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RATIONAL APPROXIMATION OF e^{-x} ON THE POSITIVE REAL AXIS

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In this paper we obtain error bounds to approximations of e^{-x} on $[0;\infty)$ by rational functions having zeros and poles only on the negative real axis.

Our main concern in this paper is the question of approximating e^{-x} on the positive real axis by reciprocals of polynomials and by rational functions, especially by those which have all their zeros and poles on the negative real axis.

NOTATION. Let π_n represent the set of all polynomials of degree $\leqq n$. Let π_n^* represent the set of all polynomials in π_n all of whose zeros are in the left half plane and π_n^{**} represent the set of all polynomials in π_n^* all of whose zeros are real and negative. Similarly let $\rho_n, \rho_n^*, \rho_n^{**}$ represent the sets of rational functions of total degree n whose numerators and denominators are in $\pi_n, \pi_n^*, \pi_n^{**}$ respectively. Let $\| \quad \|$ denote $\| \quad \|_{L_\infty[0,\infty)}$. Then we define

$$\lambda_{0,n}(f) = \inf_{p\in\pi_n} \left\| f - \frac{1}{p} \right\|,$$

$$\lambda_{0,n}^*(f) = \inf_{p\in\pi_n^*} \left\| f - \frac{1}{p} \right\|,$$

$$\lambda_{0,n}^{**}(f) = \inf_{p\in\pi_n^{**}} \left\| f - \frac{1}{p} \right\|,$$

$$\lambda_n(f) = \inf_{r\in\rho_n} \| f - r \|,$$

$$\lambda_n^*(f) = \inf_{r\in\rho_n^*} \| f - r \|,$$

$$\lambda_n^{**}(f) = \inf_{r\in\rho_n^{**}} \| f - r \|.$$

LEMMA (Newman [1], Theorem 2). *Let $p \in \pi_n^{**}$ where $n \geqq 2$, then*

$$\| e^x - p \|_{L_\infty[0,1]} \geqq (16n+1)^{-1}.$$

We obtain the following results.

(Theorems 1, 2): $(17e^2n)^{-1} \leqq \lambda_{0,n}^{**}(e^{-x}) \leqq (en)^{-1}$, $n \geqq 2$.

(Theorem 3): $\lambda^*_{0,2n}(e^{-x}) \leqq 2(ne)^{-2}$, $n \geqq 1$.
(Theorems 4, 5): $e^{-6\sqrt{n}} \leqq \lambda^{**}_n(e^{-x}) \leqq n^{-c \log n}$, $n \geqq 2$.
(Theorem 6): $e^{-5n^{2/3}} \leqq \lambda^*_n(e^{-x})$, $n \geqq 2$.

THEOREM 1. *For all $n \geqq 1$,*

$$\left\| e^{-x} - \left(1 + \frac{x}{n}\right)^{-n} \right\| \leqq \frac{1}{ne}. \tag{1}$$

Proof. For all $x \geqq 0$ and $n \geqq 1$ we have

$$0 \leqq \left(1 + \frac{x}{n}\right)^n \leqq e^x.$$

Hence

$$0 \leqq \left(1 + \frac{x}{n}\right)^{-n} - e^{-x} \leqq \left(1 + \frac{x}{n}\right)^{-n} - \left(1 + \frac{x}{n}\right)^{-n-1} \leqq \frac{1}{ne} \quad \text{for all} \quad x \geqq 0,$$

because, $(1 + (x/n))^{-n} - e^{-x}$ attains its maximum when $e^x = (1 + (x/n))^{n+1}$. Hence (1) follows.

THEOREM 2. *For all $n \geqq 2$ we have*

$$\lambda^{**}_{0,n}(e^{-x}) \geqq (17e^2n)^{-1}. \tag{2}$$

Proof. Set

$$\left\| e^{-x} - \frac{1}{p_n(x)} \right\| = \epsilon. \tag{3}$$

Then

$$| e^x - p_n(x) \|_{L_\infty[0,1]} \leqq \epsilon e p_n(1),$$

since $p_n(x)$ has only nonnegative coefficients. From (3), we get

$$[p_n(1)]^{-1} \geqq e^{-1} - \epsilon = \frac{1 - \epsilon e}{e}. \tag{4}$$

From (3) and (4), we have

$$\| e^x - p_n(x) \|_{L_\infty[0,1]} \leqq \frac{\epsilon e^2}{1 - \epsilon e}. \tag{5}$$

On the other hand we have from the lemma that

(6) $$\|e^x - p_n(x)\|_{L_\infty[0,1]} \geqq (16n+1)^{-1}.$$

From (5) and (6), we get

$$\epsilon e^2(16n+1) \geqq 1 - \epsilon e.$$

Hence (2) follows.

THEOREM 3. *For all even n*

$$\left\| e^{-x} - \left(1 + \frac{2x}{n} + \frac{2x^2}{n^2}\right)^{-n/2} \right\| \leqq 8(ne)^{-2}.$$

Proof. For all $x \geqq 0$, $n \geqq 1$, we have

$$\exp\left(\frac{2x}{n}\right) \geqq \left(1 + \frac{2x}{n} + \frac{2x^2}{n^2}\right).$$

We also know that

$1 + x + x^2/2!$ has zeros only in the left half plane.

The function

$$\left(1 + \frac{2x}{n} + \frac{2x^2}{n^2}\right)^{-n/2} - e^{-x}$$

attains its maximum when

$$e^x = \left(1 + \frac{2x}{n} + \frac{2x^2}{n^2}\right)^{n/2+1} \left(1 + \frac{2x}{n}\right)^{-1}.$$

Therefore

$$0 \leqq \left(1 + \frac{2x}{n} + \frac{2x^2}{n^2}\right)^{-n/2} - e^{-x} \leqq \frac{2x^2}{n^2 e^x} \leqq \frac{8}{n^2 e^2}.$$

Hence the theorem is proved.

THEOREM 4. *There is a constant $c > 0$ so that for all $n \geqq 2$, we have*

(7) $$\lambda_n^{**}(e^{-x}) \leqq n^{-c \log n}.$$

Proof. We use the following formula.

$$(8)\qquad \sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{1}{k+s}=\frac{m!}{s(s+1)(s+2)\cdots(s+m)}.$$

Set $N=\text{l.c.m}[1,2,\cdots,m]$, $t=N/s\geqq 0$ and $\epsilon=(m!)^2N^{-m}$. Then using the fact that $t^m\leqq m!e^t$, we get

$$(9)\quad 1+\sum_{k=1}^{m}(-1)^k\frac{N}{k}\binom{m}{k}\frac{1}{t+\frac{N}{k}}=\frac{m!t^m}{(N+t)(N+2t)\cdots(N+mt)}$$
$$\leqq\frac{m!t^m}{N^m+m!t^m}$$
$$\leqq\frac{(m!)^2e^t}{N^m+(m!)^2e^t}\leqq\frac{\epsilon e^t}{1+\epsilon e^t}.$$

By integrating (9) with respect to t from 0 to x we get

$$(10)\qquad 0\leqq x+\log R(x)\leqq\log(1+\epsilon e^x),$$

where $R(x)=\Pi_{k=1}^{m}(1+(xk/N))^{(-1)^k(N/k)\binom{m}{k}}$. From (10), we get

$$0\leqq e^xR(x)-1\leqq\epsilon e^x.$$

That is

$$0\leqq R(x)-e^{-x}\leqq\epsilon.$$

From prime number theory we know that there exist positive constants α, β so that $e^{\alpha m}<N<e^{\beta m}$ for all $m\geqq 1$. Hence $\deg R(x)\leqq N2^m\leqq n$ if we choose $\gamma\log n<m<\delta\log n$, where γ, δ are positive constants. From this choice of m, we obtain (7). That is,

$$\epsilon\leqq n^{-c\log n}\quad\text{as required.}$$

THEOREM 5. *For all $n\geqq 2$ we have*

$$(11)\qquad \lambda_n^{**}(e^{-x})\geqq e^{-6\sqrt{n}}.$$

Proof. In (8) set $s=m(1+t)$ and integrate, then we get

$$(12)\qquad \sum_{k=0}^{m}(-1)^k\binom{m}{k}\log\left(1+\frac{mUA}{m+k}\right)$$
$$=\frac{1}{\binom{2m}{m}}\int_0^{UA}\frac{dt}{(1+t)\left(1+\frac{tm}{m+1}\right)\cdots\left(1+\frac{tm}{2m}\right)},$$

and observe that for $U, A > 0$ the right side of (12) is bounded by

$$\frac{1}{\binom{2m}{m}} \int_0^\infty \frac{dt}{\left(1+\frac{t}{2}\right)^{m+1}} = \frac{2}{m\binom{2m}{m}}. \tag{13}$$

Again (8) with $s = m$ give us

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{mA}{m+k} = \frac{A}{\binom{2m}{m}}. \tag{14}$$

Assume there is a rational function of total degree n, set.

$$r(x) = e^{-c} \prod_{k=1}^{n} (1 + xu_i)^{\epsilon_i}, \qquad \epsilon_i = \pm 1, \qquad u_i \geqq 0,$$

such that

$$\| e^{-x} - r(x) \| = \epsilon,$$

thus

$$\| e^x r(x) - 1 \|_{L_\infty[0, A]} \leqq \epsilon e^A. \tag{15}$$

From (15), we obtain

$$c - \sum_{i=1}^{n} \epsilon_i \log(1 + xu_i) + x \leqq \log(1 + \epsilon e^A) < \epsilon e^A, \quad \text{for} \quad 0 \leqq x \leqq A.$$

Now set $x = mA/(m+k)$ to get

$$c - \sum_{i=1}^{n} \epsilon_i \log\left(1 + \frac{mAu_i}{m+k}\right) + \frac{mA}{m+k} < \epsilon e^A, \qquad k = 0, 1, 2, \cdots, m. \tag{16}$$

Applying the difference operator m times on both sides of (16). We get in view of (13) and (14),

$$-\frac{2n}{m} + A \leqq \binom{2m}{m} 2^m \epsilon e^A. \tag{17}$$

Now choose $m = [\sqrt{n}]$, $A = 3\sqrt{n}$ then

$$\epsilon \geqq \sqrt{n}(2e)^{-3\sqrt{n}} > e^{-6\sqrt{n}}, \qquad \text{as required.}$$

THEOREM 6. *For all $n \geqq 2$, we have*

$$\lambda_n^*(e^{-x}) \geqq e^{-5n^{2/3}}.$$

Proof. The proof of this theorem is not very different from the proof of Theorem 5, except that we use $t = ve^{i\theta}$, $|\theta| \leqq \pi/2$, and obtain

$$\begin{aligned}(18)\qquad &\sum_{k=0}^{m} (-1)^k \binom{m}{k} \log\left(1 + \frac{muA}{m+k}\right) \\ &= \frac{1}{\binom{2m}{m}} \int_0^{UA} \frac{dt}{(1+t)\left(1+\frac{tm}{m+1}\right)\cdots\left(1+\frac{tm}{2m}\right)} \\ &\leqq \frac{1}{\binom{2m}{m}} \int_0^{\infty} \frac{dv}{\left(1+\frac{v^2}{2}\right)^{m/2}} \leqq \frac{1}{\binom{2m}{m}\sqrt{m}}.\end{aligned}$$

Now by using (18) instead of (12) and (13), we obtain as in the case of Theorem 5,

$$-\frac{n}{\sqrt{m}} + A \leqq 2^m e^A \binom{2m}{m} \epsilon.$$

Choose $m = [n^{2/3}]$, $A = 2n^{2/3}$ then we get

$$\epsilon \geqq n^{2/3} 8^{-n^{2/3}} e^{-2n^{2/3}} > e^{-5n^{2/3}} \quad \text{as required.}$$

REFERENCE

1. D. J. Newman, *Rational approximation to e^x with negative zeros and poles*, J. Approximation Theory (to appear).

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