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ROOTS OF THE EULER POLYNOMIALS

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In this paper we prove some new theorems about the real and complex roots of the Euler polynomials. For each n we show how the real roots of $E_n(x)$ are distributed in the closed interval [1, 3]. We also show how the real roots of $E_n(x)$ are distributed in the arbitrary interval $[m, m+1]$ for n sufficiently large. Finally, we prove that if a and b are nonzero rational numbers and c is a square-free integer, then $E_n(x)$ has no roots of the form $a\sqrt{c}$, $c \neq 1$, or $a + b\sqrt{c}, c$ even, or $a + bi$, a and b integers.

1. Introduction. The Euler polynomial $E_n(x)$ degree n can be defined as the unique polynomial satisfying

$$E_n(x+1) + E_n(x) = 2x^n \qquad (n \geqq 0). \tag{1.1}$$

These polynomials have been extensively studied; see [**3**, Chapter VI] and [**4**, Chapter II] for example. The first fifteen Euler polynomials are listed in [**5**, p. 477].

In this paper we are primarily concerned with the real roots of $E_n(x)$, though we also prove a few results about the complex roots. It is well known that if n is even, $n > 0$, then the only real roots of $E_n(x)$ in the closed interval [0, 1] are 0 and 1, while if n is odd the only real root in [0, 1] is 1/2. Brillhart [**1**] has pointed out that these are the only complex roots in the "critical strip" of all complex numbers $x + iy$, $0 \leqq x \leqq 1$. In the same paper Brillhart proved that $E_5(x)$ is the only Euler polynomial with a multiple root and that the Euler polynomials have no rational roots other than 0, 1, 1/2.

The main results in this paper are:

(1) On the closed interval [1, 3] we show how the real roots of $E_n(x)$ are distributed for each n.

(2) On each interval $[m, m+1]$, $m > 0$, we show how the real roots of $E_n(x)$ are distributed for n sufficiently large.

(3) Let a and b be nonzero rational numbers and let c and d be square-free integers. The polynomial $E_n(x)$ has no roots of the form $a\sqrt{c}$, $(c \neq 1)$, $a + b\sqrt{c}$ (c even), $a\sqrt{d} + b\sqrt{c}\, i$ (c and d of different parity); or $a + bi$ (a, b integers).

It is pointed out that results similar to (3) are also true for the Bernoulli polynomials.

2. Preliminaries. Throughout this paper we use the notation of Nörlund [4]. The following are well-known identities:

$$E'_n(x) = nE_{n-1}(x) \qquad (n > 0), \tag{2.1}$$

$$E_n(1-x) = (-1)^n E_n(x), \tag{2.2}$$

$$E_n(x) = \sum_{s=0}^{n} \binom{n}{s} 2^{-s} C_s x^{n-s} \tag{2.3}$$

where

$$C_{s-1} = \frac{2^s(1-2^s)}{s} B_s.$$

In formula (2.3), B_s is the s'th Bernoulli number (see [4, pp. 17–23]). If s is odd, $s > 1$, then $B_s = 0$. If s is even, $s > 0$, then the denominator of B_s is even and square-free.

The Euler polynomials are related to, and often studied in conjunction with, the Bernoulli polynomials $B_n(x)$ [3, Chapter V], [4, Chapter II]. The Euler and Bernoulli polynomials are related by

$$nE_{n-1}(x) = 2^n \left[B_n\left(\frac{x+1}{2}\right) - B_n\left(\frac{x}{2}\right) \right]. \tag{2.4}$$

The numbers E_{2k} defined by

$$E_{2k} = 2^{2k} E_{2k}(1/2) \tag{2.5}$$

are known as the Euler numbers and have the following properties:

$$(-1)^k E_{2k} > 0, \tag{2.6}$$

$$(-1)^k (2\pi)^{2k+1} E_{2k} = 2^{4k+3}(2k)! \sum_{n=0}^{\infty} (-1)^n (2n+1)^{-2k-1}. \tag{2.7}$$

The first sixty Euler numbers, as well as the first sixty Bernoulli numbers and the first fifteen Bernoulli polynomials are listed in [5, pp. 477–479].

From (2.7) and inequalities proved in [3, pp. 294–295, 302], it follows that for $k > 0$

$$(2k-1)!/4^{2k-1} < |E_{2k-1}(0)| < 2(2k-1)!/3^{2k-1}, \tag{2.8}$$

$$(2k)!/2^{2k} < |E_{2k}|, \tag{2.9}$$

$$(2\pi)^2 |E_{2k}| > 16(2k)(2k-1)|E_{2k-2}|. \tag{2.10}$$

Finally, we shall use the following formulas which are derived by expanding $E_n(x)$ into a series about $x = a$ and then using (2.3).

(2.11) $$E_{2k}(a + b\sqrt{d})/(2k)!$$
$$= \sum_{r=0}^{k}\sum_{s=0}^{2r} d^{k-r}b^{2k-2r}a^{2r-s}C_s/2^s(2k-2r)!\,s!\,(2r-s)!$$
$$+ \sqrt{d}\sum_{r=0}^{k-1}\sum_{s=0}^{2r+1} d^{k-r-1}b^{2k-2r-1}a^{2r+1-s}C_s/2^s(2k-2r-1)!\,s!\,(2r+1-s)!$$

(2.12) $$E_{2k+1}(a + b\sqrt{d})/(2k+1)!$$
$$= \sum_{r=0}^{k}\sum_{s=0}^{2r+1} d^{k-r}b^{2k-2r}a^{2r+1-s}C_s/2^s(2k-2r)!\,s!\,(2r+1-s)!$$
$$+ \sqrt{d}\sum_{r=0}^{k}\sum_{s=0}^{2r} d^{k-r}b^{2k+1-2r}a^{2r-s}C_s/2^s(2k+1-2r)!\,s!\,(2r-s)!$$

The numbers C_s in (2.11) and (2.12) are defined by (2.3).

3. Distribution of the real roots of $E_n(x)$. Inkeri **[2]** has shown how the positive real roots of the Bernoulli polynomials are distributed outside of the interval [0, 1]. To the author's knowledge this has not been attempted for the Euler polynomials. By (2.2), if we restrict our attention to the positive real roots we can determine how *all* the roots are distributed. Thus we shall only consider the positive real roots and we shall use (1.1), which tells us that if $E_n(a) < 0$ then $E_n(1 + a) > 0$.

First we note that if m is a positive integer we have, by (1.1),

(3.1) $$E_n(m) = (-1)^m E_n(0) + 2\sum_{k=0}^{m-2}(-1)^k(m-1-k)^n,$$

(3.2) $$E_n(m + 1/2) = (-1)^m E_n(1/2) + 2\sum_{k=0}^{m-1}(-1)^k(m-k-1/2)^n.$$

Since $E_n(0) = 0$ if n is even and $E_n(1/2) = 0$ if n is odd, we see that

(3.3) $$E_n(m) > 0 \quad \text{if } n \text{ is even},$$

(3.4) $$E_n(m + 1/2) > 0 \quad \text{if } n \text{ is odd}.$$

Furthermore, by (2.3) and (3.1),

(3.5) $$E_{4k+1}(m) > 0 \quad \text{if } m \text{ is odd},$$

(3.6) $$E_{4k+3}(m) > 0 \quad \text{if } m \text{ is even.}$$

By (2.6) and (3.2), we see that

(3.7) $$E_{4k+2}(m + 1/2) > 0 \quad \text{if } m \text{ is odd,}$$

(3.8) $$E_{4k+4}(m + 1/2) > 0 \quad \text{if } m \text{ is even.}$$

THEOREM 3.1. *Let $k > 0$. Then $E_{4k}(x)$ has exactly one real root α_1 in the open interval $(1,2)$ and $3/2 < \alpha_1 < 2$;*
$E_{4k+1}(x)$ has exactly one real root α_2 in $(1,2)$ and $3/2 < \alpha_2 < 2$;
$E_{4k+2}(x)$ has no real roots in $(1,2)$;
$E_{4k+3}(x)$ has exactly one real root α_3 in $(1,2)$ and $1 < \alpha_3 < 3/2$.

Proof. The proof for $E_{4k}(x)$ is due to Brillhart **[1]**. By (3.1), (3.3) and (3.8), we know that $E_{4k}(1) = 0$, $E_{4k}(2) = 2$, $E_{4k}(3/2) < 0$. Furthermore, since $E_{4k-2}(x) < 0$ for $0 < x < 1$, we know $E_{4k-2}(x) > 0$ for $1 < x < 2$. Thus, by (2.1), $E_{4k}(x)$ is concave up for $1 < x < 2$ and has exactly one real root α_1 in $(1,2)$, $3/2 < \alpha_1 < 2$.

Now the theorem is true for $E_5(x) = (x - 1/2)(x^2 - x - 1)^2$, so we examine $E_{4k+1}(x)$ for $k \geqq 2$. We know that $E_{4k+1}(1) > 0$, $E_{4k+1}(3/2) > 0$ and $E_{4k+1}(2) = 2 + E_{4k+1}(0)$. Since by (2.3) $E_{4k+1}(0) < 0$ and since $E_9(0) = -15.5$, we see from (2.8) that $E_{4k+1}(2) < 0$. We know there is exactly one number α_1 in $(1,2)$ such that $E'_{4k+1}(\alpha_1) = 0$. Hence $E_{4k+1}(x)$ has exactly one real root α_2 in $(1,2)$ and $\alpha_2 > 3/2$.

We know $E_{4k+2}(x) > 0$ for $1 < x < 2$ since $E_{4k+2}(x) < 0$ for $0 < x < 1$.

We know $E_{4k+3}(1) < 0$, $E_{4k+3}(3/2) > 0$, $E_{4k+3}(2) > 0$. Also $E'_{4k+3}(x) > 0$ for $1 < x < 2$. Hence $E_{4k+3}(x)$ has exactly one real root α_3 in $(1,2)$ and $\alpha_3 < 3/2$.

It is clear from this proof that $\alpha_3 < \alpha_2 < \alpha_1$.

THEOREM 3.2. *Let $k \geqq 4$. Then $E_{4k+1}(x)$ has exactly one real root α_{4k+1} in the closed interval $[2,3]$ and $2 < \alpha_{4k+1} < 5/2$;*
$E_{4k+2}(x)$ has exactly two real roots $\alpha^{(1)}_{4k+2}$, $\alpha^{(2)}_{4k+2}$ in $[2,3]$ and $\alpha^{(1)}_{4k+2} < 5/2 < \alpha^{(2)}_{4k+2}$;
$E_{4k+3}(x)$ has exactly one real root α_{4k+3} in $[2,3]$ and $5/2 < \alpha_{4k+3}$;
$E_{4k+4}(x) > 0$ for $2 \leqq x \leqq 3$.
Furthermore, $\alpha^{(1)}_{4k+2} < \alpha_{4k+1} < 5/2 < \alpha_{4k+3} < \alpha^{(2)}_{4k+2}$.

Proof. We know $E_{4k+1}(2) < 0$, $E_{4k+1}(5/2) > 0$, $E_{4k+1}(3) > 0$. By Theorem 3.1 we know that $E_{4k}(x) < 0$ for $1 < x < \alpha_1$ and thus $E'_{4k+1}(x) > 0$ for $2 < x < 1 + \alpha_1$. Since $E_{4k+1}(x) < 0$ for $\alpha_2 < x \leqq 2$ and since $\alpha_2 < \alpha_1$, we see that $E_{4k+1}(x) > 0$ for $1 + \alpha_1 < x \leqq 3$. Thus $E_{4k+1}(x)$ has exactly one real root α_{4k+1} in $[2,3]$ and $2 < \alpha_{4k+1} < 5/2$.

We know that $E_{4k+2}(2) > 0$, $E_{4k+2}(3) > 0$ and we now show that for

$k \geqq 3$, $E_{4k+2}(5/2) < 0$. We shall use (3.2) and (2.10). We first observe that $-E_{14} = 199{,}360{,}981 > 2 \cdot 3^{14}$, so $E_{14}(5/2) < 0$. Now by (2.10) we see that if $|E_{2t}| > 2(3^{2t} - 1)$ then $|E_{2t+2}| > 2(3^{2t+2} - 1)$. Thus we have $E_{4k+2}(5/2) < 0$ for $k \geqq 3$. Now since $E'_{4k+2}(x) < 0$ for $2 < x < \alpha_{4k+1}$ and $E'_{4k+2}(x) > 0$ for $\alpha_{4k+1} < x < 3$, we see that $E_{4k+2}(x)$ has exactly two real roots $\alpha^{(1)}_{4k+2}$, $\alpha^{(2)}_{4k+2}$ in $[2,3]$ and $\alpha^{(1)}_{4k+2} < \alpha_{4k+1} < 5/2 < \alpha^{(2)}_{4k+2}$.

We know that $E_{4k+3}(2) > 0$, $E_{4k+3}(5/2) > 0$, and we now show that $E_{4k+3}(3) < 0$ for $k \geqq 4$. We shall use (3.1) and (2.8). We first note (by using tables) that $E_{19}(0) > 2^{20}$, so by (3.1) $E_{19}(3) < 0$. For $k = 5$ we use (2.8) and we see that $|E_{23}(0)| > 2^{24}$, and it is clear that for $k > 5$ we have $|E_{4k+3}(0)| > 2^{4k+4}$. Thus by (3.1) we see that $E_{4k+3}(3) < 0$ for $k \geqq 4$. We know that $E'_{4k+3}(x) > 0$ for $2 < x < \alpha^{(1)}_{4k+2}$ and for $\alpha^{(2)}_{4k+2} < x < 3$, while $E'_{4k+3}(x) < 0$ for $\alpha^{(1)}_{4k+2} < x < \alpha^{(2)}_{4k+2}$. It follows that $E_{4k+3}(x)$ has exactly one real root α_{4k+3} in $[2,3]$ and $5/2 < \alpha_{4k+3} < \alpha^{(2)}_{4k+2}$.

We know $E_{4k+4}(2) > 0$, $E_{4k+4}(5/2) > 0$, $E_{4k+4}(3) > 0$. Furthermore $E'_{4k+4}(x) > 0$ for $2 < x < \alpha_{4k+3}$ and $E'_{4k+4}(x) < 0$ for $\alpha_{4k+3} < x < 3$. It follows that $E_{4k+4}(x) > 0$ for $2 \leqq x \leqq 3$.

Since we assume $k \geqq 4$ in Theorem 3.2, we now look at the Euler polynomials $E_n(x)$ for $2 \leqq x \leqq 3$ and $n < 17$. If $n \leqq 8$, $E_n(x)$ is a positive increasing function on $[2, \infty)$. With the aid of (2.1), (3.1)–(3.8) and an electronic calculator, we have the following results for $9 \leqq n \leqq 16$ and the interval $[2,3]$:

$E_9(x)$ has one real root $\alpha < 5/2$ and is a positive, increasing function for $x > \alpha$.

$E_{10}(x)$ has two real roots α, β such that $\alpha < \beta < 5/2$ and $E_{10}(x)$ is a positive increasing function for $x > \beta$.

$E_{11}(x) > 0$ and is a positive, increasing function for $x > 5/2$.

$E_{12}(x)$ is a positive, increasing function for $x \geqq 2$.

$E_{13}(x)$ has one real root $\alpha < 5/2$ and is a positive, increasing function for $x > \alpha$.

$E_{14}(x)$ has two real roots α, β such that $\alpha < 5/2 < \beta$ and $E_{14}(x)$ is a positive increasing function for $x > \beta$.

$E_{15}(x)$ has two real roots α, β such that $5/2 < \alpha < \beta$ and $E_{15}(x)$ is a positive, increasing function for $x > \beta$.

$E_{16}(x) > 0$ and is a positive, increasing function for $x > 3$.

In examining the real roots of $E_n(x)$ on a fixed positive interval $[m, m+1]$ we shall use the fact that if n is sufficiently large, $E_n(0)$ and $E_n(1/2)$ dominate (3.1) and (3.2).

THEOREM 3.3. *If $k > m^2$, then on the interval $[m, m+1]$:*

$E_{4k+1}(x)$ has exactly one real root $\alpha_{4k+1}(\alpha_{4k+1} < m + 1/2)$ if m is even. $E_{4k+1}(x)$ has exactly one real root β_{4k+1} $(m + 1/2 < \beta_{4k+1})$ if m is odd.

$E_{4k+2}(x)$ has exactly two real roots $\alpha^{(1)}_{4k+2}$, $\alpha^{(2)}_{4k+2}$ $(\alpha^{(1)}_{4k+2} < m + 1/2 < \alpha^{(2)}_{4k+2})$ if m is even. $E_{4k+2}(x) > 0$ if m is odd.

$E_{4k+3}(x)$ has exactly one real root α_{4k+3} $(m+1/2<\alpha_{4k+3})$ if m is even. $E_{4k+3}(x)$ has exactly one real root β_{4k+3} $(m+1/2<\beta_{4k+3})$ if m is odd.

$E_{4k+4}(x)>0$ if m is even. $E_{4k+4}(x)$ has exactly two real roots $\beta^{(1)}_{4k+4}$, $\beta^{(2)}_{4k+4}$ $(\beta^{(1)}_{4k+4}<m+1/2<\beta^{(2)}_{4k+4})$ if m is odd. Furthermore, $\alpha^{(1)}_{4k+2}<\alpha_{4k+1}<\alpha_{4k+3}<\alpha^{(2)}_{4k+2}$ and $\beta^{(1)}_{4k+4}<\beta_{4k+3}<\beta_{4k+1}<\beta^{(2)}_{4k}$.

Proof. We have proved the theorem for $m=2$. Assume the theorem is true for any integer t such that $2\leqq t<m$.

Case 1. m odd. We first examine the interval $[m-1,m]$; since $k>m^2$, it is clear that $k-1>(m-1)^2$. Thus, by our induction hypothesis, $E_{4k-3}(x)$ has one real root α_{4k-3} in $[m-1,m]$ and $\alpha_{4k-3}<m-1/2$. Also $E_{4k-3}(m)>0$, so $E_{4k-3}(m-1)<0$. Hence $E_{4k-3}(x)>0$ for $m\leqq x\leqq 1+\alpha_{4k-3}$. Also by our induction hypothesis, $E_{4k-2}(x)$ has two real roots $\alpha^{(1)}_{4k-2}$, $\alpha^{(2)}_{4k-2}$ in $[m-1,m]$ such that $\alpha^{(1)}_{4k-2}<\alpha_{4k-3}<m-1/2<\alpha^{(2)}_{4k-2}$, and since $E_{4k-2}(m)>0$, $E_{4k-2}(m-1)>0$, we have $E_{4k-2}(x)>0$ for $1+\alpha^{(1)}_{4k-2}\leqq x\leqq 1+\alpha^{(2)}_{4k-2}$. Also by our induction hypothesis, $E_{4k-1}(x)$ has one real root α_{4k-1} in $[m-1,m]$ such that $m-1/2<\alpha_{4k-1}<\alpha^{(2)}_{4k-2}$. Also, $E_{4k-1}(m-1)>0$, so $E_{4k-1}(m)<0$. Thus $E_{4k-1}(x)>0$ for $1+\alpha_{4k-1}\leqq x\leqq m+1$. Furthermore, $E_{4k-1}(x)$ is concave up for $m\leqq x\leqq 1+\alpha_{4k-3}$ and is increasing for $1+\alpha^{(1)}_{4k-2}\leqq x\leqq 1+\alpha^{(2)}_{4k-2}$ with $\alpha^{(1)}_{4k-2}<\alpha_{4k-3}<\alpha_{4k-1}<\alpha^{(2)}_{4k-2}$. Hence $E_{4k-1}(x)$ has exactly one real root β_{4k-1} in $[m,m+1]$ and $\beta_{4k-1}<m+1/2$. Also, $E_{4k-1}(x)<0$ for $m\leqq x<\beta_{4k-1}$, $E_{4k-1}(x)>0$ for $\beta_{4k-1}<x\leqq m+1$.

Now that we know the behavior of $E_{4k-1}(x)$ on $[m,m+1]$ we are ready to prove the theorem. We know that $E_{4k}(m)>0$, $E_{4k}(m+1)>0$ and by (3.2) and (2.9) we have $E_{4k}(m+1/2)<0$. This last inequality follows from the fact that if $k\geqq m^2$ then $(4k)!>2(4m)^{4k}$, which can be proved in a straightforward elementary way. Also $E'_{4k}(x)<0$ for $m\leqq x<\beta_{4k-1}$ and $E'_{4k}(x)>0$ for $\beta_{4k-1}<x\leqq m+1$. It follows that $E_{4k}(x)$ has exactly two real roots $\beta^{(1)}_{4k}$, $\beta^{(2)}_{4k}$ such that $\beta^{(1)}_{4k}<\beta_{4k-1}<m+1/2<\beta^{(2)}_{4k}$.

We now continue in the same way for $E_{4k+1}(x)$. We have $E_{4k+1}(m)>0$, $E_{4k+1}(m+1/2)>0$ and $E_{4k+1}(m+1)<0$. Also $E'_{4k+1}(x)>0$ for $m\leqq x<\beta^{(1)}_{4k}$ and $\beta^{(2)}_{4k}<x\leqq m+1$, while $E'_{4k+1}(x)<0$ for $\beta^{(1)}_{4k}<x<\beta^{(2)}_{4k}$. Thus $E_{4k+1}(x)$ has exactly one real root β_{4k+1} in $[m,m+1]$ and $m+1/2<\beta_{4k+1}<\beta^{(2)}_{4k}$. We know that $E_{4k+2}(m)>0$, $E_{4k+2}(m+1/2)>0$, $E_{4k+2}(m+1)>0$, $E'_{4k+2}(x)>0$ for $m\leqq x<\beta_{4k+1}$, $E'_{4k+2}(x)<0$ for $\beta_{4k+1}<x\leqq m+1$. Thus $E_{4k+2}(x)>0$ for $m\leqq x\leqq m+1$. We know that $E_{4k+3}(m)<0$, $E_{4k+3}(m+1/2)>0$, $E_{4k+3}(m+1)>0$, $E'_{4k+3}(x)>0$ for $m\leqq x\leqq m+1$. Thus $E_{4k+3}(x)$ has exactly one real root β_{4k+3} in $[m,m+1]$ and $\beta_{4k+3}<m+1/2$. We know $E_{4k+4}(m)>0$, $E_{4k+4}(m+1/2)<0$, $E_{4k+4}(m+1)>0$, $E'_{4k+4}(x)<0$ for $m\leqq x<\beta_{4k+3}$, $E'_{4k+4}(x)>0$ for $\beta_{4k+3}<$

$x \leqq m+1$. Hence $E_{4k+4}(x)$ has exactly two real roots $\beta^{(1)}_{4k+4}$, $\beta^{(2)}_{4k+4}$ in $[m, m+1]$ and $\beta^{(1)}_{4k+4} < \beta_{4k+3} < m + 1/2 < \beta_{4k+1} < \beta^{(2)}_{4k}$.

Case 2. m even. In this case we first prove the theorem for $E_{4k+1}(x)$, treating $E_{4k+1}(x)$ in exactly the same way we treated $E_{4k-1}(x)$ when m was odd. The rest of the proof is entirely analogous to the proof of Case 1. That is, we first examine $E_{4k-1}(x)$ and $E_{4k}(x)$ on the interval $[m-1, m]$ and then show $E_{4k+1}(x)$ satisfies the theorem on $[m, m+1]$. Once we know the behavior of E_{4k+1} on $[m, m+1]$ we can easily determine the behavior of $E_{4k+2}(x)$, $E_{4k+3}(x)$ and $E_{4k+4}(x)$ on $[m, m+1]$.

It is known that $E_n(x)$ is a positive increasing function when x is sufficiently large, i.e., $x > x_0$. The next theorem gives us an upper bound for x_0.

THEOREM 3.4. *The polynomials* $E_{4k+s}(x)$, $s = 1, 2, 3, 4$, *are positive increasing functions on* $[k+1, \infty)$.

Proof. We have seen that the theorem is true for $k = 1, 2, 3$. Assume it is true for all $m < k$, and suppose k is even. By (3.3) and (3.5) we see that $E_{4k+s}(k+1) > 0$ for $s = 1, 2, 4$ and we are assuming $E_{4k}(k+1)$ is a positive increasing function on $[k, \infty)$. Thus the only difficulty is to show that $E_{4k+3}(k+1) > 0$. We shall use (3.1), inequality (2.8) and the inequality

$$2(k-1)^{4k+3} < 2 \sum_{r=0}^{k-1} (-1)^r (k-r)^{4k+3}.$$

Thus if we can show that

$$(4k+3)! < [3(k-1)]^{4k+3}, \qquad k \geqq 4, \tag{3.9}$$

then it follows that $E_{4k+3}(k+1) > 0$. We prove (3.9) by first verifying the case $k = 4$ from tables and then observing that

$$(3k-2+a)(k+6-a) < (3k-3)^2$$

for $a = 0, 1, \cdots, k+5$, with $k \geqq 5$. The proof for k odd is very similar.

Theorem 3.4 can almost certainly be improved. In fact we conjecture that the polynomials $E_{8k+s}(x)$, $1 \leqq s \leqq 8$, are positive, increasing functions on $[k+2, \infty)$.

Because of (2.4), we see that if $B_n(x)$ has no root in (m, ∞) then $E_{n-1}(x)$ has no root in $(2m, \infty)$. Inkeri **[2]** has shown that if $(M, M+1)$ is the largest interval in which $B_n(x)$ has real roots then $M \sim n/2e\pi$ as n approaches ∞.

4. Restrictions on the roots of $E_n(x)$. Inkeri [2] has shown that the only possible rational roots of $E_n(x)$ are 0, 1, and 1/2. In this section we show that other types of real and complex numbers cannot be roots of $En(x)$. We shall use the following lemma.

LEMMA 4.1. *Suppose $f(x)$ is a polynomial and*

$$f(a+b\sqrt{c}) = (a_1 + \cdots + a_k) + \sqrt{c}(b_1 + \cdots + b_k),$$

where each a_i and b_i is a rational number and c is a square-free integer, $c > 1$ or $c < 0$. Suppose there is a prime number p and positive integers j and m such that either

(a) $p^m a_j \not\equiv 0 \pmod p$ *and* $p^m a_h \equiv 0 \pmod p$ *for* $h \neq j$

or

(b) $p^m b_j \not\equiv 0 \pmod p$ *and* $p^m b_h \equiv 0 \pmod p$ *for* $h \neq j$.

Then we can conclude that $f(a+b\sqrt{c}) \neq 0$.

THEOREM 4.1. *If a is a nonzero rational number and c is a nonzero integer, $c \neq 1$, then $E_n(a\sqrt{c}) \neq 0$.*

Proof. Brillhart [1] has proved that $E_n(x)$ has no roots of the form αi where α is real, so we may assume $|c| > 1$. By (2.3) we see that if n is even the only nonzero term of $E_n(x)$ with an even exponent is x^n. Dividing $E_n(a\sqrt{c})$ into its rational and irrational parts, we see that the rational part is $a^n c^{n/2} \neq 0$. If n is odd, then x^n is the only term of $E_n(x)$ with an odd exponent and in this case the irrational part of $E_n(a\sqrt{c}) \neq 0$.

THEOREM 4.2. *If a and b are nonzero rational numbers and c is an even square-free integer, then $E_n(a+b\sqrt{c}) \neq 0$.*

Proof. First suppose $c > 0$. If $n = 2k$ we use (2.11) to break $E_{2k}(a+b\sqrt{c})/(2k)!$ into its rational and irrational parts. Let $b^2c = b_1/b_2 2^q$, $a = a_1/a_2 2^z$, g.c.d. $(b_1, b_2) = 1 =$ g.c.d. (a_1, a_2) (a negative value of q or z indicates a power of 2 in the numerator). Note that q must be odd. We now use Lemma 4.1 with $p = 2$.

Case 1. $z < 0$, $q < 0$. From (2.11) we see that the maximum power of 2 occurs in the denominator of the irrational part of

$$E_{2k}(a+b\sqrt{c})/(2k)! \quad \text{when} \quad r = k-1, \qquad s = 2k-1.$$

To see this, first replace C_s in (2.11) by $2^{s+1}(1-2^{s+1})B_{s+1}/(s+1)$, keeping in mind that $2B_{2m} \equiv 1 \pmod 2$ for $m > 0$. Since q and z are both negative, we see that $(b^2c)^{k-r}$ and a^{2r+1-s} contribute the smallest possible power of 2 to the numerator when $r = k - 1$ and $s = 2k - 1$. Notice that in this case the power of 2 dividing the product $(s+1)!\,(2k-2r-1)!\,(2r+1-s)!$ in the denominator is maximum. This is the kind of reasoning we use in the remaining two cases and in Theorems 4.3 and 4.4.

Case 2. $z > 0$, $2z > q$. The maximum power of 2 occurs in the denominator of the rational part when $r = k$, $s = 0$.

Case 3. $q > 0$, $q > 2z$. The maximum power of 2 occurs in the denominator of the rational part when $r = 0$, $s = 0$.

When $n = 2k + 1$ we use the irrational part of (2.12) and the proof is similar. If $c < 0$ we divide $E_n(a + b\sqrt{c})/n!$ into its real and imaginary parts and proceed as before.

THEOREM 4.3. *Suppose c is an odd square-free integer, $c \neq 1$, and suppose a and b are rational numbers reduced to their lowest terms, $a = a_1/a_2$, $b = b_1/b_2$. If $E_n(a + b\sqrt{c}) = 0$ then $a_2 = b_2$ and* g.c.d. $(a_2, c) = 1 =$ g.c.d. (b_2, c).

Proof. We shall use the notation $p^x \| y$ to mean p^x divides y while p^{x+1} does not divide y. First suppose $n = 2k$. Suppose p is a prime number and $p^z \| a_2$, $z > 0$. We want to show that $p^z \| b_2$ and g.c.d. $(a_2, c) = 1$. Suppose $p^q \| b_2^2c^{-1}$. We shall show that $q = 2z$, so p does not divide c.

Case 1. $2z > q$. Using (2.11), we examine the rational (or real) part of $E_{2k}(a + b\sqrt{c})/(2k)!$, and we see that the maximum power of p in the denominator occurs when $r = k$, $s = 0$. Note that in this case if $p^m \| (2k)!$ then $p^m \| (s+1)!\,(2k-2r)!\,(2r-s)!$. If $p^h \| 2k+1$, there *are* some terms having the property that if $p^m \| (2k+1)!$ then $p^m \| (s+1)!\,(2k-2r)!\,(2r-s)!$ For terms of this type the highest power of p in the denominator occurs when $r = k$, $s = p^h - 1$, but this power of p is still less than the power occurring when $r = k$, $s = 0$.

Case 2. $q > 2z$. The maximum power of p occurs in the denominator of the rational (or real) part of $E_{2k}(a + b\sqrt{c})/(2k)!$ when $r = 0$, $s = 0$.

Thus, by Lemma 4.1, if $p^z \| a_2$, $z > 0$, we must have g.c.d. $(a_2, c) = 1$. Also we have shown that $p^z \| b_2$. Now suppose $p^q \| b_2c^{-1}$ $q > 0$. We want to show $p^q \| a_2$.

Case 1. $2z > q$. The maximum power of p in the denominator of the rational (or real) part of $E_{2k}(a + b\sqrt{c})/(2k)!$ occurs when $r = k$, $s = 0$.

Case 2. $q > 2z$. The maximum power of p in the denominator of the rational (or real) part of $E_k(a + b\sqrt{c})/(2k)!$ occurs when $r = 0$, $s = 0$. Thus by Lemma 4.1 we must have $z = q$. If $n = 2k + 1$ we examine the irrational (or complex) part of $E_{2k+1}(a + b\sqrt{c})/(2k + 1)!$ and the proof is similar.

It is perhaps worth noting that $E_3(x)$ has the roots $(1 \pm \sqrt{3})/2$ and $E_4(x)$ and $E_5(x)$ both have the roots $(1 \pm \sqrt{5})/2$. Thus there *are* polynomials $E_n(x)$ having roots of the form $a + b\sqrt{c}$, c odd.

THEOREM 4.4. *If a and b are nonzero integers then* $E_n(a + bi) \neq 0$.

Proof. Suppose $E_{2k}(a + bi) = 0$ and let $a = a_1 2^z$, $b = b_1 2^q$, a_1 and b_1 odd. Again we use Lemma 4.1.

Case 1. $q = 0$. We can assume $z > 0$ by (2.2). Examining the real part of $E_{2k}(a + bi)/(2k)!$, we see that the highest power of 2 occurs in the denominator when $r = 0$, $s = 0$.

Case 2. $q > 0$. Again, by (2.2), we can assume $z > 0$. We look at the imaginary part of $E_{2k}(a + bi)/(2k)!$ and the highest power of 2 occurs in the denominator when $r = k - 1$, $s = 2k - 1$. The proof for $E_{2k+1}(a + bi)$ is similar.

Using the same method, we can prove the following theorem.

THEOREM 4.5. *If a and b are rational numbers and c and d are square-free positive integers of different parity, then* $E_n(a\sqrt{d} + b\sqrt{c}\,i) \neq 0$.

It should be pointed out that Theorems 4.1, 4.2, 4.4 and 4.5 also hold for the Bernoulli polynomials $B_n(x)$. The proofs are entirely analogous to the proofs in this paper.

Of course many questions remain unanswered. We have not been able to determine, for example, whether or not $a + bi$ can be a root of $E_n(x)$ if a and b are rational numbers. The writer also feels that Theorem 3.4 and the lower bound m^2 in Theorem 3.3 can both be improved. It would also be interesting to know how the roots of $E_n(x)$ are distributed in the last interval for which it has real roots.

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1967, pp. 663–688) contains a more extensive listing of Euler and Bernoulli numbers than the reference cited in this paper.

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