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ONE PARAMETER GROUPS OF ISOMETRIES ON CERTAIN BANACH SPACES

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Banach spaces of class $\mathcal{S}$ were introduced by Fleming and Jamison. This broad class includes all Banach spaces having hyperorthogonal Schauder bases and, in particular, $\mathcal{S}$ includes all Orlicz spaces L^{Φ} on an atomic measure space such that the characteristic functions of the atoms form a basis for L^{Φ}. The main theorem gives the structure of one parameter strongly continuous (or (C_0)) groups of isometries on Banach spaces of class $\mathcal{S}$. Other results correct and complement the work of Goldstein on groups of isometries on Orlicz spaces over atomic measure spaces.

1. In [**4**, p. 390] the following theorem was stated.

THEOREM 1. *Let $X = L^{\Phi}(\Omega, \Sigma, \mu)$ be an Orlicz space on an atomic measure space. Let $T = \{T^t \mid t \in \mathbf{R} = (-\infty, \infty)\}$ be a (C_0) group of isometries on X with infinitesimal generator A, and suppose $(*)$ and $(**)$ hold.*[1] *Suppose X is not a Hilbert space, i.e. $\Phi(s)$ is not of the form $\Phi(s) \equiv$* const $\times\, s^2$.

(I) *If X is a real space, then $T^t = I$ for each $t \in \mathbf{R}$.*

(II) *If X is a complex space, then there exists a function $g : \Omega \to \mathbf{R}$ such that $(T^t f)(\omega) = \exp\{itg(\omega)\} f(\omega)$ for each $f \in X$ and each $\omega \in \Omega$.*

The idea of the proof in [**4**] is as follows. Write $\Omega = \{\omega_i\}$ and view A as a matrix whose ij entry is $A(\delta(\omega_i))(\omega_j)$, where $\delta(\omega_i)$ is the function which is 1 at ω_i and 0 everywhere else. A is a diagonal matrix if and only if T satisfies (I) or (II). It was assumed that A was not diagonal, and the proof given in [**4**] established the existence of a τ, $0 < \tau \leqq \infty$, such that

$$\Phi(s) = \text{const} \times s^2 \quad \text{for} \quad 0 \leqq s < \tau.$$

It was asserted in [**4**] that $\tau = \infty$, which means that X is a Hilbert space. But in fact this is not correct as the following example shows.

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[1] $(*)$ and $(**)$ are mild technical conditions on Φ and A. (Cf. [**1**, p. 389] for the precise statement.) X cannot be an infinite dimensional L^{∞} space, but otherwise $(*)$ and $(**)$ are not very restrictive.

EXAMPLE 1. Let (Ω, Σ, μ) be an atomic measure space defined as follows. Ω consists of three points $\omega_1, \omega_2, \omega_3$, and $\mu(\omega_1) = \mu(\omega_2) = 1/2$, $\mu(\omega_3) = 1/200$. Let Φ and Ψ be defined as follows:

$$\Phi(s) = \begin{cases} s^2/2, & s \in [0, 4] \\ (s^3 + 32)/12, & s \in [4, \infty), \end{cases}$$

$$\Psi(s) = \begin{cases} s^2/2, & s \in [0, 4] \\ (4s^{3/2} - 8)/3, & s \in [4, \infty). \end{cases}$$

Since Φ' and Ψ' are continuous and $\Phi(1) + \Psi(1) = 1$, all the conditions of **[4]** are satisfied. If we define

$$(Hf)(\omega) = \begin{cases} f(\omega_2), & \omega = \omega_1 \\ f(\omega_1), & \omega = \omega_2 \\ 2f(\omega_3), & \omega = \omega_3, \end{cases}$$

then H is an Hermitian operator (in the sense of Lumer **[5]**) and $\{e^{itH} \mid t \in \mathbf{R}\}$ is a strongly continuous group of isometries on the complex space $L^{\Phi}(\Omega, \Sigma, \mu)$. A little computation yields

$$e^{itH} = \begin{pmatrix} \cos t & i \sin t & 0 \\ i \sin t & \cos t & 0 \\ 0 & 0 & e^{2it} \end{pmatrix},$$

where the matrix is given relative to the basis $\{\delta(\omega_j)\}_{j=1}^{3}$ consisting of the characteristic functions of the atoms. This group of isometries is clearly not of the form described in Theorem 1.

The correct version of the theorem is

THEOREM 2. *Let $X = L^{\Phi}(\Omega, \Sigma, \mu)$ be an Orlicz space on an atomic measure space. Let $T = \{T^t \mid t \in \mathbf{R}\}$ be a (C_0) group of isometries on X with generator A, and suppose $(*)$ and $(**)$ hold. Then either there exists a τ, $0 < \tau \leqq \infty$, such that $\Phi(s) = cs^2$ for $0 \leqq s < \tau$, where c is a positive constant, or else:*

(I) *If X is a real space, $T^t = I$ for each $t \in \mathbf{R}$.*

(II) *If X is a complex space, there exists a function $g: \Omega \to \mathbf{R}$ such that $(T^t f)(\omega) = \exp\{itg(\omega)\} f(\omega)$ for each $f \in X$ and each $\omega \in \Omega$.*

REMARK 1. Let T on X be not of the form (I) or (II). Then the τ constructed in the proof of Theorem 1 **[4**, p. 391] is

$$\tau = \sup \Phi(|f(\omega)|/\|f\|_\Phi),$$

the supremum being over the set of all functions $f = \alpha_1\delta(\omega_1) + \alpha_2\delta(\omega_2)$ where α_1, α_2 are nonzero scalars, $\delta(\omega_i)$ is as before the characteristic function of ω_i, and ω_1, ω_2 are distinct members of Ω such that $A(\delta(\omega_1))$ does not vanish at ω_2. The existence of such a pair ω_1, ω_2 follows from the fact that A is not a diagonal matrix since T is not of the form (I) or (II).

REMARK 2. Theorem 2 does not single out Hilbert space among the Orlicz spaces, as asserted in **[4]**, but it does single out Hilbert spaces among a class of Orlicz spaces including the Lebesgue L^p spaces (over atomic measure spaces).

The results of §4 of **[4]** which were asserted to be consequences of Theorem 1 are in fact consequences of Theorem 2, so they are correct as stated.

REMARK 3. Observe that the space in Example 1 can be written as a direct sum $L^\Phi(\Omega, \Sigma, \mu) = X_1 + X_2$ where $X_1 = \{f \mid f(\omega_3) = 0\}$ and $X_2 = \{f \mid f(\omega_i) = 0 \text{ for } i = 1, 2\}$. Also note that the operators in the group can be written as

$$T^t = \begin{pmatrix} T_{11}^t & 0 \\ 0 & T_{22}^t \end{pmatrix},$$

where $\{T_{jj}^t \mid t \in \mathbf{R}\}$ is a (C_0) group of isometries on X_j, $j = 1, 2$. Furthermore, X_1, X_2 are two dimensional and one dimensional Hilbert spaces respectively.

It will follow from Example 5 and Theorem 3 below that any counterexample to Theorem 1 is essentially of the same nature as is Example 1.

2. We now consider a class of spaces which includes the Orlicz spaces $L^\Phi(\Omega, \Sigma, \mu)$ on atomic measure spaces, and we characterize the (C_0) groups of isometries on spaces of this class.

DEFINITION. A Banach space (X, ν) is said to be a *member of class* $\mathcal{S}$ if there exists a Banach space of sequences (E, μ) with absolute norm such that (i) (E, μ) possesses a sufficiently l^p-like semi inner product; (ii) there exists a sequence $\{X_i\}$ of (not necessarily separable) Hilbert spaces such that given $x \in X$ there exists a unique sequence $\{x_i\}$ $(x_i \in X_i)$ for which $x = \sum_{i=1}^n x_i$ if $\{X_i\}$ is a finite sequence or $x = \lim_{n\to\infty} \sum_{i=1}^n x_i$ in the infinite case; (iii) if $x = \sum x_i \in X$, then $(\|x_i\|) \in E$ and $\nu(x) = \mu[(\|x_i\|)]$. In this case we write $X = \{X, \{X_i\}, E, \mu\}$.

The exact meaning of a sufficiently l^p-like semi inner product will not be used explicitly in what follows and therefore will not be given. (See **[2]** for the definition.) Rather, we will give several examples of spaces of class $\mathcal{S}$. As the examples will show, the class $\mathcal{S}$ is quite broad. In fact, $\mathcal{S}$ contains all Banach spaces which have hyperorthogonal Schauder bases **[2]**.

EXAMPLE 2. Let $1 \leqq p < \infty$ and let $\{X_i\}$ be a sequence of (not necessarily separable) Hilbert spaces. Let $l^p(X_i) = \{(x_\kappa) \mid x_\kappa \in X_\kappa, \Sigma \| x_\kappa \|^p < \infty\}$ and let $\nu[(x_k)] = (\Sigma \| x_k \|^p)^{1/p}$. This space is a member of class $\mathcal{S}$ and the associated sequence space (E, μ) is just $(l^p, \| \cdot \|_p)$. Since the $\{X_i\}$ can all be one dimensional, $l^p \in \mathcal{S}$.

EXAMPLE 3. Let $1 \leqq p < \infty$. The Lorentz sequence spaces $d(a, p)$ **[1]** are members of class $\mathcal{S}$. They are defined as follows: For any $a = (a_i) \in c_0 \backslash l^1$, $a_1 \geqq a_2 \geqq a_3 \geqq \cdots > 0$,

$$d(a,p) = \left\{ (x_\kappa) \in c_0 \colon \sup_{\sigma \in \pi} \sum_{i=1}^{\infty} | x_{\sigma(i)} |^p a_i < \infty \right\}$$

where π is the set of all permutations of the natural numbers. The linear space $d(a, p)$ becomes a Banach space if endowed with the norm $\nu(x) = [\sup_{\sigma \in \pi} \Sigma_{i=1}^{\infty} | x_{\sigma(i)} |^p a_i]^{1/p}$. For this space it can be shown that all of the X_i are just the one dimensional spaces spanned by the basis elements e_i and $(E, \mu) = (d(a, p), \nu)$.

EXAMPLE 4. Let (p_i) denote a sequence of real numbers with $1 < b \leqq p_i \leqq c < \infty$ for all i. If $x = (x_i)$ is a sequence of complex numbers, let $M(x) = \Sigma_{i=1}^{\infty} | x_i |^{p_i} / p_i$. Now let $l(p_i) = \{(x_i) \mid M(\lambda x) < \infty$ for some $\lambda > 0\}$ and define $\nu(x) = \inf\{1/\epsilon \mid M(\epsilon x) \leqq 1\}$. Under these hypotheses the space is a uniformly convex Banach space **[7]** and is a member of class $\mathcal{S}$. It can be shown that in this case all of the X_i's are one dimensional except possibly one which is the $\overline{\text{span}}\{e_i \mid p_i = 2\}$, where e_i denotes the element which is 1 in the ith coordinate and zero otherwise. The associated sequence space (E, μ) is $l(\hat{p}_i)$ where $\hat{p}_i = 2$ for at most one i and $\hat{p}_i = p_i$ for $p_i \neq 2$.

EXAMPLE 5. Let (Ω, Σ, μ) be a purely atomic measure space and let $L^\Phi(\Omega, \Sigma, \mu)$ be an Orlicz space over it. If the characteristic functions of the atoms form a basis for $L^\Phi(\Omega, \Sigma, \mu)$, then $L^\Phi(\Omega, \Sigma, \mu) \in \mathcal{S}$. In this case, the X_i's are all one dimensional except possibly one. This can occur whenever $\Phi(s) = cs^2$ for $s \in [0, \tau]$, and in this case it is given by $\overline{\text{span}}\{\delta(\omega_i) \mid \tau \geqq (c\mu(\omega_i))^{-\frac{1}{2}}\}$, where $\delta(\omega_i)$ is the characteristic function of the atom ω_i.

The proof of the main theorem (Theorem 3 below) will be based on the structure of isometries of spaces in class $\mathcal{S}$. Every bounded linear operator A on $X \in \mathcal{S}$ can be represented by an operator matrix (A_{ij}) where $A_{ij}: X_j \to X_i$. In **[3]** it was shown that every isometry U of X onto X can be represented as $(U_{i\pi(i)})$ where $UX_{\pi(i)} = X_i$, $U_{i\pi(i)}$ is unitary, and π is a permutation of the positive integers (or of $\{1, 2, 3, \cdots, n\}$ in the case there are only finitely many of the X_i's).

EXAMPLE 6. If $X = l(p_i)$ and the p_i's are distinct then U is an isometry of X onto X if and only if there exists a sequence of real numbers $\{\theta_i\}$ such that $(Ux)(\kappa) = e^{i\theta_\kappa}x(\kappa)$ for each κ. In this case only the identity permutation is allowed. If $1 < p_i = p < \infty$, $p \neq 2$, then $(Ux)(\kappa) = e^{i\theta_\kappa}x(\pi(\kappa))$ where π is *any* permutation of the positive integers.

We are now in a position to state and prove the main result.

THEOREM 3. *Let* $X = (X, \{X_i\}, E, \mu) \in \mathcal{S}$. $U = \{U^t \mid t \in \mathbf{R}\}$ *is a* (C_0) *group of isometries on X if and only if* (i) *for each* $t \in \mathbf{R}$, U^t *has a diagonal operator matrix* $(U^t_{\kappa\kappa})$, *and* (ii) *for each* κ, $\{U^t_{\kappa\kappa}\}$ *is a* (C_0) *group of isometries acting on the Hilbert subspace* X_κ.

Proof. (Necessity) For each $t \in \mathbf{R}$, U^t is an isometry. Hence by Theorem 1 of **[3]**, there is a permutation π_t such that

$$(1) \qquad U^t X_{\pi_t(i)} = X_i \quad \text{for each positive integer} \quad i,$$

$$(2) \qquad (U^t x)(i) = U_{i\pi_t(i)} x(\pi_t(i)) \quad \text{for each} \quad i,$$

where $x = \Sigma x_i \in X$ and $U_{i\pi_t(i)}$ is a unitary map of $X_{\pi_t(i)}$ onto X_i.

Since $U = \{U^t \mid t \in \mathbf{R}\}$ is a group we have $U^t U^s = U^{t+s}$ for all $s, t \in \mathbf{R}$. Let s, t be fixed and $x \in X$. Then

$$(U^s U^t x)(i) = U^s_{i\pi_s(i)} U^t_{\pi_s(i)\pi_t \cdot \pi_s(i)} x(\pi_t \cdot \pi_s(i))$$

and

$$(U^{s+t}x)(i) = U^{s+t}_{i\pi_{s+t}(i)} x(\pi_{s+t}(i)).$$

Hence for every $x \in X$ we must have

$$(3) \qquad U^s_{i\pi_s(i)} U^t_{\pi_s(i)\pi_t \cdot \pi_s(i)} x(\pi_t \cdot \pi_s(i)) = U_{i\pi_{s+t}(i)} x(\pi_{s+t}(i)).$$

Let i be given and suppose $\pi_{s+t}(i) = j$. Let x_j denote a fixed nonzero element of the subspace X_j. Let $x(j) = x_j$, $x(\kappa) = 0$, $\kappa \neq j$. Then with this x we see that the right hand side of (3) is not zero. If $\pi_s \cdot \pi_t(i) \neq j$ the left hand side of (3) is zero. Hence $\pi_s \cdot \pi_t(i) = \pi_{s+t}(i)$ and since i was arbitrary,

$$\pi_s \cdot \pi_t = \pi_{s+t}. \tag{4}$$

We now proceed to show that π_t = identity for each t. We use the strong continuity of the group at zero. For each $x \in X$,

$$U^t x \to x \quad \text{as} \quad t \to 0. \tag{5}$$

Let $i = 1$ and let x_1 be a nonzero element of X_1. Set $x(i) = x_1$ if $i = 1$, $x(i) = 0$ if $i \neq 1$. Then from (5), there exists a neighborhood V_1 of $t = 0$ such that $\nu(U^t x - x) < \|x_1\|$ for $t \in V_1$. Now

$$(U^t x)(i) = U^t_{i\pi_t(i)} x(\pi_t(i)) = \begin{cases} 0, & \pi_t(i) \neq 1 \\ U^t_{i1} x_1, & \pi_t(i) = 1. \end{cases}$$

Consequently, $\nu(U^t x - x) \geqq \|x_1\| = \nu(x)$ if $\pi_t(1) \neq 1$, for in this case, $U^t x - x = x_1 + U^t_{i1} x_1$. Thus $\pi_t(1) = 1$ for all $t \in V_1$. In the same way, for given N, there exists a neighborhood V_N of $t = 0$ such that $\pi_t(\kappa) = \kappa$ for all $\kappa \leqq N$ and all $t \in V_N$. Now if $t_0 \in \mathbf{R}$, there exists a positive integer n such that $t_0/n \in V_N$. Therefore, $\pi_{t_0/n}(\kappa) = \kappa$ for all $\kappa \leqq N$ and by (4) $\pi_{t_0} = \pi_{n(t_0/n)} = [\pi_{t_0/n}]^n$. So that $\pi_{t_0}(\kappa) = \kappa$ for all $\kappa \leqq N$. Since N was arbitrary we conclude that π_t = identity permutation for each $t \in \mathbf{R}$.

This establishes that the operator matrix for U^t is diagonal for each t. If follows from (3) that for each i,

$$U^s_{ii} U^t_{ii} x(i) = U^{s+t}_{ii} x(i) \tag{6}$$

for all $s, t \in \mathbf{R}$ and we have that each diagonal element is a member of a group of isometries acting on the corresponding Hilbert subspace. The continuity property of the diagonal element follows from the continuity of U^t.

For the converse, the only difficulty is in showing the continuity.

Let $x \in X$ and $\epsilon > 0$ be given. Recall that $x = \Sigma x_j$ and $\nu(x) = \mu[(\|x_j\|)] = \mu[(\|x_1\|, \|x_2\|, \cdots)]$. Hence there exists an N such that

$$\mu[(0, 0, \cdots, 0, \|x_{N+1}\|, \|x_{N+2}\|, \cdots)] < \epsilon/4.$$

Now,

$$(7)\begin{cases}\nu(U^t x - x) = \mu[(\|U^t_{ii}x(i) - x(i)\|)] \\ \qquad = \mu[(\|U^t_{11}x(1) - x(1)\|, \cdots, \|U^t_{NN}x(N) - x(N)\|, 0, 0, \cdots)] \\ \qquad\quad + \mu[(0, 0, \cdots, 0, \|U^t_{N+1\,N+1}x(N+1) - x(N+1)\|, \cdots)].\end{cases}$$

But $\|U^t_{\kappa\kappa}x(\kappa) - x(\kappa)\| \leqq 2\|x(\kappa)\|$ for all $t \in \mathbf{R}$ and every positive integer κ. Then from (7)

$$\begin{aligned}\nu(U^t x - x) &\leqq \mu[(\|U^t_{11}x(1) - x(1)\|, \cdots, \|U^t_{NN}x(N) - x(N)\|, 0, 0, \cdots)] \\ &\quad + 2\mu[(0, 0, 0, \cdots, \|x(N+1)\|, \cdots)].\end{aligned} \tag{8}$$

Choose V such that $t \in V$ implies $\|U^t_{ii}x(i) - x(i)\| < \epsilon/4M$ where $M = \mu[(1, \cdots, 1, 0, 0, \cdots)]$, exactly N 1's occurring inside the parentheses. Then $\nu(U^t x - x) < \epsilon$ for all $t \in V$. The proof is complete.

REMARK 4. If the Hilbert subspaces X_i are all one dimensional, then there exists a function $g: \mathbf{Z}^+ \to \mathbf{R}$ such that $U^t x(\kappa) = \exp\{itg(\kappa)\}x(\kappa)$ for each $x \in X$ and $\kappa \in \mathbf{Z}^+$.

REMARK 5. If Theorem 3 is applied to the Orlicz space of Example 1 we find that every (C_0) group of isometries on that space can be written as $U^t = \begin{bmatrix} U^t_{11} & 0 \\ 0 & U^t_{22} \end{bmatrix}$ where $\{U^t_{ii} \mid t \in \mathbf{R}\}$ is a (C_0) group of isometries on X_i, $i = 1, 2$ respectively.

REMARK 6. Theorem 3 indirectly characterizes self conjugate operators (in the sense of Palmer **[6]**) on spaces of class $\mathcal{S}$. For example if $X = l(p_i)$, the self conjugate operators correspond to multiplications by real sequences. The bounded self conjugate operators are just the Hermitian operators and correspond to multiplication by bounded sequences.

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