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ABSOLUTE NÖRLUND SUMMABILITY FACTORS FOR FOURIER SERIES

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In this paper a general theorem on the absolute Nörlund summability factors of a Lebesgue-Fourier series at a given point has been established. The theorem exhibits the potency of a Nörlund method as a tool to study absolute summability and the absolute convergence of Fourier series. Several existing results in the field are deduced as special cases. This also shows some sort of continuity amongst these theorems which otherwise seem apparently to be disconnected.

1. Let $\{p_n\}$ be a sequence of constants such that

$$P_n = p_0 + p_1 + \cdots + p_n \neq 0, \quad \text{for} \quad n = 0, 1, 2, \cdots .$$

Given a series $\sum u_n$ we define $\{t_n\}$ of its Nörlund means by

$$t_n = \frac{1}{P_n}\sum_0^n P_k u_{n-k} .$$

The series Σu_n is said to be summable $|N, p_n|$ and we write $\Sigma u_n \in |N, p_n|$, if the sequence $\{t_n\} \in bv$, that is $\Sigma|\Delta t_n| \equiv \Sigma|t_n - t_{n+1}|$ is convergent.

In the special case when $p_n = \Gamma(n+k)/\Gamma(k)\Gamma(n+1)$, $k > 0$, the summability $|N, p_n|$ reduces to the familiar Cesàro summability $|C, k|$ and when $p_n = 1/(n+1)$, it is the same as the absolute harmonic summability.

2. Let f be a periodic function with period 2π and let $f \in L(-\pi, \pi)$. Let the Fourier series of f at a point x be given by

$$f(x) \sim \frac{1}{2}a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx) \equiv \sum_0^\infty A_n(x) .$$

Throughout the paper we use the following notations:

$$\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\} ,$$

$$\Phi_0(t) = \phi(t) ,$$

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\phi(u)du,\ \alpha > 0 ,$$

$$\phi_\alpha(t) = \Gamma(\alpha+1)t^{-\alpha}\Phi_\alpha(t),\ \alpha \geqq 0 ,$$

$$G(n, t) = P_n^{-1}\sum_0^{n-1} p_k \varepsilon_{n-k} e^{i(n-k)t} ,$$

$$g(n, t) = \operatorname{Im}(G(n, t)),$$

$$H(n, t) = \frac{P_\tau}{P_n}\Big(|\varepsilon_{n-m}| + \frac{1}{n}\sum_{n-m}^{n+1}|\varepsilon_k|\Big) + \frac{p_m}{tP_n},$$

where $\tau = [1/t]$ and $m = [(1/2)n]$,

$$J(n, t) = \frac{1}{\Gamma(1-\alpha)}\int_t^\pi (w-t)^{-\alpha}\frac{d}{dw}g(n, w)dw, \ 0 \leqq \alpha < 1,$$

$$L(n, t) = \frac{1}{\Gamma(\alpha+1)}\int_t^\pi u^\alpha \frac{d}{du}J(n, u)du, \ 0 \leqq \alpha < 1,$$

$$V(n, t) = \frac{1}{\Gamma(\alpha+1)}\int_0^t u^\alpha \frac{d}{du}J(n, u)du, \ 0 \leqq \alpha < 1.$$

$K, K_1, K_2, \cdots$, denote absolute constants not necessarily the same at different occurrences.

B denotes the class of bounded sequences and

$\mathscr{M}$ denotes a class of positive and nonincreasing sequences:

$$\mathscr{M} = \left\{s_n : s_n > 0, \ \frac{s_{n+1}}{s_n} \leqq \frac{s_{n+2}}{s_{n+1}} \leqq 1\right\}.$$

3. Since the publication of the classical theorems of Bosanquet ([3], [4]) on absolute Cesàro summability of a Fourier series in 1936, various results have been worked out on absolute Cesàro summability, absolute harmonic summability and absolute Nörlund summability of Fourier series and series related with it. The purpose of this paper is to furnish a general theorem on the absolute Nörlund summability of a Fourier series from which we deduce several known and unknown results.

We establish the following theorem:

THEOREM. *Let α satisfy $0 \leqq \alpha < 1$ and let $\{p_n\} \in \mathscr{M}$ and $\{\varepsilon_n\} \in bv$ be such that*

(i) $\sum_n^\infty |\Delta\varepsilon_k| = O(|\varepsilon_n|)$ *and*

(ii) $\left\{\dfrac{P_n}{n^\alpha}\sum_n^\infty \dfrac{|\varepsilon_k|}{k^{1-\alpha}P_k}\right\} \in B.$

If $\phi_\alpha(t) \in BV(0, \pi)$, then $\sum A_n(x)\varepsilon_n \in |N, p_n|$.

4. We use the results in the following lemmas towards the proof of our theorem.

LEMMA 1 (McFadden [13]). *If $\{p_n\}$ is a nonnegative nonincreasing sequence then for $0 < t \leqq \pi$ and for any n, a and b*

$$\left|\sum_a^b p_k e^{i(n-k)t}\right| \leqq KP_{[1/t]} .$$

LEMMA 2. *Let $\{p_n\}$ be a nonnegative nonincreasing sequence and $\{\varepsilon_n\} \in bv$ be such that $\sum_n^{2n} |\Delta\varepsilon_k| = O(|\varepsilon_n|)$. Then for $1/n < t \leqq \pi$, $\tau = [1/t]$ and $m = [(1/2)n]$,*

$$\text{(i)} \qquad G(n, t) = O\left(\frac{P_\tau |\varepsilon_{n-m}|}{P_n}\right) + O\left(\frac{p_m}{tP_n}\right) ;$$

$$\text{(ii)} \qquad \frac{d}{dt}G(n, t) = O(nH(n, t)) ;$$

and

$$\text{(iii)} \qquad L(n, t) = O(n^\alpha t^\alpha H(n, t)) .$$

Proof. (i) Let

$$\begin{aligned} P_n G(n, t) &= \left(\sum_0^m + \sum_{m+1}^{n-1}\right) p_k \varepsilon_{n-k} e^{i(n-k)t} \\ &= S_1 + S_2 . \end{aligned}$$

Then by Lemma 1

$$\begin{aligned} S_1 &= O\left(P_\tau\left\{|\varepsilon_{n-m}| + \sum_0^{m-1} |\Delta\varepsilon_{n-k}|\right\}\right) \\ &= O(P_\tau |\varepsilon_{n-m}|) , \end{aligned}$$

and

$$\begin{aligned} S_2 &= \sum_1^{n-m-1} p_{n-k}\varepsilon_k e^{ikt} \\ &= O\left(\frac{1}{t}\right)\left\{p_{m+1} + \sum_1^{n-m-2} |\Delta(\varepsilon_k p_{n-k})|\right\} \\ &= O\left(\frac{1}{t}\right)\left\{p_{m+1} + \sum_1^{n-m-2} |\varepsilon_k|(p_{n-k-1} - p_{n-k}) + \sum_1^{n-m-2} p_{n-k-1}|\Delta\varepsilon_k|\right\} \\ &= O\left(\frac{p_m}{t}\right) , \end{aligned}$$

by hypotheses.

(ii) Let

$$\begin{aligned} P_n \frac{d}{dt}G(n, t) &= \left(\sum_0^m + \sum_{m+1}^{n-1}\right) p_k \varepsilon_{n-k}(n-k) i e^{i(n-k)t} \\ &= S_3 + S_4 . \end{aligned}$$

Then, proceeding as above in (i)

$$
\begin{aligned}
S_3 &= O(P_\tau)\Big\{(n-m)|\varepsilon_{n-m}| + \sum_0^{m-1} |\Delta((n-k)\varepsilon_{n-k})|\Big\} \\
&= O(P_\tau)\Big\{n|\varepsilon_{n-m}| + \sum_{n-m}^{n-1} k|\Delta\varepsilon_k| + \sum_{n-m}^{n-1} |\varepsilon_{k+1}|\Big\} \\
&= O(P_\tau)\Big\{n|\varepsilon_{n-m}| + \sum_{n-m}^{n} |\varepsilon_k|\Big\} ; \\
S_4 &= \sum_1^{n-m-1} k\varepsilon_k p_{n-k} i e^{ikt} \\
&= O(1/t)\Big\{(n-m-1)|\varepsilon_{n-m-1}|p_{m+1} + \sum_1^{n-m-2} |\Delta(k\varepsilon_k p_{n-k})|\Big\} \\
&= O(1/t)\Big\{np_m + \sum_1^{n-m-2} k|\Delta(\varepsilon_k p_{n-k})| + \sum_1^{n-m-1} p_{n-k}|\varepsilon_k|\Big\} \\
&= O(n/t)\Big\{p_m + \sum_1^{n-m-2} |\Delta(\varepsilon_k p_{n-k})|\Big\} \\
&= O(np_m/t) .
\end{aligned}
$$

(iii) We have

$$
\begin{aligned}
\Gamma(1-\alpha)J(n,t) &= \Big(\int_t^{t+1/n} + \int_{t+1/n}^{\pi}\Big)(u-t)^{-\alpha}\frac{d}{du}g(n,u)du \\
&= I_1 + I_2, \quad \text{say} . \\
I_1 &= O(nH(n,t))\int_t^{t+1/n}(u-t)^{-\alpha}du, \quad \text{by (ii)} , \\
&= O(n^\alpha H(n,t)) ,
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= O(n^\alpha)|[g(n,u)]_{t+1/n}^{\pi'}|,\ t+\frac{1}{n} < \pi' \leqq \pi , \\
&= O\Big(\frac{n^\alpha}{P_n}\Big)\Big\{P_\tau|\varepsilon_{n-m}| + \frac{p_m}{t}\Big\}, \quad \text{by (i)} .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\Gamma(\alpha+1)|L(n,t)| &= |[u^\alpha J(n,u)]_t^\pi - \alpha\int_t^\pi u^{\alpha-1}J(n,u)du| \\
&= \Big|t^\alpha J(n,t) + \frac{\alpha}{\Gamma(1-\alpha)}\int_t^\pi u^{\alpha-1}\int_u^\pi (w-u)^{-\alpha}\frac{d}{dw}g(n,w)dwdu\Big| \\
&= \Big|t^\alpha J(n,t) + \frac{\alpha}{\Gamma(1-\alpha)}\int_t^\pi \frac{d}{dw}g(n,w)dw\int_t^w (w-u)^{-\alpha}u^{\alpha-1}du\Big| \\
&= \Big|t^\alpha J(n,t) + \frac{\alpha}{\Gamma(1-\alpha)}\int_t^\pi \frac{d}{dw}g(n,w)dw\int_{t/w}^1 u^{\alpha-1}(1-u)^{-\alpha}du\Big| \\
&\leqq K_1 t^\alpha|J(n,t)| + K_2|g(n,t')|,\ t \leqq t' < \pi .
\end{aligned}
$$

$\Rightarrow L(n, t) = O(n^{\alpha} t^{\alpha} H(n, t))$.

LEMMA 3 (Das [6]). *Let* $\{p_n\} \in \mathscr{M}$. *Then* $\sum u_n \in |N, p_n|$, *if and only if*

$$\sum_1^\infty \frac{1}{nP_n} \left| \sum_1^n p_{n-k} k u_k \right| < \infty .$$

5. *Proof of the Theorem.* Let

$$T_n \equiv T_n(x) = \frac{1}{P_n} \sum_1^n p_{n-k} k A_k(x) \varepsilon_k .$$

Then by Lemma 3, it is sufficient to show that

$$\sum \frac{|T_n|}{n} < \infty .$$

As

$$\begin{aligned}
kA_k(x) &= \frac{2}{\pi} \int_0^\pi \phi(t) \frac{d}{dt} \sin ktdt , \\
\frac{\pi}{2} T_n &= \frac{1}{P_n} \sum_1^n p_{n-k} \varepsilon_k \int_0^\pi \phi(t) \frac{d}{dt} \sin ktdt \\
&= \int_0^\pi \phi(t) \frac{d}{dt} g(n, t) dt \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^\pi \frac{d}{dt} g(n, t) \int_0^t (t-u)^{-\alpha} d\Phi_\alpha(u) dt, \ 0 < \alpha < 1 , \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^\pi d\Phi_\alpha(u) \int_u^\pi (t-u)^{-\alpha} \frac{d}{dt} g(n, t) dt \\
&= - \int_0^\pi \Phi_\alpha(u) \frac{d}{du} J(n, u) du, \ 0 \leqq \alpha < 1 , \\
&= -[\phi_\alpha(u) V(n, d)]_0^\pi + \int_0^\pi V(n, u) d\phi_\alpha(u) \\
&= -\phi_\alpha(\pi) V(n, \pi) + \int_0^\pi V(n, u) d\phi_\alpha(u) .
\end{aligned}$$

If in particular we choose $\phi(t) = 1$, then $\phi_\alpha(t) = 1$ and $T_n = 0$ for every n. Hence

$$V(n, \pi) = 0 .$$

Thus

$$\frac{\pi}{2} T_n = \int_0^\pi V(n, u) d\phi_\alpha(u) .$$

As $\phi_\alpha(t) \in BV(0, \pi)$, to complete the poof it is sufficient to show that uniformly in t, $0 < t \leqq \pi$

$$\sum \frac{|V(n, t)|}{n} \leqq K .$$

Since $V(n, t) + L(n, t) = 0$, we have

$$\sum \frac{|V(n, t)|}{n} \leqq \sum_{1}^{\tau} \frac{|V(n, t)|}{n} + \sum_{\tau+1}^{\infty} \frac{|L(n, t)|}{n} .$$

As

$$|g(n, t)| \leqq \frac{1}{P_n} \sum_{0}^{n-1} p_k |\varepsilon_{n-k}| \leqq K ,$$

and

$$\left|\frac{d}{dt} g(n, t)\right| \leqq \frac{1}{P_n} \sum_{0}^{n-1} p_k (n-k) |\varepsilon_{n-k}| \leqq Kn ,$$

we have

$$\begin{aligned} \Gamma(1-\alpha) J(n, t) &= \left(\int_{t}^{t+1/n} + \int_{t+1/n)}^{\pi}\right), (u-t)^{-\alpha} \frac{d}{du} g(n, u) du \\ &= \text{etc.} \\ &= O(n^\alpha) . \end{aligned}$$

Hence, for $0 < \alpha < 1$,

$$\Gamma(1+\alpha) |V(n, t)| = \left| [u^\alpha J(n, u)]_0^t - \alpha \int_0^t u^{\alpha-1} J(n, u) du \right| \leqq K n^\alpha t^\alpha ,$$

and

$$\sum_{1}^{\tau} \frac{|V(n, t)|}{n} \leqq K t^\alpha \sum_{1}^{\tau} n^{\alpha-1} \leqq K .$$

For $\alpha = 0$, we note that

$$|V(n, t)| = |g(n, t)| \leqq \frac{1}{P_n} \sum_{0}^{n-1} p_k |\varepsilon_{n-k}| (n-k) t \leqq nt ,$$

and thus again

$$\sum_{1}^{\tau} \frac{|V(n, t)|}{n} \leqq K .$$

Therefore, after Lemma 2(iii), it is sufficient to show that

$$t^\alpha \sum_{\tau+1}^{\infty} n^{\alpha-1} |H(n, t)| \leqq K ,$$

uniformly in t, $0 < t \leqq \pi$. However

$$
\begin{aligned}
& t^{\alpha} \sum_{\tau+1}^{\infty} n^{\alpha-1} |H(n, t)| \\
& \quad = P_{\tau} t^{\alpha} \sum_{\tau+1}^{\infty} \frac{n^{\alpha-1}}{P_n} |\varepsilon_{n-m}| + P_{\tau} t^{\alpha} \sum_{\tau+1}^{\infty} \frac{n^{\alpha-2}}{P_n} \sum_{n-m}^{n+1} |\varepsilon_k| + t^{\alpha-1} \sum_{\tau+1}^{\infty} \frac{n^{\alpha-1} p_m}{P_n} \\
& \quad \leqq K_1 + P_{\tau} t^{\alpha} \sum_{[(\tau+1)/2]}^{\infty} |\varepsilon_k| \sum_{n=k-1}^{2k} \frac{n^{\alpha-2}}{P_n} + K_2 t^{\alpha-1} \sum_{\tau+1}^{\infty} n^{\alpha-2} \\
& \quad \leqq K_1 + K_2 P_{\tau} t^{\alpha} \sum_{[(\tau+1)/2]}^{\infty} \frac{|\varepsilon_k|}{k^{1-\alpha} P_k} \\
& \quad \leqq K .
\end{aligned}
$$

This completes the proof of the theorem.

6. Corollaries.

6.1. Taking $p_n = \Gamma(n + \beta)/\Gamma(\beta)\Gamma(n + 1)$, $1 \geqq \beta > 0$, we get

COROLLARY 1. *Let α satisfy $0 \leqq \alpha < 1$, and let $\{\varepsilon_n\} \in bv$ be such that*

(i) $\sum_n^{\infty} |\Delta\varepsilon_k| = O(|\varepsilon_n|)$ *and*

(ii) $\left\{ n^{\beta-\alpha} \sum_n^{\infty} \frac{|\varepsilon_k|}{k^{1+\beta-\alpha}} \right\} \in B.$

If $\phi_\alpha(t) \in BV(0, \pi)$, then $\sum A_n(x)\varepsilon_n \in |C, \beta|$, $\beta \geqq \alpha$.

The case $\{\varepsilon_n\} \equiv \{1\}$, and $\beta > \alpha$, furnishes corresponding results due to Bosanquet ([3], [4]). Taking $0 < \alpha = \beta < 1$ and specializing $\{\varepsilon_n\}$ to $\{(\log (n + 1))^{-1-\varepsilon}\}$, $\varepsilon > 0$, we get a result due to Cheng ([5]). Again in the case $\alpha = \beta$, taking $\{\varepsilon_n\}$ to be a convex sequence (see Zygmund [18], p. 93, for the definition and certain properties as needed) we obtain a result due to Prasad and Bhatt ([14]). The case $\beta = \alpha = 0$ is covered in Corollary 3 below.

6.2. The case $p_n = 1/(n + 1)$ furnishes the following result on absolute harmonic summability:

COROLLARY 2. *Let α satisfy $0 \leqq \alpha < 1$, and let $\{\varepsilon_n\} \in bv$ be such that*

(i) $\sum_n^{\infty} |\Delta\varepsilon_k| = O(|\varepsilon_n|)$ *and*

(ii) $\left\{ \frac{\log n}{n^{\alpha}} \sum_n^{\infty} \frac{|\varepsilon_k|}{k^{1-\alpha} \log k} \right\} \in B$.

If $\phi_\alpha(t) \in BV(0, \pi)$, then $\sum A_n(x)\varepsilon_n \in |N, 1/n + 1|$.

The case $\alpha = 0$, includes a well known theorem due to Varshney ([17], Varshney has proved the result for $\{\varepsilon_n\} = \{1/\log(n+2)\}$). Specialising $\{\varepsilon_n\}$ to be $\{\log(n+1)\lambda_n/n^\alpha\}$, where $\{\lambda_n\}$ is a convex sequence, we get the result due to Bhatt [1].

6.3. It is now known (see Dikshit [8]) that if $\{p_n\} \in \mathscr{M}$ and $\{P_n\} \in B$ then the method $|N, p_n|$ is ineffective, in the sense that it sums only absolutely convergent series. Thus the extra hypothesis that $\{P_n\} \in B$ in the theorem yields the following result on absolute convergence factors for Fourier series.

COROLLARY 3. *Let α satisfy $0 \leqq \alpha < 1$, and let $\{\varepsilon_n\} \in bv$ be such that*

(i) $\sum_n^\infty |\Delta\varepsilon_k| = O(|\varepsilon_n|)$ *and*

(ii) $\left\{\frac{1}{n^\alpha}\sum_n^\infty \frac{|\varepsilon_k|}{k^{1-\alpha}}\right\} \in B$.

If $\phi_\alpha(t) \in BV(0, \pi)$, then $\sum A_n(x)\varepsilon_n$ is absolutely convergent.

Results in somewhat weaker form are eventually known in as much as they could be deduced from the theorem of Cheng [5], Prased and Bhatt [14], or the Corollary 2, with an application of a result of Kogbetliantz ([11]).

6.4. The case $\{\varepsilon_n\} \equiv \{1\}$ yields the following:

COROLLARY 4. *Let $0 \leqq \alpha < 1$ and let $\{p_n\} \in \mathscr{M}$ and be such that*

$$\left\{\frac{P_n}{n^\alpha}\sum_n^\infty \frac{1}{k^{1-\alpha}P_k}\right\} \in B .$$

If $\phi_\alpha(t) \in BV(0, \pi)$, then $\sum A_n(x) \in |N, p_n|$.

A more general result in this direction is also known and is given elsewhere (Dikshit [7], [9]; see also Lal [12]).

6.5. Writing $\{\varepsilon_n\} = \{P_n\lambda_n/n^\alpha\}$ we obtain:

COROLLARY 5. *Let α satisfy $0 \leqq \alpha < 1$, and let $\{p_n\} \in \mathscr{M}$ and $\{\lambda_n\}$ be a sequence such that*

(i) $\left\{\frac{P_n\lambda_n}{n^\alpha}\right\} \in bv$,

(ii) $\sum_n^\infty \left|\Delta\left(\frac{P_k\lambda_k}{k^\alpha}\right)\right| = O\left(\frac{P_n|\lambda_n|}{n^\alpha}\right)$ *and*

(iii) $\left\{\frac{P_n}{n^\alpha}\sum_n^\infty \frac{|\lambda_k|}{k}\right\} \in B$.

If $\phi_\alpha(t) \in BV(0, \pi)$, *then* $\sum A_n(x)P_n\lambda_n/n^\alpha \in |N, p_n|$.

It is worthwhile to compare the result of this Corollary for $\alpha = 0$ with one due to T. Singh [16] and L. B. Singh [15] and for $0 < \alpha < 1$, with those due to Nand Kishore [10] and Bhatt [2].

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