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HOMOTOPIES AND INTERSECTION SEQUENCES

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For $\gamma_t: S^1 \to \mathbf{C}$, a smooth homotopy of closed curves, the changing configuration of vertices and cusps is studied by considering the set in $I \times S^1 \times S^1$ given by $(\gamma_t(z) - \gamma_t(\zeta))/(z - \zeta) = 0$. The main tool is oriented intersection theory from differential topology. The results relate to previous work by Whitney and Titus on normal curves and intersection sequences.

Consider a closed curve as a smooth map $\gamma: S^1 \to \mathbf{C}$. Let γ_t for $t \in I$ be a smooth homotopy of closed curves. A vertex of γ_t is a point w such that $w = \gamma_t(z) = \gamma_t(\zeta)$ for $z \neq \zeta$. A cusp is a point where the tangent vanishes and changes direction. Let $X = I \times S^1 \times S^1$. We study the changing configuration of vertices and cusps of γ_t by studying the set $Z = \{x \in X \mid G(x) = 0\}$ where $G(t, z, \zeta) = (\gamma_t(z) - \gamma_t(\zeta))/(z - \zeta)$, and the limiting value is taken when $z = \zeta$. If 0 is a regular value for G, then Z has the structure of an oriented 1-submanifold of X. If for fixed t, Z intersects $t \times S^1 \times S^1$ transversely, then the oriented intersection gives a set of pairs in $S^1 \times S^1$ with corresponding orientation numbers $+1$ or -1. If γ_t is a normal immersion, these pairs and their orientation numbers give the Titus intersection sequence of γ_t. The changes in the intersection sequence are reflected in the behavior of Z. If Z crosses $I \times \Delta$, where Δ is the diagonal of $S^1 \times S^1$, then we have a cusp and a change in the tangent winding number. The difference between the tangent winding numbers of γ_0 and γ_1 is just $N(Z, I \times \Delta)$, the total number of oriented intersections of Z with $I \times \Delta$.

1. Intersection sequences. In the complex plane, let S^1 be the set $|z| = 1$. Consider S^1 as a 1-manifold with functions $\theta \to e^{i\theta}$ giving local coordinate systems. The tangent vector $d/d\theta$ is defined independently of the choice of coordinate system. On $T(S^1)$, the tangent space, let $d/d\theta$ give the positive orientation at each point. This gives S^1 the structure of an oriented 1-manifold.

Suppose $\gamma: S^1 \to \mathbf{C}$ is a smooth (C^∞) map. Let $\beta(z) = (d\gamma/d\theta)(z)$ be the tangent at $\gamma(z)$. Let $S^1 \times S^1 = Y$ and let the maps $(\theta, \phi) \to (e^{i\theta}, e^{i\phi})$ give local coordinate systems for Y. Let $S^1 \times S^1$ have the product orientation, i.e., $T(S^1 \times S^1)$ has positive orientation given by the ordered basis $\{\partial/\partial\theta, \partial/\partial\phi\}$ at each point. Let $\Delta \subseteq Y = \{(z, \zeta) \mid z = \zeta\}$.

Let $\theta \to (e^{i\theta}, e^{i\theta})$ be local coordinate systems on Δ and let positive orientation be given on Δ by $d/d\theta$. Thus Δ is an oriented 1-submanifold of Y. Now we define $g\colon Y \to \mathbf{C}$ as follows

$$g(z,\zeta) = \begin{cases} \dfrac{\gamma(z)-\gamma(\zeta)}{z-\zeta}, & z \neq \zeta \\[2ex] \dfrac{-i\beta(z)}{z}, & z = \zeta. \end{cases}$$

We can check that g is a smooth function on Y.

Letting $y = (z,\zeta)$, we compute that for $y \in g^{-1}(0)$ we have

$$dg_y = \begin{cases} \dfrac{\beta(z)d\theta - \beta(\zeta)d\phi}{z-\zeta}, & z \neq \zeta \\[2ex] \dfrac{1}{iz}\dfrac{d\beta}{d\theta}(z)(d\theta + d\phi), & z = \zeta. \end{cases} \tag{1}$$

Now let $y = (z,\zeta) \in g^{-1}(0)$, and consider dg_y as a linear map from $T_y(Y)$ to $T_0(\mathbf{C})$. Then from (1):

(a) If $z \neq \zeta$, then dg_y has rank 2 iff the tangents $\beta(z)$ and $\beta(\zeta)$ are linearly independent. In this case, dg_y preserves orientation iff $\{\beta(z), -\beta(\zeta)\}$ is a positively oriented basis of $\mathbf{C}$ (where $\mathbf{C}$ has the usual orientation).

(b) If $z = \zeta$, then $\beta(z) = 0$ and dg_y has rank 1 iff $(d\beta/d\theta)(z) \neq 0$. Otherwise dg_y has rank 0. We may check that if $(d\beta/d\theta)(z) \neq 0$, then there is a cusp at $\gamma(z)$ and the limiting tangential directions at $\gamma(z)$ are the directions of $\pm(d\beta/d\theta)(z)$.

The point $0 \in \mathbf{C}$ is said to be a regular value for g if dg_y has rank 2 at every point of $g^{-1}(0)$. By remarks (a) and (b) above we see that 0 is a regular value for g iff γ is an immersion ($\beta(z) \neq 0$ for $z \in S^1$), and the tangents $\beta(z)$ and $\beta(\zeta)$ are linearly independent for each point $(z,\zeta) \in g^{-1}(0)$. Also if 0 is a regular value of g, $g^{-1}(0)$ is a finite subset of the compact set Y (a torus). In this case if $y \in g^{-1}(0)$ we set $\lambda(y) = +1$ if dg_y preserves orientation and $\lambda(y) = -1$ if dg_y reverses orientation. We say that $g^{-1}(0)$ with the sign λ gives the set of signed intersection pairs for γ.

We say that γ is a normal immersion if γ is an immersion, each point of $\mathbf{C}$ has at most two preimages under γ, and the tangents are linearly independent at each double point. Another way to say this is that 0 is a regular value for g, and projection on the first coordinate is one-to-one on $g^{-1}(0)$. ($g^{-1}(0)$ as a set of ordered pairs is a function.) If γ is a normal immersion, let $\{z_1, \cdots, z_{2n}\}$ be the preimages under γ of the double points, numbered sequentially along S^1 in a counterclockwise direction from a point z_0 on S^1, not a preimage of a double point. Then $g^{-1}(0)$ defines an involution * on the integers $1, \cdots, 2n$, such that $(z_j, z_{j^*}) \in g^{-1}(0)$

for $j = 1, \cdots, 2n$. Now define the sign ν by $\nu(j) = -\lambda((z_j, z_{j^*}))$. We say that the involution * together with the sign ν defines the intersection sequence of γ with respect to z_0. Usually z_0 is chosen so that $\gamma(z_0)$ is on the outer boundary, i.e., the boundary of the component of $\mathbf{C} - \gamma(S^1)$ containing ∞. In this case ν and * give the Titus intersection sequence (see Titus [5] or Francis [1]). We remark that signed intersection pairs are defined if 0 is a regular value for g. To define the intersection sequence also, we need in addition that $g^{-1}(0)$ is a function.

2. The fundamental theorem. In this context, we would like to prove what we call the fundamental theorem on intersection sequences. The use of intersection pairs allows a slightly more general statement than that of Whitney [6] and Titus [5]. Let γ be a normal immersion and let $[\gamma]$ denote the image of γ. For $a \in \mathbf{C} - \gamma(R)$ we define j_a on $S_a = S^1 - \gamma^{-1}(a)$ by $j_a = (\gamma - a)/|\gamma - a|$. We define

$$\omega(\gamma, a) = \frac{1}{2\pi i} \int_{S_a} \frac{dj_a}{j_a}.$$

If $a \notin \gamma$, this is just the winding number of γ about a. If $a \in [\gamma]$, we may check that $\omega(\gamma, a)$ is the average of the winding numbers of γ on the components near $\gamma(a)$.

Now, for fixed $z_0 \in S^1$, consider $z_0 \times S^1$ and $S^1 \times z_0$ as subsets of Y. Let $\theta \to (z_0, e^{i\theta})$ and $\phi \to (e^{i\phi}, z_0)$ be coordinate systems on $z_0 \times S^1$ and let these define the orientations. Thus, $z_0 \times S^1$ and $S^1 \times z_0$ have the structures of oriented 1-submanifolds of Y. Now $W = z_0 \times S^1 + S^1 \times z_0 - \Delta$ divides the torus Y into 2 simply connected 2-manifolds with boundary, Y^+ and Y^-. Here Y^+ denotes the one for which W is a positively oriented boundary and Y^- the one for which W is a negatively oriented boundary (see Fig. 1).

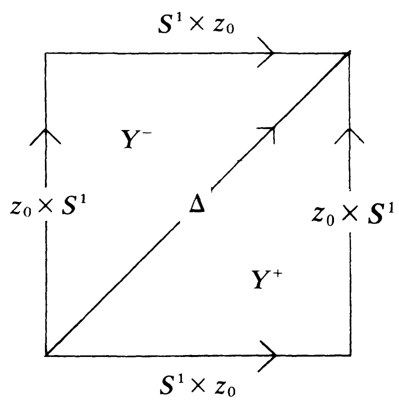

FIG. 1

If γ is an immersion, and $\beta = d\gamma/d\theta$ is the tangent, then the tangent winding number, *twn* γ, is defined to be

$$\frac{1}{2\pi i}\int_{S^1}\frac{d\beta}{\beta}.$$

We now have

THEOREM 1 (Titus–Whitney). *If* 0 *is a regular value for* $g, z_0 \in S^1$, *and* Y^+ *is the oriented* 2-*submanifold of* $S^1 \times S^1$ *with positively oriented boundary* $z_0 \times S^1 + S^1 \times z_0 - \Delta$, *then*

$$twn\,\gamma = -\sum_{y\in Y^+\cap g^{-1}(0)}\lambda(y) + 2\omega(\gamma, \gamma(z_0)).$$

Proof. Let $g^{-1}(0)\cap Y^+ = \{y_1, \cdots, y_n\}$. Let $D_1, \cdots, D_n$ be closed disjoint coordinate discs in Y^+ such that $D_j \cap g^{-1}(0) = Y_j$ for $j = 1, \cdots, n$. Let these have orientation inherited from Y and let ∂D_j be the oriented boundary of D_j for $j = 1, \cdots, n$. Recall that for $j = 1, \cdots, n$, $\lambda(y_j) = +1$ iff dg preserves orientation at y_j. Therefore we may choose each D_j so that

$$\frac{1}{2\pi i}\int_{D_j}\frac{dg}{g} = \lambda(y_j).$$

Now dg/g is closed on $Y^+ - \bigcup_{j=1}^n D_j$ so the integral of dg/g over its boundary is 0. The boundary is the cycle $z_0 \times S^1 + S^1 \times z_0 - \Delta - \Sigma_{j=1}^n \partial D_j$. From the definition of g,

$$\frac{1}{2\pi i}\int_{z_0\times S^1}\frac{dg}{g} = \frac{1}{2\pi i}\int_{S^1\times z_0}\frac{dg}{g} = \omega(\gamma, \gamma(z_0)) - 1/2$$

and $(1/2\pi i)\int_\Delta dg/g = twn\,\gamma - 1$. The theorem now follows. We remark that if $\gamma(z_0)$ is on the outer boundary of γ and its image is not a multiple point of γ, then $\omega(\gamma, \gamma(z_0)) = \pm\frac{1}{2}$. In this case, if γ is a normal immersion, then Theorem 1 is Lemma 3 of Titus **[5]**.

3. Homotopies. Let $I = [0, 1]$ considered as an oriented 1-manifold with boundary having the usual orientation. Let $I \times S^1$ be an oriented 2-manifold with boundary with the product orientation. A smooth map $F: I \times S^1 \to \mathbf{C}$ is called a homotopy. Let $\gamma_t(z) = F(t, z)$ and $\beta_t(z) = (d\gamma_t/d\theta)(z)$. Let $X = I \times S^1 \times S^1$ and $Y_t = t \times S^1 \times S^1 \subseteq X$ where both are given the product orientations. Define $G: X \to \mathbf{C}$ by

$$G(t,z,\zeta)=\begin{cases}\dfrac{F(t,z)-F(t,\zeta)}{z-\zeta}, & z\neq\zeta\\[2ex] \dfrac{-i\beta_t(z)}{z}, & z=\zeta.\end{cases}$$

Define $g_t\colon S^1\times S^1\to\mathbf{C}$ by $g_t(z,\zeta)=G(t,z,\zeta)$. Let $Z=\{x\in X\mid G(x)=0\}$. We say 0 is a regular value for G if dG has rank 2 everywhere on Z. In this case, by the implicit function theorem, Z has the structure of a 1-submanifold of X, with boundary. We intend to study the change in the intersection sequence under the homotopy F by looking at the smooth manifold $Z\subseteq X$, therefore we will make the assumption that 0 is a regular value for G.

To justify this assumption, we prove the following lemma.

LEMMA 1. *If $F(t,z)=\gamma_t(z)$ is a smooth homotopy of closed curves and $G(F)\colon I\times S^1\times S^1\to\mathbf{C}$ is defined by $G(F)(t,z,\zeta)=(F(t,z)-F(t,\zeta))/(z-\zeta)$, then F may be deformed by an arbitrarily small amount into a homotopy F for which 0 is a regular value for $G(F)$.*

Proof. Let D be the open disc $|w|<1$. For $w\in D$, define $F_w(t,z)=F(t,z)+wz$. Note that $F_0(t,z)=F(t,z)$. Then $G(F_w)(t,z,\zeta)=G(F)(t,z,\zeta)+w$. Clearly the map $(t,z,\zeta,w)\to G(F_w)(t,z,\zeta)+w$ from $(I\times S^1\times S^1)\times D$ to $\mathbf{C}$ is a submersion, and therefore 0 is a regular value for this function. By the transversality theorem (Guillemin and Pollack **[3]** p. 68), 0 is a regular value of $G(F_w)$ for almost all $w\in D$. This proves the lemma.

4. The orientation on Z. Assume that 0 is a regular value of G so that Z is a 1-manifold with boundary. We will define an orientation on Z such that we get a set of signed intersection pairs for γ_t by intersecting Z with Y_t. At each intersection point, the sign will be defined by the orientation of Z and Y_t.

First we indicate how to define a direct sum orientation on vector spaces. If V and W are oriented subspaces of a vector space and if the ordered bases $\{v_1,\cdots,v_n\}$ and $\{w_1,\cdots,w_m\}$ define positive orientation of V and W respectively, then the sum orientation on $V\oplus W$ (in that order) is defined by the ordered basis $\{v_1,\cdots,v_n,w_1,\cdots,w_m\}$.

We now orient Z as follows: If $x\in Z$, write $T_x(X)=T_x(Z)\oplus H$. Then $dG_x\colon H\to T_0(\mathbf{C})$ and the mapping is a vector space isomorphism. In a natural way, this isomorphism induces an orientation on H from the usual orientation on $T_0(\mathbf{C})$. We now choose an orientation on $T_x(Z)$ so that the sum orientation agrees with the prescribed orientation on $T_x(X)$. In this way Z is given the structure of an oriented 1-manifold.

Now as before let $Y_t = t \times S^1 \times S^1$ with the product orientation. Suppose $x = (t, z, \zeta) \in Z \cap Y_t$ and $d(g_t)_{(z,\zeta)}$ preserves orientation. Then dG_x preserves orientation on $T_x(Y_t)$. Now we can can write $T_x(X) = T_x(Z) \oplus T_x(Y_t)$ where by definition, the orientations sum to the prescribed orientation on $T_x(X)$. In this case the intersection number at $x \in Z \cap Y_t$ is said to be $+1$ (here the order in which we list Z and Y_t is important (see Guillemin and Pollack [3])). Likewise if $d(g_t)_{(z,\zeta)}$ reverses orientation, the intersection number of $x \in Z \cap Y_t$ is -1. Thus if $d(g_t)_{(z,\zeta)}$ has rank 2 at each point $x \in Z$ then the set $Z \cap Y_t$ along with the intersection number at each point gives us the set of signed intersection pairs for γ_t.

5. The change in the intersection sequences. The configuration of the oriented 1-manifold Z as a submanifold of X indicates how the intersection pairs and the intersection sequence changes under the homotopy F. (We may take the intersection sequence with respect to a continuously moving point whose image stays on the outer boundary.) We mention here only some general considerations:

(a) Z is symmetric with respect to $I \times \Delta$, i.e., $(t, z, \zeta) \in Z$ iff $(t, \zeta, z) \in Z$.

(b) The components of Z are oriented 1-manifolds homeomorphic to either S^1 or I (see Guillemin and Pollack [3] Appendix 2 or Milnor [4] Appendix).

(c) Each component either crosses $I \times \Delta$ and is symmetric with respect to $I \times \Delta$ or has another component symmetric to it with respect to $I \times \Delta$ (see Fig. 2).

(d) When a component of Z crosses $I \times \Delta$ we have a change in $twn\,\gamma_t$. We will describe this fully in the next section.

(e) Each component of Z represents a continuously moving vertex on γ_t. Components homeomorphic to I and joining points on Y_0 represent vertices lost in homotopy. Components homeomorphic to I and joining points in Y_1 represent vertices gained.

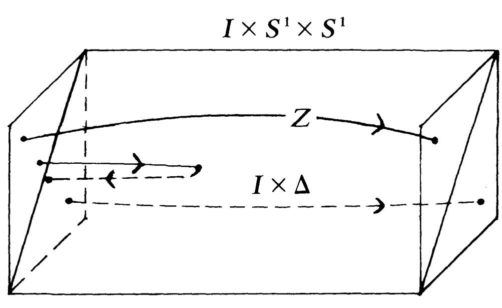

FIG. 2

Finally, suppose that $\Pi: X = I \times S^1 \times S^1 \to I \times S^1$ is the projection on the first two coordinates. Then $\Pi(Z) \subseteq I \times S^1$ consists of smooth curves. If the intersection sequence of γ_t changes at t_0, then either some vertices coincide, in which case $\Pi(Z)$ crosses itself at a point (t_0, z) or else a vertex appears or disappears, in which case the real valued function t on Z has a relative maximum or minimum at a point (t_0, z, ζ) on Z.

6. Change in $twn\ \gamma_t$.

Let $I \times \Delta \subseteq X$ have the usual product orientation. Say Z intersects $I \times \Delta$ transversely if $T_x(Z) \oplus T_x(I \times \Delta) = T_x(X)$ at each point $x \in Z \cap (I \times \Delta)$. Let $N(Z, I \times \Delta)$ be the intersection multiplicity of Z with $I \times \Delta$, i.e., the sum of the intersection numbers at points of $Z \cap (I \times \Delta)$. We prove the following theorem concerning the change in $twn\ \gamma_t$ for the homotopy.

THEOREM 2. *If Z intersects $I \times \Delta$ transversely, then $twn\ \gamma_1 - twn\ \gamma_0 = N(Z, I \times \Delta)$.*

Proof. Let $Z \cap (I \times \Delta) = \{y_1, \cdots, y_n\}$. At $y = y_j$ write $T_y(X) = T_y(Z) \oplus T_y(I \times \Delta)$. By definition of the intersection number at y_j and by definition of the orientation of Z we see that the intersection number at $y = y_j$ is $+1$ iff dG_y preserves orientation on $T_y(I \times \Delta)$. Now we can choose closed disjoint coordinate discs $D_1, \cdots, D_n$ in $I \times \Delta$ such that $D_j \cap Z = y_j$ for $j = 1, \cdots, n$ and $(1/2\pi i)\int_{\partial D_j} dG/G =$ the orientation number at $y_j \in Z \cap (I \times \Delta)$. Now dG/G is closed on $I \times \Delta - \bigcap_{j=1}^n D_j$ and the boundary is $1 \times \Delta - 0 \times \Delta - \Sigma_{j=1}^n \partial D_j$. Now $(1/2\pi i)\int_{0 \times \Delta} dG/G = twn\ \gamma_0$ and $(1/2\pi i)\int_{1 \times \Delta} dG/G = twn\ \gamma_1$, and integration of dG/G over the boundary gives 0. This proves the theorem.

We have the following well-known:

COROLLARY 1. *Regular homotopies preserve the tangent winding number.*

Proof. In this case $Z \cap (I \times \Delta) = \varnothing$.

Finally, we remark that the fundamental theorem of Titus and Whitney becomes in this context:

THEOREM 3. *Suppose for fixed $t \in I$ and $z_0 \in S^1$, Y_t^+ is the oriented submanifold of $I \times S^1 \times S^1$ with positively oriented boundary $t \times z_0 \times S^1 + t \times S^1 \times z_0 - t \times \Delta$. If Z intersects Y_t^+ transversely,*

$$N(Z, Y_t^+) = twn\, \gamma_t - 2\omega(\gamma_t, \gamma_t(z_0)).$$

Proof. We observe that if $x = (t, z, \zeta) \in Z \cap Y_t$ then the intersection number is $+1$ iff $d(g_t)_{(z,\zeta)}$ preserves orientation. Now the theorem follows from Theorem 1.

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