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PERIODIC JACOBI-PERRON ALGORITHMS AND FUNDAMENTAL UNITS

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In this paper the author states a class of infinitely many real cubic fields for which the Jacobi-Perron algorithm of a properly chosen vector becomes periodic and calculates explicitly a fundamental unit for each field. The main results of this paper are: Let $m = a^6 + 3a^3 + 3$, $\omega = \sqrt[3]{m}, m$ cube free $a \in N$; then the Jacobi-Perron algorithm of $a^{(0)} = (\omega, \omega^2)$ is periodic. The length of the primitive preperiod is four and the length of the primitive period is three. A fundamental unit in $Q(\omega)$ is given by $e = a^3 + 1 - a\omega$.

1. Introduction. The Jacobi algorithm **[9]** which was generalized by Perron **[11]** for any dimension $n \geqq 3$ proceeds as follows. Let $a^{(0)}$ be a vector in R_{n-1}; then the sequence $\langle a^{(v)} \rangle$ is called the Jacobi-Perron algorithm, if, for $a^{(v)} = (a_1^{(v)}, \cdots, a_{n-1}^{(v)})$, $(v = 0, 1, \cdots)$

$$
\begin{aligned}
&a^{(v+1)} = \frac{1}{a_1^{(v)} - b_1^{(v)}} (a_2^{(v)} - b_2^{(v)}, \cdots, a_{n-1}^{(v)} - b_{n-1}^{(v)}), \ (b_1^{(v)} \neq a_1^{(v)}; \ v = 0, 1, \cdots) \\
(1.1) \quad & \\
&b_i^{(v)} = [a_i^{(v)}], \qquad (i = 1, \cdots, n-1; \ V = 0, 1, \cdots).
\end{aligned}
$$

For notation see Bernstein's book [7, pp. 11–18].

The Jacobi-Perron algorithm of a vector $Q^{(0)} \in R_{n-1}$ is called periodic, if there exist two rational integers L and M, $L \geqq 0$, $M \geqq 1$, such that

$$(1.2) \qquad a^{(M+V)} = a^{(V)}, \qquad (V = L, L+1, \cdots).$$

If $\min L = l$, $\min M = m$, then the sequence of vectors

$$(1.3) \qquad a^{(0)}, a^{(1)}, \cdots, a^{(L-1)}$$

is called the primitive preperiod of the Jacobi-Perron algorithm, and the sequence of vectors

$$(1.4) \qquad a^{(L)}, a^{(L+1)}, \cdots, a^{(L+M-1)}$$

is called primitive period. The l and m are called respectively the lengths of the primitive preperiod and period. If $l = 0$, the algorithm is said to be purely periodic. By definition, from any periodic Jacobi-

Perron algorithm a purely periodic one can be derived. Perron **[11]** proved that if the Jacobi-Perron algorithm of $a^{(0)}$ is periodic, then all the components of $a^{(v)}$ $(v = 0, 1, \cdots)$ are algebraic numbers belonging to a field of degree $\leqq n$. In **[1–5]**, Bernstein has stated a few classes of infinitely many real algebraic fields, for which the Jacobi-Perron algorithm of a properly chosen vector $a^{(0)}$ becomes periodic.

For later purposes, we need the following two important results about units in the field $Q(a_1^{(0)}, a_2^{(0)}, \cdots, a_{n-1}^{(0)})$:

THEOREM 1. (8). *If the Jacobi-Perron algorithm of* $a^{(0)} = (a_1^{(0)}, \cdots, a_{n-1}^{(0)})$ *becomes periodic, with length l of the primitive preperiod and length m of the primitive period, then*

$$e = \prod_{i=l}^{l+m-1} a_{n-1}^{(i)} \omega \tag{1.5}$$

is a unit in $Q(a_1^{(0)}, \cdots, a_{n-1}^{(0)})$.

THEOREM 2. (6). *Let the denominators of* $a^{(v)}$ $(V = 0, 1, \cdots)$ *be rationalized, that is,*

$$\begin{aligned} a_i^{(v)} &= \frac{C_{i,1}^{(v)} + C_{i,2}^{(v)}\omega + \cdots + C_{i,n}^{(v)}\omega^{n-1}}{M_v} \\ i &= 1, \cdots, n-1;\ v = 0, 1, \cdots;\ C_{i,j}^{(v)} \in Z \qquad (j = 1, \cdots, n) \\ M_v &\in N; \qquad Q(\omega) = Q(a_1^{(0)}, \cdots, a_{n-1}^{(0)}). \end{aligned} \tag{1.6}$$

If there exists a $v \geqq 1$ *such that*

$$M_v = 1 \tag{1.7}$$

and if the $a_i^{(v)}$ $(i = 1, \cdots, n-1)$ *are algebraic integers, then*

$$e = A_0^{(v)} + a_1^{(v)}(\omega) A_0^{(v+1)} + \cdots + a_{n-1}^{(v)}(\omega) A_0^{(v+n-1)} \tag{1.8}$$

is a unit in $Q(\omega)$. *The* $A_0^{(j)}$ *are calculated by the recurrence formula*

$$\begin{aligned} &A_0^{(0)} = 1; \qquad A_0^{(1)} = A_0^{(2)} = \cdots = A_0^{(n-1)} = 0, \\ &A_0^{(n+k)} = A_0^{(k)} + b_1^{(k)} A_0^{(k+1)} + b_2^{(k)} A_0^{(k+2)} + \cdots \\ &\qquad + b_{n-1}^{(k)} A_0^{(k+n-1)} \qquad (k = 0, 1, \cdots). \end{aligned} \tag{1.9}$$

2. A new periodic Jacobi-Perron algorithm.

Let

$$m = a^6 + 3a^3 + 3, \qquad \omega^3 = m \tag{2.1}$$

and

$$a^{(0)}(\omega) = (\omega, \omega^2), \qquad a \in N, \qquad a \geqq 2.$$

It seems that the first step is to give a sufficiently good approximation for ω. Since

$$a \geqq 2, \qquad a^6 > 3a^3 + 3, \tag{2.2}$$

we can rewrite

$$\omega = (a^6 + 3a^3 + 3)^{1/3} = a^2 \left(1 + \frac{3(a^3+1)}{a^6}\right)^{1/3}$$

as

$$\omega = a^2 \left(1 + \frac{a^3+1}{a^6} - \frac{(a^3+1)^2}{a^{12}} + \cdots\right) = a^2 \left(1 + \frac{1}{a^3} - \frac{(2a^3+1)}{a^{12}} + \cdots\right)$$

and we have approximately

$$\omega = a^2 + \frac{1}{a} - \frac{(2a^3+1)}{a^{10}}. \tag{2.3}$$

Now, since $0 < 1/a - (2a^3+1)/a^{10} < 1$, we obtain

$$[\omega] = a^2. \tag{2.4}$$

In the following calculations, we shall use as the approximation for ω,

$$\omega = a^2 + \frac{1}{a} \tag{2.5}$$

since the remainder is comparatively very small. It should be noted that $a^2 + 1/a > \omega$. We further obtain the approximation of ω^2:

$$\omega^2 = a^4 \left(1 + \frac{2(a^3+1)}{a^6} - \frac{(a^3+1)^2}{a^{12}} + \cdots\right),$$

which is approximately

$$\omega^2 = a^4 \left(1 + \frac{2}{a^3} + \frac{a^6 - 2a^3 - 1}{a^{12}}\right),$$

$$\omega^2 = a^4 + 2a + \frac{a^6 - 2a^3 - 1}{a^8}. \tag{2.6}$$

Since $0 < (a^6 - 2a^3 - 1)/a^8 < 1$, we obtain

$$[\omega^2] = a^4 + 2a. \tag{2.7}$$

We shall cautiously use the following approximation

$$\omega^2 = a^4 + 2a + 1/a^2, \tag{2.8}$$

keeping in mind that $a^4 + 2a + 1/a^2 > \omega^2$. We now have the beginning of the JPA for

$$a^{(0)} = (\omega, \omega^2),$$
$$b^{(0)} = (a^2, a^4 + 2a),$$

and obtain, by definition,

$$a^{(1)} = \left(\frac{\omega^2 - (a^4 + 2a)}{\omega - a^2}, \frac{1}{\omega - a^2}\right). \tag{2.9}$$

We can now obtain the rationalization of the denominator directly, keeping in mind that

(2.10) $\omega^3 - a^6 = 3(a^3 + 1)$,

$$a^{(1)}(\omega) = \left(\frac{(\omega^2 - (a^4 + 2a))(\omega^2 + a^2\omega + a^4)}{(\omega - a^2)(\omega^2 + a^2\omega + a^4)}, \frac{(\omega^2 + a^2\omega + a^4)}{(\omega - a^2)(\omega^2 + a^2\omega + a^4)}\right),$$
$$= \left(\frac{-2a\omega^2 + (a^3 + 3)\omega + a^2(a^3 + 3)}{\omega^3 - a^6}, \frac{\omega^2 + a^2\omega + a^4}{\omega^3 - a^6}\right),$$

$$a^{(1)}(\omega) = \left(\frac{-2a\omega^2 + (a^3 + 3)\omega + a^2(a^3 + 3)}{3(a^3 + 1)}, \frac{\omega^2 + a^2\omega + a^4}{3(a^3 + 1)}\right). \tag{2.11}$$

Now, using the approximation formulas (2.5) and (2.8), we can write $-2a\omega^2 + (a^3 + 3)\omega + a^2(a^3 + 3)$ as

$$-2a\left(a^4 + 2a + \frac{1}{a^2}\right) + (a^3 + 3)\left(a^2 + \frac{1}{a}\right) + a^5 + 3a^2 = 2a^2 + a + \frac{1}{a}.$$

Therefore,

$$b_1^{(1)} = \left[\frac{2a^2 + a + \frac{1}{a}}{3(a^3 + 1)}\right] = 0. \tag{2.12}$$

We further obtain,

$$\omega^2 + a^2 + a^4 = a^4 + 2a + \frac{1}{a^2} + a^4 + a + a^4 = 3a^4 + 3a + \frac{1}{a^2}.$$

Hence

$$b_2^{(1)} = \left[\frac{3a(a^3+1) + \dfrac{1}{a^2}}{3(a^3+1)}\right] = a. \tag{2.13}$$

We write for the sake of convenience, as will also be done in the sequel, $a^{(1)}(\omega)$ and $b^{(1)}$.

$$a^{(1)}(\omega) = \left(\frac{-2a\omega^2 + \omega(a^3+3) + a^2(a^3+3)}{3(a^3+1)}, \frac{\omega^2 + a^2\omega + a^4}{3(a^3+1)}\right),$$

$$b^{(1)} = (0, a).$$

We obtain the next vector, by definition,

$$a^{(2)}(\omega) = \left(\frac{\omega^2 + a^2\omega - (2a^4 + 3a)}{-2a\omega^2 + (a^3+3)\omega + a^2(a^3+3)}, \frac{3(a^3+1)}{-2a\omega^2 + (a^3+3)\omega + a^2(a^3+3)}\right):$$

$$a^{(2)}(\omega) = \left(\frac{-a\omega^2 + (2a^3+3)\omega - a^2(a^3+2)}{3a^6 + 10a^3 + 9}, \frac{(a^3+3)\omega^2 + a^2(a^3+1)\omega + a(a^3+2)(a^3+3)}{3a^6 + 10a^3 + 9}\right). \tag{2.14}$$

For the calculation of $b_1^{(2)}$ and $b_2^{(2)}$, we obtain

$$\begin{aligned} -a\omega^2 + (2a^3+3)\omega - a^2(a^3+2) \\ = -a\left(a^4 + 2a + \frac{1}{a^2}\right) + (2a^3+3)\left(a^2 + \frac{1}{a}\right) - a^2(a^3+2), \end{aligned}$$

which simplifies to $a^2 + 2/a$. Hence

$$b_1^{(2)} = \left[\frac{a^2 + \dfrac{2}{a}}{3a^6 + 10a^3 + 9}\right] = 0. \tag{2.15}$$

Now

$$\begin{aligned} &(a^3+3)\omega^2 + a^2(a^3+1)\omega + a(a^3+2)(a^3+3) \\ &= (a^3+3)\left(a^4 + 2a + \frac{1}{a^2}\right) + (a^5 + a^2)\left(a^2 + \frac{1}{a}\right) + a^7 + 5a^4 + 6a \\ &= 3a^7 + 12a^4 + 14a + \frac{3}{a^2}. \end{aligned}$$

Therefore,

$$b_2^{(2)} = \left[\frac{3a^7 + 10a^4 + 9a}{3a^6 + 10a^3 + 9} + \frac{2a^4 + 5a + \dfrac{3}{a^2}}{3a^6 + 10a^3 + 9}\right].$$

$$b_2^{(2)} = a. \tag{2.16}$$

Now,

$$a^{(2)}(\omega) = \left(\frac{-a\omega^2 + (2a^3+3)\omega - a^2(a^3+2)}{3a^6 + 10a^3 + 9},\right.$$
$$\left.\frac{(a^3+3)\omega^2 + a^2(a^3+1)\omega + a(a^3+2)(a^3+3)}{3a^6 + 10a^3 + 9}\right),$$

$$b^{(2)} = (0, a).$$

Therefore,

$$a^{(3)}(\omega) = \left(\frac{(a^3+3)\omega^2 + a^2(a^3+1)\omega - a(2a^6 + 5a^3 + 3)}{-a\omega^2 + (2a^3+3)\omega - a^2(a^3+2)},\right. \tag{2.17}$$
$$\left.\frac{3a^6 + 10a^3 + 9}{-a\omega^2 + (2a^3+3)\omega - a^2(a^3+2)}\right),$$

$$a^{(3)}(\omega) = (\omega + a^2, \omega^2 + a^2\omega + a(a^3+1)). \tag{2.18}$$

The reader can now verify the continuation of the algorithm and prove

$$a^{(7)}(\omega) = a^{(4)}(\omega). \tag{2.19}$$

This important result can now be expressed in the following theorem:

THEOREM 2.1. *Let m be a cube-free natural number of the form $m = a^6 + 3a^3 + 3$, where a is a natural number greater than or equal to 2. Let $\omega^3 = m$. Then the JPA of the vector $a^{(0)}(\omega) = (\omega, \omega^2)$ is periodic. The length of the primitive preperiod is four and has the form*

$$a^{(0)}(\omega) = (\omega, \omega^2),$$

$$a^{(1)}(\omega) = \left(\frac{-2a\omega^2 + (a^3+3)\omega + a^2(a^3+3)}{3(a^3+1)}, \frac{\omega^2 + a^2\omega + a^4}{3(a^3+1)}\right),$$

$$a^{(2)}(\omega) = \left(\frac{-a\omega^2 + (2a^3+3)\omega - a^2(a^3+2)}{3a^6 + 10a^3 + 9},\right.$$
$$\left.\frac{(a^3+3)\omega^2 + a^2(a^3+1)\omega + a(a^3+2)(a^3+3)}{3a^6 + 10a^3 + 9}\right),$$

$$a^{(3)}(\omega) = (\omega + a^2, \omega^2 + a^2\omega + a(a^3+1))$$

$$b^{(3)} = (2a^2, 3a^4 + 4a).$$

The period of the JPA of $a^{(0)}(\omega)$ has length three and is of the form

$$
\begin{aligned}
a^{(4)}(\omega) &= \left(\frac{-3a\omega^2+3\omega+3a^2(a^3+2)}{3(a^3+1)}, \frac{\omega^2+a^2\omega+a^4}{3(a^3+1)}\right),\\
b^{(4)} &= (0, a),\\
a^{(5)}(\omega) &= \left(\frac{\omega-a^2}{3(a^3+1)}, \frac{\omega^2+a^2\omega+a(a^3+3)}{3(a^3+1)}\right),\\
b^{(5)} &= (0, a),\\
a^{(6)}(\omega) &= (\omega+2a^2, \omega^2+a^2\omega+a(a^3+3))\\
b^{(6)} &= (3a^2, 3a^3, +3a).
\end{aligned}
$$

In the above theorem, we have excluded $a = 1$ and this is done because, if $a = 1$, then $m = 7 = 2^3 - 1 = D^3 - 1$, $D = 2$ and this form appears to be a special case of Bernstein's periodic Jacobi-Perron algorithm as stated in Theorem 3.3, where $m = D^3 - d$, $d \mid D$. Here $D = 2$, $d = 1$. But his form is not exactly a special case because $D \geqq 2d(n-1)$ is not satisfied. Yet, the JPA of $a^{(0)} = (\sqrt[3]{7}, \sqrt[3]{7^2})$ is periodic and one obtains the following:

$$
\begin{aligned}
a^{(0)}(\omega) &= (\omega, \omega^2); \qquad \omega = \sqrt[3]{7},\\
b^{(0)} &= (1, 3),\\
a^{(1)}(\omega) &= \left(\frac{-2\omega^2+4\omega+4}{6}, \frac{\omega^2+\omega+1}{6}\right),\\
b^{(1)} &= (0, 1),\\
a^{(2)}(\omega) &= \left(\frac{-\omega^2+5\omega-3}{22}, \frac{4\omega^2+2\omega+12}{22}\right),\\
b^{(2)} &= (0, 1),\\
a^{(3)}(\omega) &= (\omega+1, \omega^2+\omega+2),\\
b^{(3)} &= (2, 7),\\
a^{(4)}(\omega) &= \left(\frac{-3\omega^2+3\omega+9}{6}, \frac{\omega^2+\omega+1}{6}\right),\\
b^{(4)} &= (0, 1),\\
a^{(5)}(\omega) &= \left(\frac{\omega-1}{6}, \frac{\omega^2+\omega+4}{6}\right),\\
b^{(5)} &= (0, 1),\\
a^{(6)}(\omega) &= (\omega+2, \omega^2+\omega+1),\\
b^{(7)} &= (3, 6),\\
a^{(7)}(\omega) &= a^{(4)}(\omega).
\end{aligned}
$$

Comparing the above formulas with values obtained for $a^{(v)}(\omega)$, $(v = 1, \cdots, 6)$ in Theorem 2.1, we immediately see that Theorem 2.1 also holds for the case where $a = 1$.

We have yet to show that there are infinitely many cubic fields $Q(\omega)$, $\omega^3 = a^6 + 3a^3 + 3$ or that the equation $a^6 + 3a^3 + 3 = ty^3$, where t is a fixed number, a and y are indeterminants, has only a finite number of solutions. We obtain, multiplying by a^3,

$$(a^3 + 1)^3 - 1 = t(ya)^3;$$

denoting $a^3 + 1 = x$, $ya = z$, we obtain

$$x^3 - tz^3 = 1, \tag{2.20}$$

and this Diophantine equation, by a famous theorem of Nagell **[10]**, has at most one nontrivial solution (x_1, z_1).

3. Units in $Q(\omega)$, $\omega^3 = m = a^6 + 3a^3 + 3$. In this chapter, we will calculate units in $Q(\omega)$. Since $M_3 = 1$, we can calculate a unit using the results in Theorem 2. Also, since the JPA of $a^{(0)}(\omega) = (\omega, \omega^2)$ is periodic, another unit can be calculated with the help of Theorem 1.

Using formula (1.8) and noting that $n = 3$ and $V = 3$, we obtain

$$e = A_0^{(3)} + a_1^{(3)} A_0^{(4)} + a_2^{(3)} A_0^{(5)}, \tag{3.1}$$

and from formula (2.18),

$$e = A_0^{(3)} + (\omega + a^2) A_0^{(4)} + (\omega^2 + a^2\omega + a^4 + a) A_0^{(5)}. \tag{3.2}$$

In order to calculate $A_0^{(3)}$, $A_0^{(4)}$ and $A_0^{(5)}$, we need $b^{(1)}$ and $b^{(2)}$ which are, according to formulas (2.12), (2.13), (2.15), and (2.16),

$$\begin{aligned} b^{(1)} &= (0, a), \\ b^{(2)} &= (0, a). \end{aligned} \tag{3.3}$$

Using formula (1.9), we obtain

$$\begin{aligned} A_0^{(3)} &= A_0^{(0)} + b_1^{(1)} A_0^{(1)} + b_2^{(0)} A_0^{(2)} = 1 + b_1^{(0)} \cdot 0 + b_2^{(0)} \cdot 0 = 1, \\ A_0^{(4)} &= A_0^{(1)} + b_1^{(1)} A_0^{(2)} + b_2^{(1)} A_0^{(3)} = 0 + b_1^{(1)} \cdot 0 + a \cdot 1 = a, \\ A_0^{(5)} &= A_0^{(2)} + b_1^{(2)} A_0^{(3)} + b_2^{(2)} A_0^{(4)} = 0 + 0 \cdot 1 + a \cdot a = a^2; \end{aligned}$$

$$A_0^{(3)} = 1, \qquad A_0^{(4)} = a, \qquad A_0^{(5)} = a^2. \tag{3.4}$$

Now, from (3.2) and (3.4),

$$\begin{aligned} e &= 1 + (\omega + a^2)a(\omega^2 + a^2\omega + (a^4 + a))a^2 \\ &= a^2\omega^2 + (a^4 + a)\omega + a^6 + 2a^3 + 1, \end{aligned}$$

$$e = (1 + a^3)^2 + a(1 + a^3)\omega + a^2\omega^2. \tag{3.5}$$

This is a comparatively simple form for e. We shall now calculate e^{-1} as follows.

$$\begin{aligned} e^{-1} &= \frac{1}{(1 + a^3)^2 + a(1 + a^3)\omega + a^2\omega^2}, \\ &= \frac{1 + a^3 - a\omega}{((1 + a^3)^2 + a(1 + a^3)\omega + a^2\omega^2)(1 + a^3 - a\omega)}, \\ &= \frac{1 + a^3 - a\omega}{(1 + a^3)^3 - a^3\omega^3} = \frac{1 + a^3 - a\omega}{a^9 + 3a^6 + 3a^3 + 1 - a^9 - 3a^6 - 3a^3}, \end{aligned}$$

$$e^{-1} = 1 + a^3 - a\omega \tag{3.6}$$

which is indeed an elegant and a beautiful expression for a unit in $Q(\omega)$.

Since the JPA of (ω, ω^2) is also periodic, formula (1.5), in view of Theorem 2, becomes

$$e_1 = \prod_{i=4}^{6} a_2^{(i)} = a_2^{(4)}a_2^{(5)}a_2^{(6)}. \tag{3.7}$$

We obtain

$$\begin{aligned} a_2^{(4)}a_2^{(5)}a_2^{(6)} &= \frac{1}{\omega - a^2} \cdot \frac{3(a^3 + 1)}{-3a\omega^2 + 3\omega + 3a^2(a^3 + 2)} \cdot (\omega^2 + a^2\omega + a^4) \\ &= \frac{3(a^3 + 1)(\omega^2 + a^2\omega + a^4)}{(\omega - a^2)(-3a\omega^2 + 3\omega + 3a^2(a^3 + 2))} \\ &= \frac{3(a^3 + 1)(\omega^2 + a^2\omega + a^4)}{3(a^3 + 1)(\omega^2 + a^2\omega - (2a^4 + 3a))} \\ &= \frac{(\omega^2 + a^2\omega + a^4)(\omega - a^2)}{\omega^2 + a^2\omega - (2a^4 + 3a))(\omega - a^2)} \\ &= \frac{3(a^3 + 1)}{-3a(a^3 + 1)\omega + 3(a^3 + 1)^2} \\ &= \frac{1}{-a\omega + (a^3 + 1)}, \end{aligned}$$

$$e_1 = a_2^{(4)}a_2^{(5)}a_2^{(6)} = \frac{1}{a^3 + 1 - a\omega}.$$

Now, taking e_1^{-1}, we obtain

$$e^{-1} = a^3 + 1 - a\omega, \tag{3.8}$$

which is exactly identical to the unit given in (3.6). The question of the fundamentality of this unit is yet to be answered. We shall do that in the following chapter.

4. The fundamentality of units in $Q(\omega)$, $\omega^3 = m = a^6 + 3a^3 + 3$. In the preceding chapter, we showed that $e = 1 + a^3 - a\omega$ is a unit in $Q(\omega)$, $\omega^3 = m = a^6 + 3a^3 + 3$. This unit provides a nontrivial solution of the famous Nagell equation

$$x^3 - my^3 = 1, \qquad \omega^3 = m = a^6 + 3a^3 + 3. \tag{4.1}$$

To see this, one simply sets $x = 1 + a^3$, $y = a$. Although Nagell could not prove whether or not $x^3 - my^3 = 1$ has a solution, he did prove the following important theorem.

THEOREM 3. *If (x_1, y_1) is a nontrivial solution of $x^3 - my^3 = 1$, when m is a cube-free rational integer, then $x_1 - y\omega$, $\omega^3 = m$ is a fundamental unit of $Q(\omega)$, or the square of a fundamental unit. The latter happens only when $m = 19, 20$, or 28.*

In view of Nagell's theorem, we only need to check to see if there exists an $a \in N$ such that $a^6 + 3a^3 + 3 = 19, 20$, or 28. However, it is obvious that no such a exists and therefore we have the following important result.

THEOREM 4.2. *In the field $Q(\omega)$, where $\omega^3 = a^6 + 3a^3 + 3$ and a is a natural number, $e = 1 + a^3 - a\omega$ is a fundamental unit.*

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