\rho: C(h) \to X$, $r: C(h) \to Y$, which are also homotopy equivalences, i.e., in addition to $\rho i = 1$ and $rj = 1$, we also have $i\rho \sim 1$ and $jr \sim 1$. Now, let

$$\hat{f}: C(h) \longrightarrow C(h),\ \hat{g}: C(h) \longrightarrow C(h) \tag{5}$$

be defined by $\hat{f} = if\rho$ and $\hat{g} = jgr$. Then, according to Lemma 2.2, $N(\hat{f}) = N(f)$ and $N(\hat{g}) = N(g)$. On the other hand

$$\hat{f} = if\rho \sim jhf\rho \sim jgh\rho = jgri\rho = \hat{g}i\rho \sim \hat{g} \tag{6}$$

and by the homotopy property (a), $N(\hat{f}) = N(\hat{g})$. Consequently, we have $N(f) = N(g)$ and the Nielsen number is invariant of homotopy type.

3. Applications. We give now an application which indicates how to extend results on Nielsen numbers in locally trivial fiber spaces to more general fibrations, e.g., Hurewicz fibrations. Let $\mathscr{F} = (E, p, B)$ denote a fiber space and

$$\begin{array}{ccc} E & \xrightarrow{f} & E \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{\bar{f}} & B \end{array} \tag{7}$$

a fiber-preserving map, with B 0-connected. When $\mathscr{F}$ is locally trivial it is possible to find a lifting function [5] λ for $\mathscr{F}$ so that the translations $\tau_\alpha: p^{-1}(\alpha(0)) \to p^{-1}(\alpha(1))$ given by

$$\tau_\alpha(x) = \lambda(x, \alpha)(1) \tag{8}$$

are all *homeomorphisms*. This fact is used crucially in [2] in showing that the Nielsen number of f along the fiber, $N_F(f)$, is well-defined when $\mathscr{F}$ is orientable. Recall [2]:

DEFINITION 3.1. $N_F(f)$ is defined to be the Nielsen number $N(f_b)$ of the map

$$f_b = \tau_\alpha f: F_b \longrightarrow F_b \tag{9}$$

where α is a path from $\bar{b} = \bar{f}(b)$ back to $b \in B$, and F_b is the fiber over b.

We sketch now a proof that $N_F(f)$ is well-defined i.e., independent of α and $b \in B$, whenever $\mathscr{F}$ is a *Hurewicz fibration* which is orientable in the sense that every loop in B induces translations which are homotopic to the identity.

REMARK. Before we begin, let us emphasize that this is a rather strong orientability condition compared with the condition that the fundamental group of the base act trivially on the homology of the fiber. However, fiber bundles with o-connected structure group are orientable in this sense since, in this case, one can find a lifting function λ such that every loop in B induces translations which belong to the group of the bundle.

(i) Independence of α. Let β denote another path from $\bar{b}$ to b and β^* the reverse of β. Then,

$$\tau_\alpha f \sim \tau_\alpha \tau_{\beta*} \tau_\beta f \sim \tau_\beta f \tag{10}$$

and $N(\tau_\alpha f) = N(\tau_\beta f)$.

(ii) Independence of $b \in B$. Let $c \in B$ denote another choice, $\bar{b} = \bar{f}(b)$, $\bar{c} = \bar{f}(c)$, α a path from $\bar{b}$ to b, γ a path from c to b, and $\bar{\gamma} = \bar{f}(\gamma)$. Then, we have a homotopy commutative diagram

$$\begin{array}{ccc} F_c & \xrightarrow{\varphi} & F_c \\ {\scriptstyle \tau_\gamma}\downarrow & & \downarrow{\scriptstyle \tau_\gamma} \\ F_b & \xrightarrow{\tau_\alpha f} & F_b \end{array} \tag{11}$$

where $\varphi = \tau_{\gamma*} f \tau_\alpha \tau_\gamma$, $F_b = p^{-1}(b)$, $F_c = p^{-1}(c)$. According to Theorem 2.1, $N(\tau_\alpha f) = N(\varphi)$. Now, since $\bar{\gamma} = f(\gamma)$ a simple argument shows that $f\tau_\gamma \sim \tau_{\bar{\gamma}} f$. Thus,

$$\varphi = \tau_{\gamma*} \tau_\alpha f \tau_\gamma \sim \tau_{\gamma*} \tau_\alpha \tau_{\bar{\gamma}} f \sim \tau_\beta f \tag{12}$$

where $\beta = \bar{\gamma}\alpha\gamma^*$ is a path from $\bar{c}$ back to c. Thus, $N(\tau_\alpha f) = N(\tau_\beta f)$ and we have independence of $b \in B$.

REMARK 3.2. The fact that $N_F(f)$ is well-defined for orientable $\mathscr{F}$'s, requires only the Covering Homotopy Theorem for a class of spaces containing all the fibers. Thus, for example, $N_F(f)$ is well-defined for Serre fibrations provided all the fibers are compact polyhedra.

The fact that the Nielsen number along the fiber $N_F(f)$ is well-defined for Hurewicz fibrations can now be employed in conjunction with the Closed Fiber Smoothing Theorem of Casson and Gottlieb [4] to provide the following tool for extending results on Nielsen numbers valid in the category of fiber bundles to the category of Hurewicz fiber spaces. We first consider some lemmas.

LEMMA 3.3. *Given* $f: X \to X$ *and* $g: Y \to Y$ *such that* $N(g) = 1$, *then* $N(f) = N(f \times g)$.

Proof. The result for X and Y polyhedral is a special case of the main theorem in [3]. To extend the result to the case where X and Y are ANR's compact metric we make use recent result (West [8], Miller [7]) that there exist finite polyhedra A and B and homotopy equivalences $\varphi: X \to A$, $\psi: Y \to B$. If we let $\bar{\varphi}$ and $\bar{\psi}$ denote homotopy inverses of φ and ψ, respectively and set $f' = \varphi f \bar{\varphi}$, $g' = \psi g \bar{\varphi}$, then we have a homotopy commutative diagram

$$(13) \qquad \begin{array}{ccc} X \times Y & \xrightarrow{f \times g} & X \times Y \\ {\scriptstyle \varphi \times \psi}\downarrow & & \downarrow{\scriptstyle \varphi \times \psi} \\ A \times B & \xrightarrow{f' \times g'} & A \times B \end{array}$$

and Theorem 2.1 implies that $N(f \times g) = N(f' \times g')$, $N(f) = N(f')$ and $N(g) = N(g')$. Therefore

$$(14) \qquad N(f \times g) = N(f' \times g') = N(f')N(g') = N(f)N(g) = N(f) .$$

LEMMA 3.4. *Given a fiber homotopy equivalence*

$$(15) \qquad \begin{array}{ccc} E & \xrightarrow{h} & E' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & B & \end{array}$$

of orientable Hurewicz fibrations and a fiberpreserving map f (7). *Then h induces a fiber-preserving map $f' = hf\bar{h}$,*

$$(16) \qquad \begin{array}{ccc} E' & \xrightarrow{f'} & E' \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{\bar{f}} & B \end{array}$$

where $\bar{h}$ is a fiber homotopy inverse for h, with $N(f) = N(f')$ and $N_F(f) = N_F(f')$.

Proof. This is immediate from the two homotopy commutative diagrams below using the Theorem 2.1:

$$\begin{array}{ccc} E & \xrightarrow{f} & E \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ E' & \xrightarrow{f'} & E' \end{array} \qquad \begin{array}{ccc} F_b & \xrightarrow{\tau_\alpha f} & F_b \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ F'_b & \xrightarrow{\tau'_\alpha f'} & F'_b \end{array}$$

where τ_α and τ'_α are, respectively, translations along a path α from

$\bar{b} = f(b)$ to b, in the fiber spaces (E, p, B) and (E', p', B).

THEOREM 3.5. *Let $\mathscr{F} = (E, p, B)$ denote an orientable Hurewicz fiber space where the spaces involved are* ANR's (compact metric) *and let*

$$\begin{array}{ccc} E & \xrightarrow{f} & E \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{\bar{f}} & B \end{array} \tag{17}$$

denote a give fiber-preserving map. Then, there exists a locally trivial fiber space $\mathscr{F}' = (E', p', B')$, with both fiber and base finite polyhedra, and a fiber-preserving map

$$\begin{array}{ccc} E' & \xrightarrow{f'} & E' \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p'} \\ B' & \xrightarrow{\bar{f}'} & B' \end{array} \tag{18}$$

such that $N(f) = N(f')$, $N(\bar{f}) = N(\bar{f}')$ and $N_F(f) = N_F(f')$, where the latter are Nielsen numbers along the fibers for $\mathscr{F}$ and $\mathscr{F}'$, respectively.

Proof. We assume first that B is a finite polyhedron. Then, using [4], for some integer $n > 0$, $\mathscr{F} \times T^n = (E \times T^n, B, pp_1)$, where $p_1 =$ projection on first factor and T^n is an n-dimensional torus, is fiber homotopy equivalent to a locally trivial fiber space $\mathscr{F}' = (E', p', B)$ with compact fiber F' which is a compact smooth manifold with boundary. The fiber-preserving map (17) induces

$$\begin{array}{ccc} E \times T^n & \xrightarrow{f \times g} & E \times T^n \\ {\scriptstyle pp_1}\downarrow & & \downarrow{\scriptstyle pp_1} \\ B & \xrightarrow{\bar{f}} & B \end{array} \tag{19}$$

where $g: T^n \to T^n$ is any map such that $N(g) = 1$. For example, g may be taken as the product of self maps of the circle of degree 2. Applying Lemma 3.3, we obtain $N(f) = N(f \times g)$, $N_F(f) = N_F(f \times g)$. Now, applying Lemma 3.4, we have a fiber-preserving map $f': E' \to E'$ induced on $\mathscr{F}'$ by (19), with

$$N(f') = N(f \times g) = N(f) \text{ and } N_F(f') = N_F(f \times g) = N_F(f) \tag{20}$$

and the theorem follows for B a finite polyhedron.

In order to handle the case when B is an ANR (compact metric) we again make use of the recent result [7, 8] that B is the same homotopy type as a finite polyhedron, induced fibrations as well as the techniques already empolyed. We omit these details.

COROLLARY 3.6 (for example). *Let* $\mathscr{F} = (E, p, B)$ *denote an orientable Hurewicz fibration with* E *and* B *compact metric* ANR's, *and let* $(f, \bar{f}): \mathscr{F} \to \mathscr{F}$ *denote a fiber-preserving map* (see (7)). *If one of the following conditions is satisfied*:

(a) $\pi_1(B) = \pi_2(B) = 0$.

(b) $\pi_1(F) = 0$, *F a fiber of* $\mathscr{F}$.

(c) $\mathscr{F}$ *is fiber homotopically trivial and* $\pi_1(B) = 0$.

(d) *Letting F denote a typical fiber for $\mathscr{F}$, there is a homotopy commutative diagram*

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & F \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ E & \xrightarrow{\varphi} & F \end{array}$$

where $\varphi | p^{-1}$ (b) *is a homotopy equivalence for each* $b \in B$, *then*

$$N(f) = N(\bar{f}) N_F(f) . \tag{21}$$

Proof. We apply the main theorem of [3] in conjunction with Theorem 3.5, observing that the proof of the main theorem in [3] requires only that the base and fiber are finite polyhedra. (a) and (c) require no special attention since the fibration $\mathscr{F}'$ yielded by Theorem 3.5 has a base of the same homotopy type and in case (c) $\mathscr{F}'$ may be taken to be trivial. Case (d) requires the observation that under the given hypotheses $\mathscr{F}$ is fiber homotopic to the trivial fiber space $B \times F$ and f has the same homotopy type as $\bar{f} \times g$. Finally case (b) requires a little attention because, the new fiber F' in the replacement bunnle given by Theorem 3.5 is no longer simply connected. However, F' has the form $F' = W \times T^n$ (see [4]) and the self map of F' used to compute the Nielsen number $N_F(f')$ along the fiber is homotopic to one of the form $\varphi \times g$, $\varphi: W \to W$, $g: T^n \to T^n$, $N(g) = 1$. Since, $\pi_1(W) = 0$, $N_F(f') = 0$ or 1. In the case $N_F(f') = 1$, the proof of the main theorem in [3] shows that $N(\bar{f}') = N(f')$ and hence

$$N(f) = N(f') = N(\bar{f}') \cdot N_F(f') = N(\bar{f}) \cdot N_F(f) . \tag{22}$$

If $N_F(f') = 0$, again the same proof shows that $N(f') = 0$, and hence since $N(f) = N(f') = 0$ and $N_F(f') = N_F(f) = 0$ our conclusion follows.

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