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WEAKLY COMPACT SETS IN H^1

FREDDY DELBAEN

Suppose that A is a uniform algebra on a compact set X and that $\phi: A \rightarrow C$ is a nonzero multiplicative linear functional on A. Let M_ϕ be the set of positive representing measures for ϕ. If M_ϕ is finite dimensional, let m be a core measure of M_ϕ. The space H^1 is the closure of A in $L^1(m)$. The space H^∞ is the weak* (i.e. $\sigma(L^\infty, L^1)$) closure of A in $L^\infty(m)$. The weakly compact sets R in H^1 are then those sets such that for all $\varepsilon > 0$ there is a bounded set in H^∞ which approximates R up to ε.

It is well known (see Gamelin [1] for all details) that if m is a core measure in the finite dimensional set M_ϕ, then the annihilator N of A (or $\mathrm{Re}\, A$) in the real Banach space L^1_R is finite dimensional, and is in fact a subspace of L^∞_R (see Gamelin [1] p. 108). Since N is finite dimensional there is a constant K_1 such that $\|g\|_1 \leqq \|g\|_\infty \leqq K_1\|g\|_1$ for all $g \in N$. There also exists a linear projection P of L^1_R onto N, the kernel of P being precisely $\overline{\mathrm{Re}\, A}$.

I am very grateful to the referee who pointed out an error in the first draft of this paper and gave a simplification of the proof.

2. **Weakly compact sets in H^1.** The notation used in the proof of the following theorem is the same as in the introduction.

THEOREM. *If $R \subset H^1$ then the following are equivalent*

(1) *R is relatively weakly compact in H^1*

(2) *$\forall \varepsilon > 0$ $\exists M$ such that $\forall f \in R$ $\exists g \in H^\infty$ with $\|g\|_\infty \leqq M$ and $\|f - g\| \leqq \varepsilon$*

(3) *$\forall \varepsilon > 0$ $\exists M$ such that $\forall f \in R$ $\exists g \in A$ with $\|g\| \leqq M$ and $\|f - g\| \leqq \varepsilon$.*

Proof. (3) $\Rightarrow$ (2) obvious, (2) $\Rightarrow$ (1) follows from general arguements due to Grothendieck ([2] p. 296); (1) $\Rightarrow$ (2) is less trivial. Without loss of generality we may suppose that for all $f \in R$ we have $\|f\|_1 \leqq 1$. From now on all calculations are made with fixed f. It is clear that all bounds only depend on $\|P\|$ and K_1. Since $\log^+|f| \leqq |f|$ it is obvious that $\|\log^+|f|\|_1 \leqq \|f\|_1 \leqq 1$. Since $L^1_R = \overline{\mathrm{Re}\, A} \oplus N$ we also have uniquely determined elements $u \in \overline{\mathrm{Re}\, A}$ and $v \in N$ such that $\log^+|f| = u + v$. Since v is the image of $\log^+|f|$ by the operator P we have

$$||v||_\infty \leqq K_1||v||_1 \leqq K_1\cdot||P||\,||\log^+|f|\,|| \leqq K_1\cdot||P|| = K_2 \,.$$

The conjugation operator* is defined on $\overline{\operatorname{Re} A}$ and takes values in $L^p(0 < p < 1)$, hence $\exists K_3$ such that $||{}^*u||_{1/2} \leqq K_3||u||_1$. The function e^{u+i^*u} is well defined and $fe^{-u-i^*u} \in H^\infty$. Indeed:

$$|f|\cdot|e^{-u-i^*u}| = e^{\log|f|}e^{-u} \leqq e^{\log^+|f|}e^{-u} = e^v \leqq e^{K_2} = K_4 \,.$$

Hence $f = F\cdot e^{u+i^*u}$ with $||F||_\infty \leqq K_4$. The next step is the approximation of e^{u+i^*u} by functions in H^∞. First remark that $u = \log^+|f| - v \geqq -K_2$. Put $u_n = \min(u, n) \geqq -K_2$ and $u_n = w_n + v_n$ where $w_n \in \overline{\operatorname{Re} A}$ and $v_n \in N$. We first prove that:

(i) $||e^{w_n+i^*w_n}||_\infty \leqq M_n$ where M_n is independent of u

(ii) $||e^{w_n+iw_n} - e^{u+i^*u}||_1 \to 0$ uniformly in u as $n \to \infty$.

Proof of (i): Since $\log^+|f| = u + v$ we have

$$|u| \leqq ||v||_\infty + \log^+|f| \leqq \log^+|f| + K_2 \,.$$

Hence $e^u \leqq K_4\cdot|f|$ and so the family e^u is equally integrable (Here it is used that relatively weakly compact sets in L^1 are equally integrable (see [2] p. 295).) Consequently $e^{u_n} \to e^u$ uniformly in u. Since $v_n = P(u_n - u)$ we also have $||v_n||_\infty \leqq K_1||v_n||_1 \leqq K_1\cdot||P||\,||u_n - u||_1 \leqq K_2$ for n large enough. Indeed since $-K_2 \leqq u_n \leqq u \leqq \log^+|f| + K_2 \leqq |f| + K_2$ we have that the functions u form an equally integrable family and hence $u_n \to u$ uniformly in u. All this implies

$$|e^{w_n+i^*w_n}| = e^{w_n} = e^{u_n-v_n} \leqq K_4e^{u_n} \leqq K_4e^n = M_n \,.$$

Proof of (ii)

$$\begin{aligned}|e^{u+i^*u} - e^{w_n+i^*w_n}| &\leqq |e^{u+i^*w} - e^{u+i^*w_n}| + |e^{u+i^*w_n} - e^{u_n+i^*w_n}| \\ &\quad + |e^{u_n+i^*w_n} - e^{w_n+i^*w_n}| \\ &\leqq e^u|e^{i^*u} - e^{i^*w_n}| + |e^u - e^{u_n}| + |e^{u_n} - e^{w_n}| \\ &\leqq A_n + B_n + C_n \,.\end{aligned}$$

Here is

$$\begin{aligned}A_n &= e^u|e^{i^*u} - e^{i^*w_n}| \\ B_n &= |e^u - e^{u_n}| \\ C_n &= |e^{u_n} - e^{w_n}| \,.\end{aligned}$$

In the proof of (i) it was already observed that $||B_n||_1 \to 0$ uniformly in u. For n large enough one has

$$|e^{u_n} - e^{w_n}| = |e^{w_n+v_n} - e^{w_n}| = e^{w_n}|e^{v_n} - 1| \leqq K_4e^u|e^{v_n} - 1| \,.$$

Since $||v_n||_\infty \leqq K_2||u_n - u|| \to 0$ uniformly in u one has $||C_n||_1 \leqq$

$K_4\|e^u\|_1 \cdot \|e^{v_n} - 1\|_\infty \to 0$ uniformly in u. Remains to show that $\int A_n \to 0$.

Put $E_n = \{x \mid |{}^*u(x) - {}^*w_n(x)| \geqq \delta\}$ where $\delta > 0$ will be conveniently chosen.

$$\int A_n = \int_{E_n} A_n + \int_{E_n^c} A_n \leqq \int_{E_n} 2e^u + \int_{E_n^c} e^u |e^{i^*u} - {}^{i^*w_n}| .$$

Since

$$K_3\int |u - w_n| dm \geqq \left(\int |{}^*u - {}^*w_n|^{1/2} dm\right)^2 \geqq \left(\int_{E_n} |{}^*u - {}^*w_n|^{1/2} dm\right)^2$$
$$\geqq \delta m(E_n)^2$$

one has $m(E_n) \to 0$ uniformly in u, hence by equally integrability of e^u it follows that $\int_{E_n} 2e^u \to 0$ uniformly in u. Also

$$\int_{E_n^c} e^u |e^{i^*u} - e^{i^*w_n}| \leqq \int \delta e^u \leqq \delta \int K_4 |f| \leqq \delta K_4$$

and hence $\|A_n\|_1 \leqq \delta K_4 + 2\int_{E_n} e^u$.

The first term is made small by choosing δ, afterwards we choose n to be sure that the second term is also small enough, since this can be done uniformly in u the proof of (ii) is complete.

Fix now $\varepsilon > 0$ and let n be large enough to assure $\|e^{w_n + i^*w_n} - e^{u + i^*u}\|_1 \leqq \varepsilon / K_4$. It then follows that

$$\|f - Fe^{w_n + i^*w_n}\|_1 \leqq \|F\|_\infty \|e^{u+i^*u} - e^{w_n + i^*w_n}\|_1 \leqq \varepsilon .$$

Taking $M = M_n \cdot K_4 = K_4^2 e^n$ will do the job.

To prove that $(2) \Rightarrow (3)$ we only have to observe that the unit ball of A is dense in the unit ball of H^∞ for the L^1 norm. Since m is a core point, m is dominant and we can apply the Arens-Singer result ([1], p. 152, 153).

References

1. T. W. Gamelin, *Uniform Algebras*, Prentice-Hall, Englewood Cliffs, 1969.
2. A. Grothendieck, *Espaces Vectoriels Topolologiques*, Sao Paulo, 1964.

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BRIJE UNIVERSITEIT BRUSSELS