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THE GEOMETRY OF $p(S^1)$

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Let p be a polynomial of degree n. The image of the unit circle, $p(S^1)$, can be thought of as a subset of the real part of an algebraic curve W of degree $2n$. This paper outlines some facts about $p(S^1)$ which can be obtained using classical algebraic geometry, for example Bézout's theorem.

Introduction. We wish to study the image of the unit circle S^1 in the complex plane under mapping by a polynomial of degree n. If we let $x^2 + y^2 = 1$ be the equation of the unit circle in R^2, then if x and y vary over the complex numbers **C**, we can think of the unit circle as the real part of an algebraic variety V in $\mathbf{C}^2$. We show that similarly $p(S^1)$ can be thought of as a subset of the real part of an algebraic variety W in $\mathbf{C}^2$. We use the method of absolute coordinates as outlined in Winger **[12]** and Morley **[7]**, and we discuss W in terms of the Schwarz function as used by Davis **[2]**.

We obtain the equation for the real part of W in the form $h(\xi, \bar{\xi}) = 0$, where h is a polynomial of degree $2n$. We show that if all the zeros of p' are in $|z| < 1$, then $p(S^1)$ is actually all of the real part of W. We show that the circular points are of multiplicity n on W and that W has at most $(n-1)^2$ simple nodes. If no singular point of W is on $p(S^1)$ then p is univalent, i.e., one-to-one in $|z| < 1$. We give this condition in terms of a Hermitian form.

1. Definitions. Let **C** denote the complex numbers. In the following, we consider **C** as a subset of $\mathbf{C}^2$, identifying the complex number z with the point $(z, \bar{z}) \in \mathbf{C}^2$. We say $(z, \bar{z})$ are absolute coordinates of z (Winger **[12]** p. 324). If V is a set in $\mathbf{C}^2$ we will call $\mathbf{C} \cap V = \{(z, \zeta) \in V \mid \zeta = \bar{z}\}$ the real part of V.

Let $S^1 = \{z \mid |z| = 1\}$ be the unit circle in **C**. The equation of S^1 in absolute coordinates is $z\bar{z} = 1$, so we may consider S^1 as the real part of the variety $V \subseteq \mathbf{C}^2$ given by the equation $z\zeta = 1$.

Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial of degree n. Let $\bar{p}(z) = \overline{a_0} + \overline{a_1} z + \cdots + \overline{a_n} z^n$. We consider p as a map from **C** to **C**. Since $(z, \bar{z}) \to (p(z), \overline{p(z)})$ gives the mapping in absolute coordinates, we may look at p as the restriction to **C** of the mapping $\tilde{p}: \mathbf{C}^2 \to \mathbf{C}^2$ defined by $(z, \zeta) \to (p(z), \bar{p}(\zeta))$.

2. $\tilde{p}(V)$. We now look at $W = \tilde{p}(V)$, which is the rational curve in $\mathbf{C}^2$ given by parametric equations $\xi = p(z)$, $\eta = \bar{p}(1/z)$. We find the equation of W in ξ and η by eliminating z from $p(z)-\xi=0$ and $p^*(z)-\eta z^n=0$, where $p^*(z)=\overline{a_n}+\overline{a_{n-1}}z+\cdots+\overline{a_0}z^n=z^n\bar{p}(1/z)$. Let $h(\xi,\eta)$ be the resultant of $p(z)-\xi$ and $p^*(z)-\eta z^n$ as polynomials in z, i.e.,

$$h(\xi,\eta)=\begin{vmatrix} a_0-\xi & & \cdots & & a_n & & & \\ & a_0-\xi & \cdots & & & a_n & & \\ & & \vdots & & & & & \\ & & & a_0-\xi & & & \cdots & a_n \\ \overline{a_n} & & \cdots & & \overline{a_0}-\eta & & & \\ & \overline{a_n} & \cdots & & & \overline{a_0}-\eta & & \\ & & \vdots & & & & & \\ & & & \overline{a_n} & & & \cdots & \overline{a_0}-\eta \end{vmatrix}.$$

We see that h is of degree $2n$ and $h(\xi,\eta)=0$ is the equation of W. The real part of W, $W\cap\mathbf{C}$, is given by the equation $h(\xi,\bar{\xi})=0$ in absolute coordinates. Clearly, $p(S^1)=\{(p(z),\overline{p(z)})\,|\,|z|=1\}$ therefore $p(S^1)\subseteq W\cap\mathbf{C}$. We note also that $h(\xi,\eta)=\bar{h}(\eta,\xi)$ so that $(\xi,\eta)\in W$ iff $(\eta,\xi)\in W$.

We also remark that $h(\xi,\bar{\xi})$ may be written as the determinant of an $n\times n$ Hermitian matrix as follows. Let $g(z)=p(z)-\xi$, $g^*(z)=z^n\bar{g}(1/z)=p^*(z)-\bar{\xi}z^n$. Define the Bézout resultant (see Marden **[6]** p. 200):

$$\frac{g^*(x)\bar{g}^*(y)-g(x)\bar{g}(y)}{1-xy}=\sum_{j,k=0}^{n-1}h_{jk}x^jy^k.$$

Then $H=H(\xi,\bar{\xi})=(h_{jk})$ is a $n\times n$ Hermitian matrix and $h(\xi,\bar{\xi})=\det H(\xi,\bar{\xi})$.

The matrix H also defines a Hermitian form on $\mathbf{C}^n$ of some interest. Let $U=(u_0,\cdots,u_{n-1})$ be a row matrix, then $U\to\bar{U}HU^t$ defines a Hermitian form. Let π be the number of positive squares and ν the number of negative squares of H reduced to canonical form. If $h(\xi,\bar{\xi})\neq 0$, then π is the number of zeros of $p(z)-\xi$ in $|z|<1$ and ν is the number of zeros in $|z|>1$ (see Marden **[6]** p. 200).

3. The Schwarz function of $p(S^1)$.

Let V be the curve in $\mathbf{C}^2$ given by $z\zeta=1$, as in §1. Let $z^*=1/\bar{z}$ be the reflection of z in

S^1. We see that $V = \{(z, \overline{z^*}) \mid z \in \mathbf{C}\}$. The function $z \to \overline{z^*} = 1/z$ is called the Schwarz function for S^1 (Davis [2]), and V may be considered as the graph in $\mathbf{C}^2$ of the Schwarz function.

Likewise near a nonsingular point of $p(S^1) \subseteq W$, the function $p(z) \to p(1/\bar{z})$ is reflection in the analytic arc $p(S^1)$, and locally this function followed by conjugation is called the Schwarz function for $p(S^1)$. Writing $\eta = S(\xi)$ for the Schwarz function, we see that the complete analytic function that it determines is algebraic satisfying $h(\xi, \eta) = 0$, where h is as in the previous section. Thus W may be considered as the graph of the Schwarz function for $p(S^1)$.

4. $W \cap \mathbf{C} - p(S^1)$. We have seen that $p(S^1) \subseteq W \cap \mathbf{C}$. If $\xi = p(z) = p(1/\bar{z})$ for $|z| \neq 1$, then $\xi \in W \cap \mathbf{C}$, but ξ is not on $p(S^1)$. We may say $\xi \in W \cap \mathbf{C} - p(S^1)$ if ξ is not on $p(S^1)$ but is its own reflection in $p(S^1)$, i.e., $S(\xi) = \bar{\xi}$ and $\xi \not\in p(S^1)$. It would be interesting to know more about $W \cap \mathbf{C} - p(S^1)$, and in particular the relationship to the zeros of the derivative of p. We prove the following

THEOREM 1. *If all the zeros of $p'(z)$ are in $|z| < 1$, then $W \cap \mathbf{C} =$, $p(S^1)$.*

Proof. Suppose to the contrary that there is a complex number a such that $|a| \neq 1$ and $p(a) = p(1/\bar{a})$. Then

$$\int_{1/\bar{a}}^{a} p'(t)dt = 0$$

where the integral is over the line segment from $1/\bar{a}$ to a. Therefore $p'(z)$ is apolar to

$$q(z) = \int_{1/a}^{a} (t - z)^{n-1} dt = \frac{(z - a)^n}{n} - \frac{(z - 1/\bar{a})^n}{n}$$

(see Marden [6] p. 61). The zeros of q are on the perpendicular bisector L of the line segment joining a and $1/\bar{a}$. The distance of L from 0 is $\frac{1}{2}(1/r + r) > 1$, where $r = |a|$. Let A be the closed half-plane determined by L, and not containing the disc $|z| < 1$. By Grace's theorem (Marden [6] p. 61), A contains at least one zero of p'. But this contradicts the hypothesis of the theorem, and we have proof by contradiction.

As a consequence of the theorem, for example, the image of the unit circle under $p(z) = z^2 + z$ is

$$h(\xi,\bar{\xi}) = \begin{vmatrix} -\bar{\xi} & 1 & 1 & 0 \\ 0 & -\bar{\xi} & 1 & 1 \\ 1 & 1 & -\xi & 0 \\ 0 & 1 & 1 & -\xi \end{vmatrix}$$

$$= |1-\xi|^2 - (1-|\xi|^2)^2$$

$$= 0$$

since the only zero of the derivative is at $-1/2$.

5. Points of W on the line at ∞. We consider $\mathbf{C}^2$ as a subspace of the projective space $P_2(\mathbf{C})$ in the usual way by identifying the point (z,ζ) with the point in $P_2(\mathbf{C})$ with homogeneous coordinates $(z,\zeta,1)$. Let $\tilde{h}(\xi,\eta,\chi)$ be the ternary form defined by $\tilde{h}(\xi,\eta,\chi) = \chi^{2n}h(\xi/\chi,\eta/\chi)$. Let W^* be the projective closure of W in $P_2(\mathbf{C})$, i.e., let W^* be the projective variety given by $\tilde{h}(\xi,\eta,\chi)=0$. From the determinant expression for h in §1, we see that $\tilde{h}(\xi,\eta,0) = (\xi\eta)^n$. Therefore the points with homogeneous coordinates $(0,1,0)$ and $(1,0,0)$ are on W^*. These are just the circular points given in absolute coordinates (Winger **[12]** p. 52). We also see that $\tilde{h}(\xi,1,\chi) = (-1)^n(a_0\chi-\xi)^n +$ (forms in (χ,ξ) of degree $> n$). Thus $(0,1,0)$ is on W^* of multiplicity n. Likewise $(1,0,0)$ is on W^* of multiplicity n. The effect of this is to reduce the number of real intersections of $p(S^1)$ with curves through the circular points. For example, by Bézout's theorem (Walker **[11]** p. 111; Fulton **[3]** p. 112) W^* intersects a circle exactly $2(2n)$ times. Now $2n$ of these intersections are at circular points, therefore the number of real intersections is at most $2n$. Since $p(S^1)\subseteq W\cap\mathbf{C}$, the number of intersections of a circle with $p(S^1)$ is at most $2n$. For more on this see Quine **[10]**.

6. Multiple points of W. We investigate points of W with more than one preimage under $\tilde{p}$. Suppose that $p(\alpha)=p(\beta)$ and $\bar{p}(1/\alpha)=\bar{p}(1/\beta)$. Write

$$G(z,\zeta) = \frac{p(z)-p(\zeta)}{z-\zeta} = \sum_{k=1}^{n} a_k\phi_k(z,\zeta)$$

where ϕ_k is the form of degree $k-1$ defined by

$$\phi_k(z,\zeta) = (z^k-\zeta^k)/(z-\zeta).$$

We note that G is of degree $n-1$ and $G(z,z)=p'(z)$. Now writing

$$G^*(z,\zeta) = z^{n-1}\zeta^{n-1}\bar{G}(1/z,1/\zeta)$$
$$= \sum_{k=1}^{n} \bar{a}_k\phi_k(z,\zeta)(z\zeta)^{n-k}$$

we note that G^* is of degree $2(n-1)$. We see that (α,β) is on the intersection of the curves given by $G(z,\zeta)=0$ and $G^*(z,\zeta)=0$. By Bézout's theorem, if G and G^* have no common component, then they have at most $2(n-1)^2$ intersections. We have the following theorem

THEOREM 2. *If G and G^* have a common component, then $p(z)=q(z^k)$ where k is an integer greater than 1 and q is a polynomial.*

Proof. Make the change of variables $z=uv$, $\zeta=u$. We have $G(z,\zeta)=g(u,v)$ where

$$g(u,v) = \sum_{k=1}^{n} a_k \frac{v^k-1}{v-1} u^{k-1}$$

and $G^*(z,\zeta)=u^{n-1}g^*(u,v)$ where

$$g^*(u,v) = \sum_{k=1}^{n} \overline{a_k} v^{n-k} \frac{v^k-1}{v-1} u^{n-k}.$$

Now G and G^* have a common component iff g and g^* have a common component. Let $R(v)$ be the resultant of g and g^* as polynomials in u. From the determinant expression for the resultant we have

$$R(v) = |a_n|^{2(n-1)}\left(\frac{v^n-1}{v-1}\right)^{2(n-1)} + \cdots$$

so that R is of degree $2(n-1)^2$. Thus any common factor of g and g^* is a polynomial in v alone. Therefore let $f=f(v)$ and suppose f divides g. Then f divides $(v^n-1)/(v-1)$ and so f has as a zero some primitive kth root of unity, where k divides n. Denote this root by ω, then

$$g(u,\omega) = \frac{p(u)-p(u\omega)}{u(1-\omega)}$$

is identically 0 in u. Therefore $p(u)\equiv p(u\omega)$ hence $p(z)=q(z^k)$ for some polynomial q and the proof follows by contradiction.

If $p(z)=q(z^k)$ then $p(S^1)=q(S^1)$. Therefore without loss of generality in studying $p(S^1)$, we may assume that p is reduced so that $p(z)$ is not of the form $q(z^k)$, and we will henceforth make this assumption. We note that if $a_1=1$ the assumption holds automatically.

COROLLARY 1. *The equation* $\tilde{p}(v_1)=\tilde{p}(v_2)=w$ *for* $v_1, v_2 \in V$ *and* $v_1 \neq v_2$ *holds for at most* $(n-1)^2$ *points in* W.

COROLLARY 2. $p(S^1)$ *has at most* $(n-1)^2$ *self-intersections.*

The last corollary is sharp as we showed in Quine **[8]**. We note that self-intersections of $p(S^1)$ correspond to real singularities of the algebraic curve W.

7. Univalent polynomials. Let $p(z)=z+a_2z^2+\cdots+a_nz^n$. Let $V_n=\{(a_2,\cdots,a_n) \mid p \text{ is 1–1 in } |z|<1\}$ be the domain of variability for polynomials of degree n. Now $(a_2,\cdots,a_n)$ is in the interior of V_n iff W has no singular points on $p(S^1)$ (see Quine **[8]**). We determine the condition algebraically as follows: Let $R(z)$ be the resultant of $G(z,\zeta)$ and $G^*(z,\zeta)$ as polynomials in ζ. R is of degree $2(n-1)^2$, and the condition that $(a_2,\cdots,a_n)\in \operatorname{Int} V_n$ becomes $R(z)\neq 0$ for $|z|=1$. By the symmetry of G and G^* we see that $R(z)=0$ iff $R(1/\bar{z})=0$, therefore without loss of generality, we may assume that R is self-inversive, i.e., $z^{2(n-1)^2}\bar{R}(1/z)=R(z)$. The condition that a self-inversive polynomial have no zeros on $|z|=1$ can be expressed in terms of a Hermitian form following Krein **[5]**. Let $R_1(z)=(n-1)^2R(z)-zR'(z)$. Let

$$B(x,y)=\frac{R(x)\overline{R_1}(y)+R_1(x)\bar{R}(y)}{1-xy}$$

$$=\sum_{j,k=0}^{2(n-1)^2-1} b_{jk}x^jy^k.$$

The matrix (b_{jk}) determines a Hermitian form B on $\mathbf{C}^{2(n-1)^2}$ in the usual way. Let π be the number of positive squares and ν the number of negative squares of B reduced to canonical form. Krein showed that $R(z)$ has no zeros on $|z|=1$ iff $\pi=\nu$. Therefore we have

THEOREM 3. $(a_2,\cdots,a_n)\in \operatorname{Int} V_n$ *iff* $\pi=\nu$ *for the Hermitian form* B.

For more information on V_n, see Koessler **[4]**, Quine **[9]**, Brannan **[1]**.

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