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NONOSCILLATION THEORY OF ELLIPTIC EQUATIONS OF ORDER $2n$

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Several nonoscillation theorems are obtained for elliptic equations of order $2n$. These results extend several well known nonoscillation theorems for elliptic equations of order 2 and 4, and for ordinary differential equations of higher order.

Introduction. Several authors have considered the problem of establishing oscillation and nonoscillation criteria for elliptic equations. We refer the reader to the books by C. A. Swanson **[15]** and K. Kreith **[8]** where extensive bibliographies can be found.

Most of the interest has so far centered on second order equations, with some results also established for fourth order equations. In this paper we establish several nonoscillation theorems for elliptic equations of order $2n$. These theorems extend in particular, results of Swanson **[14]**, Piepenbrink **[12]**, Headley and Swanson **[5]** and Yoshida **[16]**.

Our proofs make extensive use of variational arguments, of extended Sobolev-type inequalities and of estimates on quadratic forms associated with elliptic equations.

The first part of the paper discusses some preliminary comparison theorems and lower estimates on quadratic forms. The second part deals with the nonoscillation of operations defined in subdomains of E^m for $m \neq 2$. In the next part, some results are established for operations defined in subdomains of E^2. The final part deals with extensions to more general cases.

Definitions and notations. Let Ω be an unbounded domain of m-dimensional Euclidean space E^m. Without loss of generality, we may assume $0 \notin \bar{\Omega}$. Points of E^m are denoted by $x = (x_1, \cdots, x_m)$ and differentiation with respect to x_i by D_i, $i = 1, \cdots, m$. Let L be the differential expression given by:

$$Lu = (-1)^n \sum_{|\alpha| = |\beta| = n} D^\alpha (a_{\alpha\beta} D^\beta u) - a_{00} u, \qquad a_{\alpha\beta} = a_{\beta\alpha}$$

whose coefficients are real defined in Ω and sufficiently regular so that all derivatives involved in L exist and are at least continuous in the closure of $\Omega - R$ for some sphere R. As is usual, we set $D^\alpha u = D_1^{\alpha(1)} \cdots D_m^{\alpha(m)} u$,

$\alpha = (\alpha(1), \cdots, \alpha(m))$, $|\alpha| = \Sigma_{i=1}^{m} \alpha(i)$, $\alpha! = \alpha(1)! \cdots \alpha(m)!$, where each $\alpha(i)$, $i = 1, \cdots, m$, is a nonnegative integer.

For any subdomain F of Ω, we denote by $H_n^0(F)$ the completion of $C_0^\infty(F)$ in the Sobolev norm: $\left\{\Sigma_{|\alpha|\leqq n} \int_F |D^\alpha u|^2 dx\right\}^{1/2}$; by $\|\cdot\|_p(F)$ the $L^p(F)$ norm; and by $(\cdot, \cdot)$ the $L^2(F)$ inner product. If F is obvious from the context, we write $\|\cdot\|_p$ for $\|\cdot\|_p(F)$, etc. L is assumed to be uniformly strongly elliptic in any bounded subdomain F, i.e. for any such F there exists a positive scalar $d(F)$ such that:

$$\sum_{|\alpha|=|\beta|=n} a_{\alpha\beta}(x)\xi^{\alpha+\beta} \geqq d(F)|\xi|^{2n},$$

for all $x \in F$ and $\xi = (\xi_1, \cdots, \xi_m) \in E^m$. It follows that by the usual means, see for example **[3]**, L can be used to define an operator, denoted by $\mathcal{L}$, whose domain is contained in $H_n^0(F)$.

A bounded domain $F \subset \Omega$ is a nodal domain for L iff there exists a nontrivial function $u \in H_n^0(F)$ such that $\mathcal{L}u = 0$. L is *nonoscillatory* in Ω iff there exists a sphere R, centered at the origin, such that L has no nodal domains in $\Omega - R$.

Preliminary theorems and reduction to simpler form. We begin by recalling the following result:

LEMMA 0. *Let F denote a bounded subdomain of Ω. Then the smallest eigenvalue of $\mathcal{L}$ is given by*:

$$\inf_{\substack{\phi \in C_0^\infty(F) \\ \phi \neq 0}} \frac{(\mathcal{L}\phi, \phi)}{(\phi, \phi)}.$$

Lemma 0 is a well known consequence of the Courant min-max theory of eigenvalues. We next employ the procedure of **[1]**, **[10]** to associate a symmetric matrix $(A_{ij})_{i,j=1,\cdots,N}$ with any symmetric expression $(a_{\alpha,\beta})_{|\alpha|=|\beta|=n}$, as follows: let σ denote a bijection from the first $N = (n+m-1)!/n!(m-1)!$ integers to the set of m-tuples whose sum is n. We define A_{ij} by $A_{ij} = a_{\sigma(i)\sigma(j)}$. Clearly the specific choice of the map σ does not affect the smallest eigenvalue of (A_{ij}).

LEMMA 1. *Let $\ell u = (-1)^n |\Sigma_{|\alpha|=|\beta|=n} D^\alpha(b_{\alpha\beta} D^\beta u)$ denote a symmetric elliptic operation with constant coefficients. Assume that*:

(i) *the symmetric matrix associated with $(a_{\alpha\beta}(x) - b_{\alpha\beta})$ is nonnegative for all $x \in \Omega$.*

Then the inequality:

$$(\phi, L\phi) \geqq \mu_0(-1)^n(\phi, \Delta^n\phi) - (a_{00}\phi, \phi),$$

is valid for all $\phi \in C_0^\infty(\Omega)$ *where* μ_0 *denotes the ellipticity constant of* ℓ, *and* Δ^n *denotes the* n*-times iterated Laplacian.*

Proof. Let $\phi \in C_0^\infty(\Omega)$, and let σ be a bijection from the first N integers to the m-tuples whose sum is n. It follows that:

$$(\phi, L\phi) - (\phi, \ell\phi) + (a_{00}\phi, \phi) = \int_{E^m} \sum_{|\alpha|=|\beta|=n} (a_{\alpha\beta} - b_{\alpha\beta}) D^\alpha \phi D^\beta \phi dx$$

$$= \int_{E^m} \sum_{i,j=1}^{N} (a_{\sigma(i)\sigma(j)} - b_{\sigma(i)\sigma(j)}) D^{\sigma(i)}\phi D^{\sigma(j)}\phi \, dx.$$

The integral on the right hand side is nonnegative by assumption (i). Next, let $\hat{\phi}$ denote the Fourier Transform of ϕ. It follows that

$$(\phi, \ell\phi) = \int_{E^m} \sum_{|\alpha|=|\beta|=n} b_{\alpha\beta} D^\alpha \phi D^\beta \phi \, dx$$

$$= \int_{E^m} |\hat{\phi}|^2 \sum_{|\alpha|=|\beta|=n} b_{\alpha\beta} \xi^\alpha \xi^\beta d\xi$$

$$\geqq \mu_0 \int_{E^m} |\hat{\phi}|^2 |\xi|^{2n}$$

$$= \mu_0(-1)^n(\phi, \Delta^n\phi).$$

We will say that L majorizes ℓ whenever condition (i) of the Lemma holds.

The conditions of Lemma 1 are clearly satisfied by any L whose leading part has constant coefficients, or in case $n = 1$ and L is uniformly elliptic in the whole of Ω. Similar arguments also show the validity of the following Corollary which gives another condition on $(a_{\alpha\beta})$ which is sufficient for an inequality analogous to that of Lemma 1 to hold.

COROLLARY 1. *Let* $\mu(x)$ *denote the least eigenvalue of* $(a_{\sigma(i)\sigma(j)}(x))_{i,j=1,\cdots,N}$. *Assume that there exists a constant* k *such that* $\mu(x) \geqq k > 0$ *for all* $x \in \Omega$. *Then,* L *majorises* $k\, \Sigma_{|\alpha|=n}(-1)^n D^{2\alpha} u$ *and for any* $\phi \in C_0^\infty(\Omega)$ *the following inequality holds*:

$$(\phi, L\phi) \geqq (-1)^n k \chi (\phi, \Delta^n \phi) - (a_{00}\phi, \phi),$$

where $\chi = \inf_{|\alpha|=n} (\alpha\,!/n\,!)$.

The next Lemma represents an extension of a lemma of Rellich **[13]** (where only the case $\alpha = 0$ was considered). Since the proof of Lemma 2 parallels that given in **[13]**, and in view of the lengthy calculations involved, the proof is only sketched.

LEMMA 2. *Let $\phi \in C_0^\infty(E^m - \{0\})$, $\alpha \in E^1$, $\alpha \leqq 0$. Then the following inequality is valid:*

$$\int_{E^m} |x|^\alpha (\Delta\phi)^2 dx \geqq K(\alpha) \int_{E^m} |x|^{\alpha-4} \phi^2 dx,$$

where: $K(\alpha) = (4 - \alpha - m)^2 (m - \alpha)^2/16 + \tau(\alpha)$,

$$\tau(\alpha) = \inf_{k \in \{0,1,\cdots\}} \left\{ (k)(k + m - 2)\left(k^2 + (m-2)k + \frac{m^2 - 4m + 4\alpha - \alpha^2}{2}\right)\right\}.$$

Proof. Let $\phi \in C_0^\infty(E^m - \{0\})$. Introducing polar coordinates, we obtain

$$\int_{E^m} r^\alpha (\Delta\phi)^2 dx = \int_0^\infty \int_\Phi (\Delta\phi)^2 dw\, r^{\alpha+m-1} dr,$$

where Φ denotes the full range of the angular variables and dw denotes the angular component of the volume element. Let $\{Y_i\}_{i=0}^\infty$ denote a complete orthonormal system of spherical harmonics. Then,

$$\int_\Phi (\Delta\phi)^2 dw = \sum_{i=0}^\infty c_i^2,$$

where $c_i = \int_\Phi (\Delta\phi) Y_i dw$. Let the order of Y_i be $k = k(i)$. It follows that:

$$c_i = \left[\frac{d^2}{dr^2} + \frac{(m-1)}{r}\frac{d}{dr} - \frac{k(k+m-2)}{r^2}\right] f_i,$$

where: $f_i = \int_\Phi Y_i \phi\, dw$. Hence:

$$(1) \quad \int_{E^m} r^\alpha (\Delta\phi)^2 dx = \sum_0^\infty \int_0^\infty r^{\alpha+m-1}\left(f_i'' + \frac{(m-1)f_i'}{r} - \frac{k(k+m-2)}{r^2} f_i\right)^2 dr.$$

Following the reasoning of the discussion preceding the Proof of Theorem 1 of [**13**, p. 93], we introduce new functions $g_i = r^{-\beta} f_i$, with $\beta = (4 - \alpha - m)/2$, and integrate by parts repeatedly to reduce the right hand side of (1) to:

$$\sum_0^\infty \int_0^\infty \Big\{ r^{\alpha+2\beta+m-1}(g_i'')^2 + (g_i')^2 r^{\alpha+2\beta+m-3}\Big(\frac{(m+\alpha-4)(m+\alpha-2)}{2}$$
$$+(m-1)(1-\alpha)+2k(k+m-2)\Big)$$
$$+ g_i^2 r^{\alpha+2\beta+m-5}\Big(\frac{(4-\alpha-m)^2(m-\alpha)^2}{16} + (k)(k+m-2)\Big(k^2$$
$$+(m-2)k + \frac{m^2-4m+4\alpha-\alpha^2}{2}\Big)\Big)\Big\}\, dr.$$

Since $\alpha \leqq 0$, the coefficient of $(g_i')^2$ is nonnegative for $i = 0, 1, \cdots$. Consequently:

$$\int_{E^m} r^\alpha (\Delta\phi)^2 dx \geqq \sum_0^\infty \int_0^\infty g_i^2 r^{\alpha+2\beta+m-5}\Big(\frac{(4-\alpha-m)^2(m-\alpha)^2}{16} + \tau(\alpha)\Big)\, dr$$
$$= K(\alpha)\sum_0^\infty \int_0^\infty f_i^2 r^{-5+\alpha+m} dr$$
$$= K(\alpha)\int_{E^m} \phi^2 r^{\alpha-4} dx.$$

Corollary 2. *$K(\alpha) \geqq 0$ and $=0$ iff for some triplet α, k and m we have*:

$$(2) \qquad (k)(k+m-2) = -\tfrac{1}{4}(m^2-4m+4\alpha-\alpha^2).$$

Proof. If we set $z = (m^2-4m+4\alpha-\alpha^2)/2$, $\eta(k) = k(k+m-2)$, then

$$\tau(\alpha) = \inf_{k\in\{0,1,\cdots\}} (\eta(k)[\eta(k)+z]).$$

Since $\eta[\eta+z]$ has a single minimum at $\eta = -z/2$ it follows that $\tau(\alpha) \geqq -z^2/4$, and at a minimum,

$$K(\alpha) = 1/16\{(4-\alpha-m)(m-\alpha)-(m^2-4m+4\alpha-\alpha^2)\}.$$
$$\cdot\{(4-\alpha-m)(m-\alpha)+(m^2-4m+4\alpha-\alpha^2\}$$
$$=0.$$

Finally, the minimum is achieved only if $\eta = -z/2$, i.e. if for some α, k, m equation (2) is satisfied.

Corollary 3.' *If $\alpha \leqq 0$ and $m \geqq 1 + \sqrt{(3-\alpha)(1-\alpha)}$ then $\tau(\alpha) \geqq 0$, and, consequently, it follows that*:

$$K(\alpha) \geqq \frac{(4-\alpha-m)^2}{16}(m-\alpha)^2.$$

Proof. Set $h(k) = \eta(k)[\eta(k)+z]$. Then $h(k)$ is a quartic in k, with zeros at $\eta(k)=0$ and $\eta(k)+z=0$. The first condition gives roots of $h(k)$ at $k=0$, $k=2-m$. The second condition gives possible roots at:

$$k = \frac{2-m \pm \sqrt{(m-2)^2-4z}}{2}.$$

The larger of the possible roots will exceed 1 iff:

$$(m-2)^2 - 4z > m^2,$$

i.e.,

$$(m-1)^2 < (\alpha-3)(\alpha-1).$$

Since this violates the condition of the Corollary, we conclude that all roots of $h(k)$ do not exceed 1. Consequently $h(k)$ is nonnegative for $k = 1, \cdots$.

The next Lemma was established as a consequence of a more general result by J. Piepenbrink **[11]**, by considerations involving Picone type identities. We give a much shorter direct proof, which will also be useful in the sequel.

LEMMA 3 **[11]**. *Let $\phi \in C_0^\infty(\Omega)$, $\alpha \in E^1$. Then the following inequality is valid:*

$$\int_{E^m} |x|^\alpha \sum_{i=1}^m (D_i\phi)^2 dx \geqq \pi(\alpha) \int_{E^m} |x|^{\alpha-2}\phi^2 dx, \tag{3}$$

where

$$\pi(\alpha) = \frac{(2-m-\alpha)^2}{4}.$$

Proof. Let $\phi = |x|^\beta\psi$, with $\beta = (2-m-\alpha)/2$. Substituting into the left hand side of (3) and using Green's formula we find

$$\int_{E^m} |x|^\alpha \sum_{i=1}^m (D_i\phi)^2 dx \geqq \int_{E^m} |x|^{\alpha+2\beta} \sum_i (D_i\psi)^2 dx$$

$$+(-\beta^2+\beta(2-m-\alpha))\int_{E^m} |x|^{\alpha+2\beta-2}\psi^2 dx,$$

and the result follows.

COROLLARY 4. *Let* $\alpha \leqq 0$, $\phi \in C_0^\infty(\Omega)$. *Then the following inequalities are valid*:

$$\text{(i)} \qquad \int_{E^m} |x|^\alpha (\Delta^t \phi)^2 dx \geqq \prod_{i=0}^{t-1} K(\alpha - 4i) \int_{E^m} |x|^{\alpha - 4t} \phi^2 dx,$$

$$\text{(ii)} \quad \int_{E^m} |x|^\alpha \sum_{i=1}^{m} [D_i(\Delta^t \phi)]^2 dx \geqq \pi(\alpha) \prod_{i=0}^{t-1} K(\alpha - 4i - 2) \int_{E^m} |x|^{\alpha - 4t - 2} \phi^2 dx.$$

Proof. Consider inequality (i). We observe that for $t = 1$, this is precisely Lemma 2. Assume next that the inequality is valid for $t = T$, then:

$$\begin{aligned} \int_{E^m} |x|^\alpha (\Delta^{T+1}\phi)^2 dx &= \int_{E^m} |x|^\alpha (\Delta(\Delta^T \phi))^2 dx \geqq K(\alpha) \int_{E^m} |x|^{\alpha-4} (\Delta^T \phi)^2 dx \\ &\geqq K(\alpha) \prod_{i=0}^{T-1} K(\alpha - 4 - 4i) \int_{E^m} |x|^{\alpha - 4T - 4} \phi^2 dx \\ &= \prod_{i=0}^{T} K(\alpha - 4i) \int_{E^m} |x|^{\alpha - 4(T+1)} \phi^2 dx. \end{aligned}$$

The result then follows by induction. The proof of (ii) is identical, except we first employ Lemma 3.

The inequalities of Corollary 4 are clearly also valid for any function in $H^0_{2t}(F)$ and $H^0_{2t+1}(F)$ respectively where F denotes a bounded domain of Ω.

Nonoscillation Criteria $(m \neq 2)$. We first consider the case $m \neq 2$. The case $m = 2$ will be considered in the next section. We begin with a Kneser-type theorem:

THEOREM 1. *Let L majorize an elliptic expression with constant coefficients and ellipticity constant* μ_0. *Assume that*:

$$\limsup_{|x| \to \infty} \{ |x|^{2n} a_{00}(x) \} < \omega$$

where

$$\omega = \mu_0 \prod_{i=0}^{n/2-1} K(-4i) \qquad (n \text{ even}), \tag{4}$$

$$= \mu_0 \pi(0) \prod_{i=0}^{(n-3)/2} K(-4i-2) \qquad (n \text{ odd}, \neq 1), \tag{5}$$

$$= \mu_0 \pi(0) \qquad (n = 1).$$

Then L is nonoscillatory.

Proof. If we assume the contrary, then given any sphere R there exists a bounded domain $F \subset \Omega - R$ and a function $u \in H_n^0(F)$ such that $(u, \mathscr{L}u) = 0$. By Lemma 1 it follows that:

$$\mu_0(-1)^n(u, \Delta^n u) - (a_{00}u, u) \leqq 0.$$

Estimating the first inner product by Corollary 4 leads to the desired contradiction.

We observe that examples involving Schrodinger equations can easily be constructed both for the cases $n = 1$ and $n = 2$ to show that in general the constants in Theorem 1 cannot be improved upon.

COROLLARY 5. *If $m > 2n$, or $2n \geqq m$ and m is odd, then the constant ω of Theorem 1 is positive. If $m \geqq 1 + \sqrt{(2n-1)(2n-3)}$ or $m = 1$, then:*

$$\omega \geqq \mu_0 4^{-n} \prod_{i=0}^{n-1} (m + 4i - 2n)^2.$$

Proof. Assume first that n is even. By Corollary 2, $K(-4i)$ will be positive for $i = 0, \cdots, n/2 - 1$ unless for some integer $k \geqq 0$ the following relations are valid:

$$(k)(k + m - 2) = -\tfrac{1}{4}(m^2 - 4m - 16i - 16i^2),$$
$$m^2 - 4m - 16i - 16i^2 < 0.$$

From the first equation we obtain the only possible value:

$$k = \frac{4(1 + i) - m}{2}.$$

If m is odd, this equation cannot be satisfied by any integer k. If m is even, then this equation again cannot be satisfied by a nonnegative integer if $m > 2n$. A similar argument shows the conclusion for n odd. If $m \geqq 1 + \sqrt{(2n-1)(2n-3)}$, then by Corollary 3, $K(-4i) \geqq (4 + 4i - m)^2(m + 4i)^2/16$. We thus obtain, for n even:

$$\begin{aligned}\prod_{i=0}^{n/2-1} K(-4i) &\geqq \prod_{i=0}^{n/2-1} \frac{(4 + 4i - m)^2(m + 4i)^2}{16} \\ &= 4^{-n} \prod_{i=0}^{n/2-1} (m + 4i - 2n)^2 \prod_{i=n/2}^{n-1} (m + 4i - 2n)^2 \\ &= 4^{-n} \prod_{i=0}^{n-1} (m + 4i - 2n)^2.\end{aligned}$$

Similarily, for n odd, it follows that:

$$\pi(0)\prod_{i=0}^{(n-3)/2} K(-4i-2) \geqq \pi(0)\prod_{i=0}^{(n-3)/2} \frac{(6+4i-m)^2}{16}(m+4i+2)^2$$
$$= (m-2)^2 4^{-n}\prod_{i=0}^{(n-3)/2}(m+4i-2n)^2$$
$$\times \prod_{i=(n-3)/2+2}^{n-1}(m+4i-2n)^2$$
$$= 4^{-n}\prod_{i=0}^{n-1}(m+4i-2n)^2.$$

Finally the result can be shown for the case $m = 1$ by a simple iteration procedure using inequality (3) of Lemma 3.

Theorem 1 and Corollary 5 reduce in special cases to several known criteria. If $n = m = 1$, then we obtain Kneser's classical result **[6]**; if $n = 2$, $m = 1$ then we have a result of Leighton and Nehari **[9]**; if $n = 1$, then we have a theorem of Headley and Swanson **[5]**; if $n = 2$ and $m > 4$ then we have a result of Yoshida **[16]**; if $m = 1$ then we have a criterion of Glazman [**4**, p. 96].

As can be observed, Theorem 1 and Corollary 5 leave unanswered the following question: What is a Kneser-type theorem for $m \leqq 2n$ and m even? This appears to be an open question.

We next establish an integral theorem which does not involve pointwise estimates on a_{00}.

THEOREM 2. *Let L majorize an elliptic expression with constant coefficients and ellipticity constant μ_0. Assume that $m > 2n$ and that for some ϵ_0, $0 \leqq \epsilon_0 < 1$, there exists a sphere R such that*:

$$\left[\epsilon_0\mu_0 4^{-n}|x|^{-2n}\prod_{i=0}^{n-1}(m+4i-2n)^2 - a_{00}\right]_- \in L^{m/2n}(\Omega - R).$$

Then L is nonoscillatory.

Proof. If we assume to the contrary that L is oscillatory, then given any sphere R_1 there exists a bounded domain $F \subset \Omega - R_1$ and a function $u \in H_n^0(F)$ such that:

$$(u,(-1)^n\Delta^n u) - (a_{00}\mu_0^{-1}u, u) \leqq 0.$$

Then, by Corollary 4,

$$(u,(-1)^n\Delta^n u) = (1-\epsilon_0)(u,(-1)^n\Delta^n u) + \epsilon_0(u,(-1)^n\Delta^n u)$$
$$\geqq (1-\epsilon_0)(u,(-1)^n\Delta^n u) + \mu_0^{-1}\omega\epsilon_0(|x|^{-2n}u, u).$$

Consequently,

$$(u,(-1)^n\Delta^n u)-\frac{\mu_0^{-1}}{1-\epsilon_0}((|x|^{-2n}\omega\epsilon_0-a_{00})_- u,u)\leqq 0.$$

We observe that, by Hölder's Inequality and generalization of Sobolev's Estimates [3, p. 24], it follows that for some constant C and any $\phi\in C_0^\infty(\Omega-R_1)$, the following inequality holds:

$$\begin{aligned}((\omega\epsilon_0|x|^{-2n}-a_{00})_-\phi,\phi)&\leqq\left(\|(\epsilon_0\omega|x|^{-2n}-a_{00})_-\|_{m/2n}(\Omega-R_1)\right)\|\phi\|^2_{2m/(m-2n)}\\&\leqq C\|(\epsilon_0\omega|x|^{-2n}-a_{00})_-\|_{m/2n}(\Omega-R_1)\\&\qquad\times(\phi,(-1)^n\Delta^n\phi).\end{aligned}$$

A simple limit argument shows that this inequality is also valid for u, *hence:*

$$0\geqq(u,(-1)^n\Delta^n u)\left[1-\frac{C}{1-\epsilon_0}\|(\epsilon_0\omega|x|^{-2n}-a_{00})_-\|_{m/2n}(\Omega-R_1)\right].$$

Choosing R_1 sufficiently large gives a contradiction.

As a corollary of Theorem 2, we obtain an extension of a theorem of [12], where the case $n=1$ was considered. We recall that L is unconditionally nonoscillatory if for any constant $\lambda>0$ the operation:

$$(-1)^n\sum_{|\alpha|=|\beta|=n}D^\alpha(a_{\alpha\beta}D^\beta u)-\lambda(a_{00})_+u=0$$

is nonoscillatory.

COROLLARY 6. *If* $2n<m$ *and* $(a_{00})_+\in L^{m/2n}(\Omega-R)$ *for some sphere* R, *then* L *is unconditionally nonoscillatory.*

Proof. We observe that, for all $\epsilon>0$, $\lambda>0$:

$$\|\{\epsilon\omega|x|^{-2n}-\lambda(a_{00})_+\}_-\|_{m/2n}(\Omega-R)\leqq|\lambda|\|(a_{00})_+\|_{m/2n}(\Omega-R)<\infty.$$

The result follows from Theorem 2.

We observe that as a consequence of the above results it follows that if $(a_{00})_+$ is bounded and the measure of Ω is finite then L is nonoscillatory.

We conclude this section with a brief heuristic reference to an alternate method which, although less general, is considerably simpler than the one we have followed. The arguments now introduced will also be useful for the case $m=2$ discussed in the next section. The basic ideas are contained in the following Lemmas. The proof of the first Lemma is immediate from the Courant min-max theory of eigenvalues.

LEMMA 4. *If* $(-1)^n \Delta^n u - (a_{00})_+ u$ *is oscillatory in* $\{x / |x| > R\}$, *then it will also be oscillatory if with a nodal domain we associate the boundary conditions*: $u = \Delta u = \cdots = \Delta^{n-1} u = 0$ *instead of the standard null conditions.*

LEMMA 5. *Let* $\mathcal{F} = \{u; u \in C^\infty(-R)$, *u positive and* $(-1)^t \Delta^t u \geqq 0$ *in* $-R$ *for* $t = 1, \cdots, n\}$ *for some sphere R. Assume L majorizes an operation with constant coefficients and ellipticity constant* μ_0 *and that, for some* $u \in \mathcal{F}$ *and all* $x \in -R$, *the following inequality holds*:

$$a_{00}(x) \leqq \mu_0 \left(\frac{(-1)^n \Delta^n u(x)}{u(x)} \right). \tag{6}$$

Then L is nonoscillatory.

Proof. Without loss of generality we may assume that equality holds in (6). If L is oscillatory, then by Lemma 4 there exists a bounded domain $F' \subset -R$ and a nontrivial function v such that:

$$(-1)^n \Delta^n v(x) - \mu_0^{-1} a_{00}(x) v(x) = 0, \quad x \in F'$$
$$(-1)^t \Delta^t v(x) = 0, \quad x \in \partial F', \quad t = 0, \cdots, n-1.$$

Let ϵ be chosen so that $\inf_{x \in F'}(u(x) - \epsilon v(x)) = 0$. In view of the boundary conditions, there exists a point $x_0 \in F'$ such that:

$$u(x_0) - \epsilon v(x_0) = 0.$$

We observe that:

$$-\Delta((-1)^{n-1} \Delta^{n-1}(u - \epsilon v)) = \frac{a_{00}}{\mu_0}(u - \epsilon v) \geqq 0.$$

It follows that $(-1)^{n-1}\Delta^{n-1}(u - \epsilon v) \geqq 0$ in F'. By induction, it follows that $-\Delta(u - \epsilon v) \geqq 0$ in F'. Since $u - \epsilon v \not\equiv 0$, then it cannot have a minimum of zero **[7]**, and the contradiction establishes the Lemma.

The proof of Lemma 5, for n even, can also be based on integral identities similar to those of Diaz and Dunninger **[2]**. Our procedure appears simpler, since only a form of the maximum principle is employed, and with slight modifications it also leads directly to the establishment of Sturmian theorems similar to those given in **[2]**.

It seems reasonable to establish nonoscillation criteria by substituting functions of the type $u = |x|^{\alpha}$ into Lemma 5 and choosing α so that a_{00} can be taken as large as possible. After a simple but lengthy calculation, this procedure leads to some of the results of Corollary 5, but only for the case $m \geqq 2n + 1$. If $m \leqq 2n$ this procedure appears to fail as the following example illustrates. Consider the operation $Lu = \Delta^2 u - (9/16)|x|^{-4}u$, defined in the complement of a sphere in E^3. From Corollary 4 it follows that L is nonoscillatory. Yet there is no value of α such that $u = |x|^{\alpha}$ substituted into Lemma 5 will give this result.

Nonoscillation Criteria ($m = 2$). The methods of the previous section appear to fail for the case $m = 2$. Our considerations are restricted to the case $m = 2$, $n = 1$.

THEOREM 3. *Let $m = 2$, $n = 1$. Assume that L majorizes an elliptic operator with ellipticity constant μ_0 and that for all $|x|$ sufficiently large,*

$$|x|^2(ln\,|x|)^2 a_{00}(x) \leqq \frac{\mu_0}{4}. \tag{7}$$

Then L is nonoscillatory.

Proof. Set $u = (ln\,|x|)^{\frac{1}{2}}$ then $-\Delta u - (1/(4|x|^2(ln\,|x|)^2))u \equiv 0$. The result now follows by Lemma 5.

The theorem may also be established by the simple variable change: $\phi = (ln\,|x|)^{1/2}\psi$, and the use of Lemma 3. This and other radial results can also be obtained by the methods of **[14]**. Simple radial examples can be constructed to show that this is the best possible result in the sense that the constant $\mu_0 4^{-1}$ cannot be improved upon.

THEOREM 4. *Let the conditions of Theorem 3 hold except for inequality* (7). *Assume that $a_{00} \in C^2(\Omega)$ and that for some $\epsilon > 0$, there exists a sphere R such that: $[a_{00}]_+|x|^{4\epsilon/(1+\epsilon)} \in L^{1+\epsilon}(\Omega - R)$. Then L is nonoscillatory.*

Proof. It suffices to show that $-\Delta u - a_{00}\mu_0^{-1}u = 0$ is nonoscillatory. Assuming the contrary we find that given any sphere R_1 there exists a bounded domain $F \subset \Omega - R_1$ and a function $\phi \in C_0^{\infty}(F)$ such that:

$$\int_{\Omega - R_1} \left\{ \sum_{i=1}^{2} (D_i\phi)^2 - \mu_0^{-1}(a_{00})_+ \phi^2 \right\} dx < 0.$$

By introducing the change of variable $y = x/|x|^2$, we conclude that there exists a deleted annular neighborhood N of 0, with diameter $\leqq 1$, and a function $\psi \in C_0^\infty(N)$ such that:

$$\int_N \left\{ \sum_{i=1}^{2} (D_i \psi)^2 - \frac{c(y)}{|y|^4} \psi^2 \right\} dy < 0,$$

where $\psi(y) = \phi(x(y))$ and $c(y) = (a_{00}(x(y)))_+ \cdot q(y) \cdot \mu_0^{-1}$, where $q(y) \in C^\infty(E^2)$, $0 \leqq q(y) \leqq 1$ and:

$$q(y) \equiv \begin{cases} 1, & y \in \operatorname{supp} \psi(y) \\ 0, & y \notin \{y \mid y \mid y|^{-2} \in F\}. \end{cases}$$

If we set $N_1 = N \cup \{0\}$ it follows that there exists a function $u \in C^2(\bar{N}_1)$ which is positive in N_1 and satisfies:

$$-\Delta u - \frac{c(y)}{|y|^4} u \leqq 0 \quad y \in N_1, \quad u = 0 \quad \text{on the boundary of } N_1.$$

Without loss of generality, we may assume $u(y) \leqq 1$ with $u(y_0) = 1$ for some $y_0 \in N_1$. Let $G(y, \xi)$ denote Green's function for the Laplacian. It follows that:

$$\begin{aligned} 1 = u(y_0) &\leqq \int_{N_1} u(\xi) \frac{c(\xi)}{|\xi|^4} G(y_0, \xi) d\xi \\ &\leqq \frac{1}{2\pi} \int_{N_1} u(\xi) \log\left(\frac{1}{|y_0 - \xi|}\right) \frac{c(\xi)}{|\xi|^4} d\xi \\ &\leqq \frac{1}{2\pi} \left\| \log\left(\frac{1}{|y_0 - \xi|}\right) \right\|_{(1+\epsilon)/\epsilon} (N_1) \cdot \| c(\xi) |\xi|^{-4} \|_{1+\epsilon} (N_1) \\ &\leqq K \| [a_{00}]_+ |x|^{4\epsilon/(1+\epsilon)} \|_{1+\epsilon} (\Omega - R_1), \end{aligned}$$

for some constant K. Choosing R_1 sufficiently large leads to the desired contradition.

It is interesting to note that we cannot take $\epsilon = 0$ in the previous result so that the result of Corollary 6 cannot be extended to this case, as the following example shows. Consider $-\Delta u - (1/|x|^2 \log^2 |x|) u = 0$ on the outside of the sphere $R = \{x \mid |x| > 2\}$. Then $\int_{E^2 - R} (1/|x|^2 \log^2 |x|) dx < \infty$ but by the arguments following Theorem 3 this equation is oscillatory.

More General Cases. We conclude by considering some extensions of the previous results to more general operators.

The above discussion dealt only with self-adjoint operators defined by operations such as L. The results however are immediately applicable to the wider class of nonselfadjoint elliptic operators for which we can conclude that if F is a nodal domain for the arbitrary operator $\mathscr{L}$ then for some nontrivial function u we have $0=(\mathscr{L}u,u)=(((\mathscr{L}+\mathscr{L}^*)/2)u,u)$, where $\mathscr{L}^*$ denotes the adjoint of $\mathscr{L}$. Consequently, conditions which are sufficient for the nonoscillation of the selfadjoint operator $(\mathscr{L}+\mathscr{L}^*)/2$ are also sufficient to guarantee the nonoscillation of $\mathscr{L}$. We illustrate these remarks with the following example:

COROLLARY 7. *The nonselfadjoint uniformly elliptic operator formally defined by*:

$$Lu = -\sum_{i,j=1}^{m} D_i(a_{ij}D_ju)+2\sum_{j=1}^{m} b_jD_ju - a_{00}u = 0, \qquad a_{ij}=a_{ji}$$

is nonoscillatory if $b_j \in C^1(\Omega)$ *for* $j=1,\cdots,m,\ m>2$, *and*

$$\left(\frac{(m-2)^2}{4|x|^2}\epsilon_0 - \frac{1}{\mu_0}\left(a_{00}+\sum_{j=1}^{m} D_j(b_j)\right)\right)_- \in L^{m/2}(\Omega - R)$$

for some sphere R *and* $1>\epsilon \geqq 0$, *where* μ_0 *denotes the ellipticity constant of* L.

Proof. We observe that if L is oscillatory then given any sphere R_1 there exists a domain $F \subset \Omega - R_1$ and a function $u \in H_1^0(F)$ such that:

$$0 = \int_F \left(\mu_0\sum_{i=1}^{m}(D_iu)^2 + 2u\sum_{i=1}^{m} b_iD_iu - a_{00}u^2\right)dx$$

$$= \int_F \left(\mu_0\sum_{i=1}^{m}(D_iu)^2 - \left(\sum_{i=1}^{m} D_i(b_i)+a_{00}\right)u^2\right)dx.$$

Consequently the selfadjoint expression:

$$-\Delta u - \left(\sum_{i=1}^{m} D_i(b_i)+a_{00}\right)\frac{1}{\mu_0}u = 0$$

is oscillatory. A contradiction now follows from Theorem 2.

Corollary 7 extends a pointwise theorem first established by Swanson **[14]**.

Finally, we consider some possible extensions to selfadjoint operators of a more general type, specifically to those generated by expressions of type:

$$Lu = \sum_{t=0}^{n} (-1)^t \sum_{|\alpha|=|\beta|=t} D^\alpha (a_{\alpha\beta} D^\beta u).$$

Assumptions can be made on the coefficients of the lower order derivatives which lead to the previously considered situations. As an illustrative example we consider the following:

COROLLARY 8. *The operator formally defined by*:

$$Lu = \Delta^2 u - \sum_{\substack{i=1\\ j=1}}^{m} D_i (a_{ij} D_j u) - a_{00} u,$$

is nonoscillatory if $m > 4$ and for some nonpositive function $-\mu_0$ we have:

(i) $(a_{ij}(x)) \geqq -\mu_0(x) I$ *for all* $|x|$ *sufficiently large where I denotes the identity matrix and,*

(ii) *for some sphere* R, $\mu_0 \in L^{m/2}(\Omega - R)$; $(a_{00})_+ \in L^{m/4}(\Omega - R)$.

Proof. If L is oscillatory, then given any sphere R_1, there exists a domain $F \subset \Omega - R_1$ and a function $u \in H_2^0(F)$ such that

$$\begin{aligned} 0 &= \int_F \left\{ (\Delta u)^2 + \sum_{i,j=1}^{m} a_{ij} D_i u D_j u - a_{00} u^2 \right\} dx \\ &\geqq \int_F \left\{ (\Delta u)^2 - \mu_0 \sum_{i=1}^{m} (D_i u)^2 - a_{00} u^2 \right\} dx. \end{aligned}$$

But, by Sobolev's inequality **[3]**,

$$\int_F \mu_0 \sum_{i=1}^{m} (D_i u)^2 \leqq C \|\mu_0\|_{m/2} (\Omega - R_1) \int_F (\Delta u)^2 dx$$

for some constant C. Consequently,

$$0 \geqq \int_F \left\{ \left[1 - C \|\mu_0\|_{m/2} (\Omega - R_1) \right] (\Delta u)^2 - a_{00} u^2 \right\} dx.$$

The result now follows from Theorem 2.

Added in proof. The nonoscillation theory of elliptic equations of higher order has also been recently considered in the paper "Nonoscillation criteria for elliptic equations of order $2m$" (submitted for publication) by E. Noussair and N. Yoshida. There is a small overlap between their results and ours.

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