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CHARACTERIZING FINSLER SPACES WHICH ARE PSEUDO-RIEMANNIAN OF CONSTANT CURVATURE

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Let M be an indefinite Finsler space. The bisector of two points of M is the set of points equidistant from these two points. A bisector is called flat if with any pair of points it contains the extremals joining this pair. In this paper it is shown that M is pseudo-Riemannian of constant curvature if and only if M locally has flat bisectors. Another result is that M is pseudo-Riemannian of constant curvature if and only if M can be reflected locally in each nonnull extremal.

1. Introduction. Blaschke **[6]** has shown that if M is a two dimensional definite Finsler space in which the bisector of two points is an extremal then M is a Riemannian space of constant curvature. Busemann **[7]** has shown that among his G-spaces the requirement that bisectors contain with each pair of points a segment joining this pair characterizes the Euclidean, hyperbolic and spherical spaces of dimension greater than one. Phadke **[8]** has investigated the flat bisector condition in two dimensional G-spaces which have a distance which is not necessarily symmetric. In **[4]** we have shown that a pseudo-Riemannian manifold locally has flat bisectors if and only if it is a space of constant sectional curvature.

In the present paper an ordinary or definite Finsler space with a symmetric distance is considered to be a special case of an indefinite Finsler space. Consequently, our arguments are valid for definite metrics as well as nondefinite metrics. The arguments are different from those of Busemann **[7]** because he does not make any differentiability assumptions and since a number of his arguments do not extend to indefinite metrics.

2. Indefinite Finsler spaces. Let M be an n dimensional connected and paracompact differentiable manifold of class C^∞. The local coordinates of a point x will be denoted $x^1, \cdots, x^n$. In the tangent space $T(x)$ to M at x take the natural basis and let $y^1, \cdots, y^n$ denote the components of a vector $Y \in T(x)$. The coordinates of Y are (x, y). Let $L(x, y)$ be a continuous function defined on the tangent bundle $T(M)$ of M which has the following properties:

(A) The function $L(x, y)$ is C^∞ for all (x, y) with $y \neq 0$.

(B) $L(x, ky) = k^2 L(x, y)$ for all $k > 0$.

(C) The metric tensor $g_{ij}(x, y) = \frac{1}{2}\partial^2 L/\partial y^i \partial y^j$ has s negative eigenvalues and $n - s$ positive eigenvalues for all (x, y) with $y \neq 0$.

(D) $L(x, -y) = L(x, y)$.

The function $L(x, y)$ is called the basic metric function. It corresponds to the square of the fundamental function $F(x, y)$ usually studied in definite Finsler spaces (compare **[10]**).

The manifold M together with the basic metric function $L(x, y)$ is called an indefinite Finsler space of signature $n - 2s$. If $L(x, y)$ is replaced with $-L(x, y)$, then M becomes a space of signature $2s - n$. In the special case $s = 0$ the manifold M is a definite Finsler space. In this paper we do not exclude the case $s = 0$.

When M has a metric tensor $g_{ij}(x, y)$ which does not depend on y, then M is called pseudo-Riemannian. A pseudo-Riemannian space is Riemannian when $s = 0$ or n. If M is R^n and the metric tensor is constant, then M is called pseudo-Euclidean.

Let W, Y, Z be three tangent vectors at $x \in M$. Using the natural basis let (x, w), (x, y) and (x, z) be the respective coordinate representations of these vectors. The scalar product of Y and Z with respect to W is defined by

$$W(Y, Z) = g_{ij}(x, w)y^i z^j.$$

If Y is a nonzero vector, then we say Y is perpendicular to Z when $Y(Y, Z) = 0$. When Y is perpendicular to Z we write $Y \dashv Z$. This relation is not, in general, symmetric. When M has dimension at least three we have shown **[5]** that perpendicularity is symmetric on M if and only if M is pseudo-Riemannian.

The norm squared of a vector Y is defined by $|Y|^2 = Y(Y, Y)$. The quantity $|Y|^2$ may be positive, negative or zero. A vector Y with $|Y|^2 = \pm 1$ is called a unit vector. If $|Y|^2 = 0$, then Y is called a null vector. A vector is nonzero as long as it is not the origin of the tangent space at which it is attached.

The indicatrix $K(x)$ consists of all of the unit vectors in $T(x)$. The light cone $C(x)$ consists of the null vectors in $T(x)$.

If $Y \in K(x)$, then $Y \dashv Z$ if and only if Z is parallel to the tangent hyperplane to $K(x)$ at Y, compare [**10**, p. 26].

3. The bisector condition. The Christoffel symbols $\gamma^i_{jk}(x, y)$ are defined in the usual way. The extremals are the solutions of the differential equations

$$\ddot{x}^i + \gamma^i_{jk}(x, \dot{x})\dot{x}^j\dot{x}^k = 0.$$

An extremal $x(t)$ with velocity vector of length zero is called a null extremal.

A result of Whitehead **[9]** implies that for each point x there is a simple convex neighborhood $U(x)$. Given two points p and q in $U(x)$ there is a unique extremal arc $\alpha(p, q)$ from p to q which lies in $U(x)$. In $U(x)$ the separation between two points p and q is defined by

$$d(p,q) = \int L^{1/2}(x, \dot{x})dt.$$

The integral is taken along $\alpha(p, q)$. The quantity $L^{1/2}(x, y)$ is either real and nonnegative or pure imaginary. Hence, $d(p, q)$ is either nonnegative or imaginary. The function d is continuous on the domain $U(x) \times U(x)$. In indefinite metric spaces the local distance function $d(p, q)$ is usually only defined for points sufficiently close together.

The bisector of p and q with respect to $U(x)$ is defined by

$$B(p,q) = \{p' \in U(x) \mid d(p,p') = d(q,p')\}.$$

We say locally M has flat bisectors if for each $x \in M$ there is a simple convex neighborhood $U(x)$ such that for all $p, q \in U(x)$ with $d(p, q) \neq 0$ the bisector $B(p, q)$ contains with any pair of points the extremals in $U(x)$ containing this pair.

4. The two dimensional case. In this section and the next we always assume M satisfies the bisector condition. If $n = 2$, then this is the assumption that $B(p, q)$ lies on an extremal of M.

PROPOSITION 1. *Let M be a two dimensional indefinite Finsler space which locally has flat bisectors. Then M is a pseudo-Riemannian space of constant curvature.*

Proof. If M has signature two or minus two, then the metric is definite and the proposition follows from the result of Blaschke **[6]** which was mentioned in the introduction.

Let M have signature zero. The metric tensor must have one negative eigenvalue and one positive eigenvalue for all (x, y) with $y \neq 0$. For each fixed $x \in M$, the light cone $C(X)$ consists of a finite number m of lines passing through the origin of the tangent space $T(x)$. When M is pseudo-Riemannian, the light cone consists of two lines. When M is an indefinite Finsler space, the number of lines m may be larger than two, see **[2]**.

Let $m > 2$ and let $U(x)$ be a simple convex neighborhood of x such that $B(p, q)$ is flat whenever $p, q \in U(x)$ with $d(p, q) \neq 0$. Each $p \in U(x)$ has at least three distinct null directions and there are three null extremals through p corresponding to these directions. At x, choose three null vectors Y_1, Y_2 and Y_3 such that any pair Y_i, Y_j for $i \neq j$ is a linearly independent set. Since the null directions through a point vary continuously with the point, each null vector Y_i attached at x may be extended to a continuous and nonvanishing null vector field Y_i defined on a neighborhood $W(x)$ with $W(x) \subset U(x)$. For each $p \in W(x)$, let $\alpha_i(p)$ where $i = 1, 2, 3$ be a null extremal through p with tangent vector Y_i at p. Assume without loss of generality that $W(x)$ and the extemals $\alpha_i(p)$ have been chosen such that each extremal has its endpoints outside of $W(x)$. Choose $q = x$. For all p sufficiently close to q we have $\alpha_i(p) \cap \alpha_j(q) \neq \phi$ when $i \neq j$, since the tangent to $\alpha_i(p)$ converges to Y_i at q as $p \to q$ and the tangent to $\alpha_j(q)$ is Y_j at q. Choose a fixed p with $\alpha_i(p) \cap \alpha_j(q) \neq \phi$ for $i \neq j$ and with $d(p,q) \neq 0$. Let $p_1 = \alpha_1(p) \cap \alpha_3(q)$ and $p_2 = \alpha_2(p) \cap \alpha_3(q)$. Since $d(p, p_i) = d(q, p_i) = 0$ it follows that $p_i \in B(p,q)$ for $i = 1, 2$. The flat bisector condition implies $d(p,r) = d(q,r) = 0$ for all $r \in \alpha(p_1, p_2)$, since $\alpha(p_1, p_2)$ lies on the null extremal $\alpha_3(q)$. For each point $r \in \alpha(p_1, p_2)$, there is a null extremal $\alpha(p,r)$ which determines a null direction at p. Since $p \notin \alpha_3(q)$, distinct points of $\alpha(p_1, p_2)$ must determine distinct directions at p. This contradicts the fact that p has only a finite number of null directions.

Assume that $m = 2$. A two dimensional indefinite Finsler manifold for which $C(x)$ always consists of two lines has been shown to be a doubly timelike surface, see [**2**, p. 1038]. Doubly timelike surfaces have been studied by the author in [**1**]. In particular, the doubly timelike surfaces which locally satisfy the flat bisector condition have been completely characterized by Theorems (IV. 36) and (VI. 17) of [**1**]. These two Theorems together with the differentiability of $L(x,y)$ imply that M is a pseudo-Riemannian manifold of constant curvature.

5. The bisector theorem. Let M have dimension at least three and satisfy the bisector condition. If $p, q \in U(x)$ with $d(p,q) \neq 0$, let r be the midpoint of $\alpha(p,q)$ so that $d(p,r) = d(q,r)$. The bisector $B(p,q)$ is a submanifold through r of codimension one. This implies that $B(p,q)$ has an $n-1$ dimensional tangent space $T_r(B(p,q))$ at r. The space $T_r(B(p,q))$ is naturally identified with an $n-1$ dimensional linear subspace of the tangent space $T(r)$.

LEMMA 2. *If r is the midpoint of the nonnull extremal $\alpha(p,q)$, then $\alpha(p,q)$ is a perpendicular to $B(p,q)$ at r.*

Proof. Let W be the unit tangent to $\alpha(p,q)$ at r and let Y be a nonzero vector at r in the hyperplane $T_r(B(p,q))$. Let $a(s)$ be the solution of the extremal equations such that $a'(0) = Y$. For each s (sufficiently small), let $x(t,s)$ represent the extremal $\alpha(p, a(s))$ for

$0 \leqq t \leqq 1$. Let $\dot{x}$ denote the partial derivative of $x(t, s)$ with respect to t. Define

$$f(x, \dot{x}) = L^{1/2}(x, \dot{x}) = [g_{ik}\dot{x}^i\dot{x}^k]^{1/2}.$$

For each fixed s, the value of $f(x, \dot{x})$ is either real or pure imaginary. Define

$$I_1(s) = \int f(x, \dot{x})dt = d(p, a(s))$$

where the integral is from $t = 0$ to $t = 1$. Differentiation of this equation with respect to s yields

$$I_1'(s) = \int \left(\frac{\partial f}{\partial x^j}\frac{\partial x^j}{\partial s} + \frac{\partial f}{\partial \dot{x}^j}\frac{\partial \dot{x}^j}{\partial s}\right) dt.$$

Integrating by parts we obtain

$$I_1'(s) = \frac{\partial f}{\partial \dot{x}^j}\frac{\partial x^j}{\partial s}\bigg|_0^1 + \int \left(\frac{\partial f}{\partial x^j} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}^j}\right)\right)\left(\frac{\partial x^j}{\partial s}\right) dt.$$

This last integral must vanish because the Euler-Langrange equations hold along each extremal. Furthermore, the derivative of x^j with respect to s is zero at $t = 0$. Hence,

$$I_1'(0) = \frac{\partial f}{\partial \dot{x}^j}\frac{\partial x^j}{\partial s}\bigg|_{t=1}.$$

The next equation (compare [**10**, p. 15]) results from the homogeneous assumption (B) together with the definition (C) of the metric tensor.

$$\frac{\partial g_{ik}}{\partial \dot{x}^j}\dot{x}^i = 0.$$

This last equation and the definition of $f(x, \dot{x})$ imply

$$\frac{\partial f}{\partial \dot{x}^j} = \frac{g_{ij}\dot{x}^i}{f(x, \dot{x})}.$$

Consequently,

$$I_1'(0) = \frac{g_{ij}\dot{x}^i}{f(x, \dot{x})}\frac{\partial x^j}{\partial s}\bigg| = |W|^{-1}W(W, Y).$$

If $I_2(s) = d(q, a(s))$, then

$$I_2'(0) = -|W|^{-1}W(W, Y).$$

The fact that $a(s) \in B(p, q)$ implies $I_1'(0) = I_2'(0)$. This implies $W \dashv Y$ and establishes the Lemma.

LEMMA 3. *Let r be the midpoint of the nonnull extremal $\alpha(p_1, q_1)$. If $p, q \in \alpha(p_1, q_1)$ and r is the midpoint of $\alpha(p, q)$, then $B(p, q) = B(p_1, q_1)$.*

Proof. From Lemma 2 it follows that both $B(p, q)$ and $B(p_1, q_1)$ consist of the union of all extremals in $U(x)$ which pass through r and have the property that $\alpha(p, q)$ is perpendicular to them at r.

Let W and Y be nonzero vectors attached at x with coordinate representations (x, w) and (x, y) respectively. Then $W \dashv Y$ if and only if $g_{ij}(x, w)w^i y^j = 0$. Since the metric tensor is nonsingular the vector W is always perpendicular to a hyperplane containing the origin of $T(x)$. This hold even if $|W|^2 = 0$ (as long as $W \neq 0$). This hyperplane varies continuously with W and may actually contain W.

LEMMA 4. *If M is an indefinite Finsler space which locally has flat bisectors, then perpendicularity is symmetric on M.*

Proof. The nonnull vectors are dense in the set of nonzero vectors and a vector W is perpendicular to a hyperplane which varies continuously with W. Consequently, it is only necessary, to verify that $W \dashv Y$ implies $Y \dashv W$ for nonnull vectors W and Y.

Let $\alpha(p, q)$ be a nonnull extremal with midpoint r and unit tangent W at r. Let Y be a nonnull vector at r with $W \dashv Y$. Using the notation of Lemma 2, we let $a(s)$ be an extremal with $a(0) = r$ and $a'(0) = Y$. The extremal $\alpha(p, q)$ has an arclength representation $b(u)$ where $-|d(p, r)| \leqq u \leqq |d(p, r)|$ and $b'(0) = W$. Choose some fixed s_0 different from zero and let $x(t, u)$ represent the extremal $\alpha(a(s_0), b(u))$ for $0 \leqq t \leqq 1$. The partial derivative of x with respect to t will be denoted by $\dot{x}$. Define

$$I_0(u) = \int f(x, \dot{x})dt = d(a(s_0), b(u)).$$

The arguments used in the proof of Lemma 2 yield

$$I_0'(0) = \frac{\partial f}{\partial \dot{x}^j}\frac{\partial x^j}{\partial u}\bigg|_{t=1} = |Y|^{-1}Y(Y, W).$$

Lemma 3 implies that $I_0(-u) = I_0(u)$. It follows that $I_0'(0) = 0$. Hence, $|Y|^{-1} Y(Y, W) = 0$. This implies $Y \dashv W$ and establishes the Lemma.

THEOREM 5. *Let M be an indefinite Finsler space. Locally M has flat bisectors if and only if M is pseudo-Riemannian of constant sectional curvature.*

Proof. If M has dimension two, then Proposition 1 yields the result.

In **[5]** we have shown that an indefinite Finsler space of dimension at least three has symmetric perpendicularity if and only if it is pseudo-Riemannian. In **[4]** we have shown that a pseudo-Riemannian manifold locally has flat bisectors if and only if it is a space of constant curvature. These two results together with the conclusion of Lemma 4 that M has symmetric perpendicularity complete the proof of the Theorem.

6. Reflections in extremals. In this section another theorem characterizing pseudo-Riemannian spaces of constant curvature is proven.

Let f be a diffeomorphism of M onto itself and let f_* denote the derivative map induced on the tangent bundle. The map f is an isometry if for all $x \in M$ and $W, Y, Z \in T(x)$ we have

$$W(Y, Z) = f_*(W)(f_* Y, f_* Z).$$

When f is a diffeomorphism of some open set U_1 of M onto an open set U_2 of M which satisfies the above equality, the map f is called a local isometry. When f is a local isometry different from the identity and such that f^2 is the identity, then f is an involution.

Let x be an interior point of the nonnull extremal α. A reflection in α near x is said to exist, if there is a neighborhood $V(x)$ and a local isometry f defined on $V(x)$ such that f is an involution and the set of fixed points of f is exactly $\alpha \cap V(x)$.

If every nonnull extremal may be reflected near each interior point, then we say M may be locally reflected in each nonnull extremal.

Let f be a reflection in α near x. The tangent map f_* is a linear map of $T(x)$ onto $T(x)$ which preserves the metric induced on $T(x)$. Hence, f_* maps the indicatrix $K(x)$ onto itself and the light cone $C(x)$ onto itself. If W is a nonzero vector tangent to α at x, then $f_* W = W$ and

$$W(W, Z) = W(W, f_* Z)$$

for all $Z \in T(x)$. This implies that if W is perpendicular to the $(n-1)$ dimensional linear subspace H of $T(x)$ then $f_* H = H$.

Let (M, g) be a pseudo-Riemannian space of constant sectional curvature. It is known (see [**11**, p. 69]) that each $x \in M$ must have a neighborhood which is isometric to an open set of one of the model spaces S^n_s, R^n_s or H^n_s. When $s = 0$, these model spaces are the classical models for spaces of constant curvature. The space S^n_0 is an n dimensional sphere, the space R^n_0 is n dimensional Euclidean space and H^n_0 is an n dimensional hyperbolic space. The groups of motions of all of the model spaces are well known, compare [**11**, pp. 65–66]. In particular, each of the model spaces may be reflected over any nonnull geodesic G. This reflection may have more than G as its set of fixed points, however, the geodesic G will have a neighborhood U such that the fixed points of U are all on G. If follows that any pseudo-Riemannian space of constant curvature may be locally reflected in any nonnull extemal. In general, pseudo-Riemannian spaces of constant curvature cannot be reflected over null extremals.

PROPOSITION 6. *If M is a two dimensional indefinite Finsler space which may be locally reflected in all nonnull extremals, then M is pseudo-Riemannian of constant curvature.*

Proof. If the metric on M is definite the result is well known, see [**7**, p. 350].

Assume the metric is not definite and let W be a nonnull vector in $T(x)$. There is a local reflection f in the extremal α determined by W. Furthermore, $f_* W = W$ and f_* is an involutoric motion on $T(x)$. Letting W vary, it follows that there exist infinitely many motions of $T(x)$ holding the origin fixed. The metric on $T(x)$ is Minkowskian and it is known [**3**, p. 533] that a two dimensional Minkowskian space has an infinite group of motions holding one point fixed if and only if the metric is the ordinary two dimensional Lorentz metric. Letting x vary, it follows that M is pseudo-Riemannian.

Let $\alpha(p, q)$ be a nonnull extemal from p to q. For each positive integer k, there is a set of equally spaced points $\{p_0, p_1, \cdots, p_k\}$ on $\alpha(p, q)$ with $d(p, p_m) = md(p, q)/k$ where $m = 1, 2, \cdots, k$. Each extremal $\alpha(p_i, p_{i+1})$ has a midpoint r_i. Let $\alpha^\perp(r_i)$ be the nonnull extremal perpendicular at r_i to $\alpha(p_i, p_{i+1})$. Let F_i be the local reflection over $\alpha^\perp(r_i)$. The map F_i takes points of $\alpha(p_i, p_{i+1})$ to points of $\alpha(p_i, p_{i+1})$. For sufficiently large k each F_i may be defined on all of $\alpha(p_i, p_{i+1})$ and this map interchanges p_i and p_{i+1}. Consequently, the composite map

$$F = F_k \circ F_{k-1} \circ \cdots \circ F_1$$

is a local isometry taking p to q whenever k is sufficiently large. It follows that M has the same curvature at p and q.

To conclude that M has the same curvature at all points we observe that any pair of points of M may be joined by a path consisting of a finite sequence of nonnull extremals. This establishes the Proposition.

LEMMA 7. *Let W be a unit vector at x which is tangent to α and let f be a reflection in α near x. Then $W \dashv Z$ implies $f_*Z = -Z$.*

Proof. Let W be perpendicular to Z. Then W is also perpendicular to f_*Z since f_* preserves the metric on $T(x)$. Assume $f_*Z \neq -Z$ and let $Y = Z + f_*Z$. Then Y is nonzero. Also, $f_*Y = f_*Z + f_*^2Z = f_*Z + Z = Y$ and $W \dashv Y$.

If $|Y|^2 \neq 0$, let β be the extremal through x with tangent Y at x. Then f leaves β pointwise fixed near x which contradicts the assumption that f only leaves $\alpha \cap V(x)$ fixed.

If $|Y|^2 = 0$, let P be the two dimensional linear subspace of $T(x)$ spanned by Y and W. The map f_* is the identity on P since $f_*Y = Y$ and $f_*W = W$. For sufficiently small positive ϵ, the vector $X = W + \epsilon Y$ is a nonnull vector in P. Letting β be an extremal tangent to X at x, it follows as before that f leaves β pointwise fixed near x. This last contradiction establishes the Lemma.

THEOREM 8. *If M is an indefinite Finsler space, then M may be reflected locally in each nonnull extremal if and only if M is a pseudo-Riemannian space of constant curvature.*

Proof. Because of Proposition 6, we only consider $n \geqq 3$.

Let W be a nonnull vector tangent to α at x. Assume that f is a local reflection in α and that Z is any vector with $W \dashv Z$. Let (x, w) and (x, z) be the respective coordinate representations of W and Z. Lemma 7 and the fact that f_* must preserve the metric induced on the tangent space $T(x)$ yield $g_{ij}(x, w + \epsilon z) = g_{ij}(x, w - \epsilon z)$ for all real ϵ. This implies the derivative of $g_{ij}(x, w + \epsilon z)$ with respect to ϵ must vanish at $\epsilon = 0$. The function $g_{ij}(x, y)$ is homogeneous of degree zero in y because of conditions (B) and (C). Thus, the derivative of $g_{ij}(x, w + \epsilon w)$ with respect to ϵ must vanish at $\epsilon = 0$. We conclude that

$$\frac{\partial g_{ij}(x, w)}{\partial \dot{x}^k} = 0$$

for all $k = 1, 2, \cdots, n$. This equation must hold for all nonnull vectors W.

Since the nonnull vectors at x are dense in $T(x)$, we find $g_{ij}(x, \dot{x})$ is independent of $\dot{x}$. Hence, M is pseudo-Riemannian.

Consider a nondegenerate two dimensional linear subspace E of $T(x)$ with sectional curvature $K(x, E)$. Let E be spanned by vectors Y and Z. The two dimensional sections of $T(x)$ have a natural topology induced from the Grassmann manifold of 2-planes in $T(x)$. If $Y_t \to Y$ and $Z_t \to Z$, then the subspace spanned by Y_t and Z_t converges to E.

If f is the reflection in the nonnull extremal α through x, then $K(x, E) = K(x, f_*E)$. In general, given two arbitrary sections E_1 and E_2 at x there may not be a reflection f such that $E_2 = f_*E_1$. In fact, it may happen that the metric is definite on one section and indefinite on the other.

Let Y' be a vector attached at x and let E' denote the section spanned by Y' and Z. If Y' is chosen sufficiently close to Y, then there is a reflection f in some nonnull extremal α such that $E' = f_*E$. It follows easily that all sections sufficiently close to E have the same curvature. This implies that two nondegenerate sections E_1 and E_2 will have the same curvature if there is a continuous family of nondegenerate sections from E_1 to E_2. It follows that the sectional curvature $K(x, E)$ is independent of E. However, when $n \geqq 3$ the sectional curvature is only constant at each x when the curvature is independent of x, see [**11**, p. 57]. Therefore, M is a space of constant curvature.

Theorems 5 and 8 yield our final Proposition.

PROPOSITION 9. *If M is an indefinite Finsler space, then the following conditions are equivalent.*

(i) *M is pseudo-Riemannian of constant curvature.*

(ii) *Locally M has flat bisectors.*

(iii) *M may be reflect locally in each nonnull extremal.*

REMARK. If M has a definite Finsler metric, then Theorems 5 and 8 may be established without using the assumption of condition (D) that the metric be symmetric. Furthermore, by making some modifications of the arguments in [**3**] and in the proof of Theorem 8, we may establish Theorem 8 for indefinite metrics without assuming condition (D).

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