PACIFIC JOURNAL OF MATHEMATICS
Vol. 64, No. 2, 1976

QUASITRIANGULAR OPERATOR ALGEBRAS

JOAN K. PLASTIRAS

Fix a sequence $\mathcal{P} = \{P_n\}_{n=1}^{\infty}$ of finite dimensional projections increasing to the identity on a separable Hilbert space $\mathcal{H}$ and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded operators on $\mathcal{H}$. The *quasitriangular algebra associated with* $\mathcal{P}$ **and denoted as $\mathcal{QT}(\mathcal{P})$ is defined to be the set of those operators T in $\mathcal{L}(\mathcal{H})$ for which $\|P_n^{\perp}TP_n\| \to 0$.**

In this paper we will examine the structure of the $\mathcal{QT}(\mathcal{P})$ algebras. Specifically, if $\mathcal{R} = \{R_n\}_{n=1}^{\infty}$ is another sequence of finite dimensional projections increasing to the identity on the same Hilbert space, when is $\mathcal{QT}(\mathcal{R})$ equal to $\mathcal{QT}(\mathcal{P})$? By an *algebraic isomorphism* **between two algebras we shall mean a bijection which preserves algebraic structure: that is to say — addition, scalar multiplication, multiplication, but we do not impose any topological condition. When are two quasitriangular algebras isomorphic?**

In [5] we asked the same questions of $\mathcal{D}(\mathcal{E}) + \mathcal{C}(\mathcal{H}) = \{T + K: T$ belongs to the commutant of E and K is compact$\}$ and answered them completely by arguments very different from those presented here; the conclusions were different too. The concept of quasitriangularity for operators was first isolated for systematic study in [3]. The quasitriangular algebra was introduced later in [1] and a formula expressing the distance from such an algebra to an arbitrary operator was obtained. We begin our discussion with an algebraic property:

DEFINITION 1. A subset $\mathcal{S}$ of $\mathcal{L}(\mathcal{H})$ is said to be inverse-closed if whenever T in $\mathcal{S}$ is invertible in $\mathcal{L}(\mathcal{H})$ then T^{-1} belongs to $\mathcal{S}$.

LEMMA 2. *$\mathcal{QT}(\mathcal{P})$ is inverse-closed for every sequence $\mathcal{P} = \{P_n\}_{n=1}^{\infty}$ of finite dimensional projections increasing to the identity on a Hilbert space.*

Before verifying Lemma 2 we remark that the assumption that the P_k be finite dimensional is essential.

Proof. From [1, Corollary following 2.2] we know that $\mathcal{QT}(\mathcal{P}) = \mathcal{T}(\mathcal{P}) + \mathcal{C}(\mathcal{H})$, where $\mathcal{T}(\mathcal{P})$ is the set of operators T such that $P_n^{\perp}TP_n = 0$ for all n. Hence, it suffices to assume that S belongs to $\mathcal{T}(\mathcal{P}) + \mathcal{C}(\mathcal{H})$ and is invertible in $\mathcal{L}(\mathcal{H})$ and show that S^{-1} belongs to $\mathcal{QT}(\mathcal{P})$. So, $S = T + C$, where $T \in \mathcal{T}(\mathcal{P})$ and $C \in \mathcal{C}(\mathcal{H})$. Since $S_m = T + P_mCP_m$ tends in norm to S, S_m is invertible for all m greater than a positive integer l. Fix $m > l$ and note that $S_m P_n = P_n S_m P_n$ for all $n \geqq m$, and since $\dim P_n < \infty$, S_m maps $P_n \mathcal{H}$ onto itself, so that $P_n S_m^{-1} P_n = S_m^{-1} P_n$ (or equivalently, $P_n^{\perp} S_m^{-1} P_n \equiv 0$). Hence, S_m^{-1} belongs to $\mathcal{Q}\mathcal{T}(\mathcal{P})$ by definition. As S_m^{-1} tends in norm to $(T+C)^{-1}$ and $\mathcal{Q}\mathcal{T}(\mathcal{P})$ is norm-closed [**1**, Proposition 2.1], we conclude that $(T+C)^{-1}$ belongs to $\mathcal{Q}\mathcal{T}(\mathcal{P})$.

THEOREM 3. *Suppose that T is an invertible operator in $\mathcal{L}(\mathcal{H})$. Then T implements an automorphism of $\mathcal{Q}\mathcal{T}(\mathcal{P})$* (i.e. $T\mathcal{Q}\mathcal{T}(\mathcal{P})T^{-1} = \mathcal{Q}\mathcal{T}(\mathcal{P})$) *if and only if T belongs to $\mathcal{Q}\mathcal{T}(\mathcal{P})$.*

Proof. $\Leftarrow$: Assume that T belongs to $\mathcal{Q}\mathcal{T}(\mathcal{P})$. To show that T implements an inner automorphism of $\mathcal{Q}\mathcal{T}(\mathcal{P})$ it will suffice to show that T^{-1} also belongs to $\mathcal{Q}\mathcal{T}(\mathcal{P})$. But that is immediate from Lemma 2.

$\Rightarrow$: Assume that T implements an automorphism of $\mathcal{Q}\mathcal{T}(\mathcal{P})$. First we conclude from [**1**, Theorem 3.3] that T admits a factorization $T = UA$, where A belongs to $\mathcal{T}(\mathcal{P})$ and U is a partial isometry. Note that $A = U^*T$ has closed range; since $\ker A = \{0\}$, A is semi-Fredholm by definition. Since A belongs to $\mathcal{Q}\mathcal{T}(\mathcal{P})$ the index of A is nonnegative [**2**] so that $\ker A^* = \{0\}$ and A is consequently invertible. This forces U to be unitary. Since $A \in \mathcal{Q}\mathcal{T}(\mathcal{P})$ is invertible, then by the previous argument, A implements an automorphism of $\mathcal{Q}\mathcal{T}(\mathcal{P})$ so that we are reduced to showing that if U is a unitary operator which implements an automorphism of $\mathcal{Q}\mathcal{T}(\mathcal{P})$, then U belongs to $\mathcal{Q}\mathcal{T}(\mathcal{P})$.

So, we assume that U does not belong to $\mathcal{Q}\mathcal{T}(\mathcal{P})$ and arrive at a contradiction. Since U does not belong to $\mathcal{Q}\mathcal{T}(\mathcal{P})$ then by the definition of $\mathcal{Q}\mathcal{T}(\mathcal{P})$ there is an $\alpha > 0$ and a subsequence $\{P_{n(k)}\}_{k=1}^{\infty}$ of $\mathcal{P}$ for which $\underline{\lim}_k \|P_{n(k)}^{\perp} U P_{n(k)}\| \geqq \alpha$. From Lemma 2 we know that U^* does not belong to $\mathcal{Q}\mathcal{T}(\{P_{n(k)}\}_{n=1}^{\infty})$, so that by definition, there is $\beta > 0$ and a subsequence $\{m(k)\}_{k=1}^{\infty}$ of $\{n(k)\}_{k=1}^{\infty}$ for which $\underline{\lim}_k \|P_{m(k)}^{\perp} U^* P_{m(k)}\| \geqq \beta$. If we let $\epsilon = \min(\alpha, \beta)/2$, then we can conclude that $\|P_n^{\perp} U P_n\|$ and $\|P_n U P_n^{\perp}\|$ $(= \|P_n^{\perp} U^* P_n\|)$ are both greater than ϵ for all n in an infinite subset M of $\mathbf{N}$.

We will obtain a sequence $\{m_i, n_i\}_{i=1}^{\infty}$ of positive integers such that $0 < m_1 < n_1 < m_2 < n_2 < \cdots$ and projections $\{F_k, E_k\}_{k=1}^{\infty}$ such that $F_k = P_{m_k} P_{n_{k-1}}^{\perp}$ and $E_k = P_{n_k} P_{m_k}^{\perp}$ for which $\|F_k U E_k\|$ and $\|E_k U F_k\|$ are both greater than $\epsilon/2$. We do so inductively.

For $k = 1$, define $F_1 = P_{m_1}$, where m_1 is the first integer in M. Let n_1 be the first integer such that $\|P_{n_1} P_{m_1}^{\perp} U P_{m_1}\|$ and $\|P_{m_1} U P_{m_1}^{\perp} P_{n_1}\|$ are both greater than $\epsilon/2$ (such an n_1 exists because $\|P_{m_1}^{\perp} U P_{m_1}\|$ and $\|P_{m_1} U P_{m_1}^{\perp}\|$ are greater than ϵ and the P_n tend strongly to the identity).

Assume that we have obtained $\{E_k, F_k\}_{k=1}^{l}$. To obtain m_{l+1} and n_{l+1}, note that UP_{n_l} and $P_{n_l}U$ are compact; hence, there is a positive integer j such that $\|P_n^{\perp} U P_{n_l}\|$ and $\|P_{n_l} U P_n^{\perp}\|$ are both less than $\epsilon/4$ for all $n \geqq j$. Let m_{l+1} be the first integer in M greater than j.

Then

$$\begin{aligned} \|P^{\perp}_{m_{l+1}} U P_{m_{l+1}} P^{\perp}_{n_l}\| &\geqq \|P^{\perp}_{m_{l+1}} U P_{m_{l+1}}\| - \|P^{\perp}_{m_{l+1}} U P_{m_{l+1}} P_{n_l}\| \\ &\geqq \epsilon - \|P^{\perp}_{m_{l+1}} U P_{n_l}\| \\ &\geqq \epsilon - \epsilon/4 = \tfrac{3}{4}\epsilon. \end{aligned}$$

Similarly, $\|P_{m_{l+1}} P^{\perp}_{n_l} U P^{\perp}_{m_{l+1}}\| \geqq 3\epsilon/4$ by the same argument. Let n_{l+1} be the first positive integer greater than m_{l+1} for which $\|P_{n_{l+1}} P^{\perp}_{m_{l+1}} U P_{m_{l+1}} P^{\perp}_{n_l}\|$ and $\|P_{m_{l+1}} P^{\perp}_{n_l} U P^{\perp}_{m_{l+1}} P_{n_{l+1}}\|$ are both greater than $\epsilon/2$. Let $F_{l+1} = P_{m_{l+1}} P^{\perp}_{n_l}$ and let $E_{l+1} = P_{n_{l+1}} P^{\perp}_{m_{l+1}}$. Continue inductively.

We select a subsequence $\{E_{i_j}, F_{i_j}\}_{j=1}^{\infty}$ of $\{E_i, F_i\}_{i=1}^{\infty}$ as follows: first, we let $\{\alpha_{ij}\}_{i,j=1}^{\infty}$ be any sequence of positive real numbers such that $\Sigma_{i,j}\alpha_{ij}^2 \leqq \epsilon^2/16$. Let $i_1 = 1$. Assuming that we have obtained i_k, let i_{k+1} be the next positive integer such that for all $l \lneqq k+1$, $\|E_{i_{k+1}} U F_{i_l}\|$ and $\|F_{i_{k+1}} U E_{i_l}\|$ are less than $\alpha_{k+1,l}$ while $\|E_{i_l} U F_{i_{k+1}}\|$ and $\|F_{i_l} U E_{i_{k+1}}\|$ are less than $\alpha_{l,k+1}$. This is possible because UF_{i_l}, $F_{i_l}U$ (respectively UE_{i_l}, $E_{i_l}U$) are compact and the E_i (respectively F_i) tend weakly to zero. Continue inductively. Now for each i_k there is a rank one partial isometry $T_{i_k} \in \mathscr{L}(E_{i_k}\mathscr{H}, F_{i_k}\mathscr{H})$ such that $\|E_{i_k} U T_{i_k} U^* F_{i_k}\| \geqq \epsilon^2/4$. Clearly, $T = \Sigma_{k=1}^{\infty} T_{i_k}$ is a partial isometry in $\mathscr{T}(\mathscr{P})$. So, for arbitrary l in $\mathbf{N}$,

$$E_{i_l}(UTU^*)F_{i_l} = \sum_{k=1}^{\infty} E_{i_l} U T_{i_k} U^* F_{i_l} = E_{i_l} U T_{i_l} U^* F_{i_l} + \sum_{\substack{k=1 \\ k \neq l}}^{\infty} E_{i_l} U T_{i_k} U^* F_{i_l}.$$

Hence,

$$\|E_{i_l}(UTU^*)F_{i_l}\| + \left\| \sum_{\substack{k=1 \\ k \neq l}}^{\infty} E_{i_l} U T_{i_k} U^* F_{i_l} \right\| \geqq \|E_{i_l} U T_{i_l} U^* F_{i_l}\|.$$

$$\|E_{i_l}(UTU^*)F_{i_l}\| + \sum_{\substack{k=1 \\ k \neq l}}^{\infty} \|E_{i_l} U T_{i_k} U^* F_{i_l}\| \geqq (\epsilon/2)^2 = \epsilon^2/4.$$

Therefore,

$$\begin{aligned} |E_{i_l}(UTU^*)F_{i_l}\| &\geqq \frac{\epsilon^2}{4} - \sum_{k \neq l} \|E_{i_l} U F_{i_k}\| \cdot \|E_{i_k} U^* F_{i_l}\| \\ &\geqq \frac{3\epsilon^2}{16}. \end{aligned}$$

Since i_l was arbitrary, it follows from the construction that

$$\frac{3\epsilon^2}{16} \leqq \|E_{i_l}(UTU^*)F_{i_l}\| \leqq \|P^{\perp}_{m_{i_l}}(UTU^*)P_{m_{i_l}}\|.$$

Hence,

$$\overline{\lim_k} \| P_k^{\perp}(UTU^*)P_k \| > 0$$

and it follows by definition of $\mathcal{QT}(\mathcal{P})$ that UTU^* does not belong to $\mathcal{QT}(\mathcal{P})$. This contradicts our assumption that U implements an automorphism of $\mathcal{QT}(\mathcal{P})$ and thus concludes the argument of the proof of Theorem 3.

DEFINITION 4. Let $\mathcal{P} = \{P_n\}_{n=1}^{\infty}$ be a sequence of finite dimensional projections increasing to the identity on a Hilbert space $\mathcal{H}$. An operator T is said to be *strictly upper triangular for* $\mathcal{P}$ if $P_n^{\perp}TP_{n+1} = 0$ for all n in $\mathbf{N}$.

REMARK 5. Note that in the proof of Theorem 3 we showed that if U does not belong to $\mathcal{QT}(\mathcal{P})$ then there is an operator T, which is strictly upper triangular for $\mathcal{P}$, and such that UTU^* does not belong to $\mathcal{QT}(\mathcal{P})$.

REMARK 6. Let $\mathcal{S} = \{S_n\}_{n=1}^{\infty}$ be any sequence of finite dimensional projections increasing to the identity on $\mathcal{H}$. Let $\mathcal{P} = \{P_n\}_{n=1}^{\infty}$ be a subsequence of $\mathcal{S}$. Then $\mathcal{QT}(\mathcal{S}) \subseteq \mathcal{QT}(\mathcal{P})$. Equality may fail; however, if T is strictly upper triangular for $\mathcal{P}$ then T belongs to $\mathcal{QT}(\mathcal{S})$.

DEFINITION 7. A sequence of projections $\mathcal{S} = \{S_n\}_{n=1}^{\infty}$ increasing to the identity on a Hilbert space $\mathcal{H}$ is said to be a *defining sequence* for a quasitriangular algebra $\mathcal{A}$ if and only if $\mathcal{A} = \{T \in \mathcal{L}(\mathcal{H}) \colon \| S_n^{\perp}TS_n \| \to 0\}$.

REMARK 8. Suppose that U is a unitary operator which implements an isomorphism $T \to UTU^*$ from $\mathcal{QT}(\mathcal{P})$ onto $\mathcal{QT}(\mathcal{S})$. Then U maps defining sequences of $\mathcal{QT}(\mathcal{P})$ to defining sequences of $\mathcal{QT}(\mathcal{S})$.

LEMMA 9. *Suppose that* $\mathcal{P} = \{P_n\}_{n=1}^{\infty}$ *and* $\mathcal{S} = \{S_n\}_{n=1}^{\infty}$ *are sequences of finite dimensional projections increasing to the identity such that* $\mathcal{P} \cup \mathcal{S}$ *is totally ordered by inclusion. Then* $\mathcal{QT}(\mathcal{P}) = \mathcal{QT}(\mathcal{S})$ *if and only if there exist positive integers* m_0 *and* n_0 *such that* $P_{m_0+k} = S_{n_0+k}$ *for all* k *in* $\mathbf{N}$.

Proof. $\Leftarrow$: This conclusion is clear.

$\Rightarrow$: Assume that $\mathcal{QT}(\mathcal{P}) = \mathcal{QT}(\mathcal{S})$. Then $\mathcal{QT}(\mathcal{P}) = \mathcal{QT}(\mathcal{P} \cup \mathcal{S})$. We assert that $\mathcal{P}$ contains all but perhaps finitely many of the projections in $\mathcal{P} \cup \mathcal{S}$. Contrapositively, assume not. Let $\mathcal{R} = \{R_n\}_{n=1}^{\infty}$ be a total ordering of $\mathcal{P} \cup \mathcal{S}$ and choose an infinite subsequence $\{n_k\}_{k=1}^{\infty}$ for which $R_{n_k} \notin \mathcal{P}$ but $R_{n_k+1} \in \mathcal{P}$. Let T_k be any rank one partial isometry with initial space $(R_{n_k} \ominus R_{n_k-1})\mathcal{H}$ and final space $(R_{n_k+1} \ominus R_{m_k})\mathcal{H}$. Then $T = \Sigma_{k=1}^{\infty} T_k$ is a partial isometry which belongs to $\mathcal{QT}(\mathcal{P})$ but not to $\mathcal{QT}(\mathcal{P} \cup \mathcal{S})$.

Hence, $\mathcal{QT}(\mathcal{P} \cup \mathcal{S}) \subsetneqq \mathcal{QT}(\mathcal{P})$. We conclude that $\mathcal{P}$ contains all but perhaps finitely many of the projections in $\mathcal{P} \cup \mathcal{S}$.

By symmetry, $\mathcal{S}$ contains all but perhaps finitely many of the projections in $\mathcal{P} \cup \mathcal{S}$. So there exists a positive integer k such that $\{P_n : \dim P_n \geqq k\} \subseteq \mathcal{S}$ and $\{S_n : \dim S_n \geqq k\} \subseteq \mathcal{P}$. Let m_0 be the first positive integer such that $\dim(P_{m_0}) \geqq k$ and let n_0 be the first integer such that $\dim(S_{n_0}) \geqq k$. Then $P_{m_0+k} = S_{n_0+k}$ for all $k \in \mathbf{N}$.

THEOREM 10. *$\mathcal{S} = \{S_n\}_{n=1}^{\infty}$ is a defining sequence for $\mathcal{QT}(\mathcal{P})$ if and only if there exist positive integers m_0 and n_0 such that $\lim_k \|P_{m_0+k} - S_{n_0+k}\| = 0$.*

Proof. $\Leftarrow$: We note that $\mathcal{QT}(\mathcal{S}) \subseteq \mathcal{QT}(\mathcal{P})$ since for T in $\mathcal{QT}(\mathcal{S})$,

$$\begin{aligned}\|P_{m_0+k}^{\perp} T P_{m_0+k}\| &\leqq \|S_{n_0+k}^{\perp} T S_{n_0+k}\| + \|(P_{m_0+k}^{\perp} - S_{n_0+k}^{\perp}) T S_{n_0+k}\| \\ &\quad + \|P_{m_0+k}^{\perp} T (P_{m_0+k} - S_{n_0+k})\| \\ &\leqq \|S_{n_0+k}^{\perp} T S_{n_0+k}\| + \|P_{m_0+k}^{\perp} - S_{n_0+k}^{\perp}\| \cdot \|T\| \\ &\quad + \|T\| \cdot \|P_{m_0+k} - S_{n_0+k}\|,\end{aligned}$$

and the other inclusion follows by symmetry.

$\Rightarrow$: We assume that $\mathcal{S} = \{S_n\}_{n=1}^{\infty}$ is a defining sequence for $\mathcal{QT}(\mathcal{P})$. Let V be any unitary operator such that $\{VS_nV^*\}_{n=1}^{\infty} \cup \{P_n\}_{n=1}^{\infty}$ is a sequence of projections totally ordered by set inclusion.

Let $\mathcal{W} = \{W_n\}_{n=1}^{\infty}$ with $W_n = VS_nV^*$ for each n. We assert that V belongs to $\mathcal{QT}(\mathcal{W})$. So assume that T is strictly upper triangular for $\mathcal{W}$; it suffices to show that VTV^* belongs to $\mathcal{QT}(\mathcal{W})$ by Remark 5. By Remark 6, T belongs to $\mathcal{QT}(\mathcal{P} \cup \mathcal{W}) \subseteq \mathcal{QT}(\mathcal{P})$ so that it remains to observe that $V\mathcal{QT}(\mathcal{P})V^* \subseteq \mathcal{QT}(\mathcal{W})$: $W_n^{\perp}(VTV^*)W_n = (VS_n^{\perp}V^*)(VTV^*)(VS_nV^*) = VS_n^{\perp}TS_nV^*$, so that $\|W_n^{\perp}(VTV^*)W_n\| = \|VS_n^{\perp}TS_nV^*\| = \|S_n^{\perp}TS_n\| \to 0$.

Hence, we conclude that V belongs to $\mathcal{QT}(\mathcal{W})$. Since $\mathcal{QT}(\mathcal{W})$ is inverse-closed by Lemma 2, it follows that $\|W_n^{\perp}VW_n\| \to 0$ and $\|W_nVW_n^{\perp}\| = \|W_n^{\perp}V^*W_n\| \to 0$.

(1) Since $W_nV = VS_n$, we have that $W_nVW_n^{\perp} = VS_nW_n^{\perp}$ so that $\|W_nVW_n^{\perp}\| = \|VS_nW_n^{\perp}\| = \|S_nW_n^{\perp}\| \to 0$ and

(2) Since $W_n^{\perp}V = VS_n^{\perp}$, we have that $W_n^{\perp}VW_n = VS_n^{\perp}W_n$ so that $\|W_n^{\perp}VW_n\| = \|VS_n^{\perp}W_n\| = \|S_n^{\perp}W_n\| \to 0$.

Since $\|S_n - W_n\| = \max\{\|S_n^{\perp}W_n\|, \|S_nW_n^{\perp}\|\}$ [**5**, Lemma 6] it follows that $\lim_n \|S_n - W_n\| = 0$ and by a previous argument that $\mathcal{W}$ is a defining sequence for $\mathcal{QT}(\mathcal{P})$. It follows from Lemma 9 that there are integers m_0 and n_0 such that $W_{n_0+k} = P_{m_0+k}$ for all k in $\mathbf{N}$. Hence

$$\lim_k \| S_{n_0+k} - P_{m_0+k} \| = 0,$$

which concludes the proof.

EXAMPLE 11. As an easy consequence of Theorem 10, it follows that there exist defining sequences $\mathcal{P} = \{P_n\}_{n=1}^{\infty}$ and $\mathcal{R} = \{R_n\}_{n=1}^{\infty}$ for a quasitriangular algebra $\mathcal{A}$ such that $\{P_n \vee R_n\}_{n=1}^{\infty}$ is not a defining sequence for $\mathcal{A}$ ("$\vee$" denotes the supremum of two projections). This phenomenon is suggested by an example in [**3**, p. 285].

We shall say that two subsets of $\mathcal{L}(\mathcal{H})$, $\mathcal{S}$ and $\mathcal{T}$, are *locally isomorphic* if each operator in $\mathcal{S}$ is unitarily equivalent to an operator in $\mathcal{T}$ and conversely. Because every quasitriangular operator is a compact perturbation of a triangular operator, it follows that any two quasitriangular algebras are locally isomorphic; from Theorem 12 we conclude that they are not necessarily isomorphic.

THEOREM 12. *Let $\mathcal{QT}(\mathcal{P})$ and $\mathcal{QT}(\mathcal{S})$ be quasitriangular algebras. Then $\mathcal{QT}(\mathcal{P})$ and $\mathcal{QT}(\mathcal{S})$ are algebraically isomorphic if and only if there exist positive integers j_0 and l_0 such that* $\dim(P_{j_0+k}) = \dim(S_{l_0+k})$ *for all k in* $\mathbf{N}$.

Proof. $\Leftarrow$: If we assume that there exist positive integers j_0 and l_0 such that $\dim(P_{j_0+k}) = \dim(S_{l_0+k})$ for all k in $\mathbf{N}$, then we can define a unitary operator U such that $UP_{j_0+k}U^* = S_{l_0+k}$ for all k in $\mathbf{N}$. We assert that U implements an isomorphism from $\mathcal{QT}(\mathcal{P})$ to $\mathcal{QT}(\mathcal{S})$.

$\Rightarrow$: Assume that there is a map α from $\mathcal{QT}(\mathcal{P})$ to $\mathcal{QT}(\mathcal{S})$ which preserves algebraic structure. Since $\mathcal{QT}(\mathcal{P})$ and $\mathcal{QT}(\mathcal{S})$ are Banach algebras, each of which contains the set of finite rank operators, it follows from [**6**, Theorem 2.5.19] that there exists an invertible operator S such that $\alpha(T) = STS^{-1}$ for all T in $\mathcal{QT}(\mathcal{P})$.

We conlude from [**1**, Theorem 3.3] that S has a factorization $S = UA$ where A belongs to $\mathcal{T}(\mathcal{P})$ and U is unitary. Then we note that $R_n = UP_nU^*$ is a defining sequence for $\mathcal{QT}(\mathcal{S})$; by Theorem 10, we note that there exist positive integers m_0 and n_0 such that $\| R_{m_0+k} - S_{n_0+k} \| \to 0$. So, there exists a positive integer d such that $\| R_{m_0+d+k} - S_{n_0+d+k} \| < 1$ for all k in $\mathbf{N}$. Hence, $\dim(R_{m_0+d+k}) = \dim(S_{n_0+d+k})$ for all k in $\mathbf{N}$. Since $\dim(P_n) = \dim(R_n)$ for all n in $\mathbf{N}$, let $j_0 = m_0 + d$ and let $l_0 = n_0 + d$ to obtain the theorem.

REFERENCES

1. W. B. Arveson, *Interpolation problems in nest algebras*, J. Functional Analysis, **20** (1975), 208–233.

2. R. G. Douglas and C. Pearcy, *A note on quasitriangular operators*, Duke Math. J., **37** (1970), 177–188.

3. P. R. Halmos, *Quasitriangular operators*, Acta Sci. Mat. (Szeged), **29** (1968), 283–293.

4. ———, *Ten problems in Hilbert space*, Bulletin Amer. Math. Soc., **76** (1970), 887–933.

5. J. K. Plastiras, *Compact perturbations of certain von Neumann algebras*, Trans. Amer. Math. Soc., (to appear).

6. C. E. Rickart, *General Theory of Banach Algebras*, D. van Nostrand Company, Inc. (1960).

7. B. Sz.-Nagy, *Specktraldarstellung linearer Transformation des Hilbertschen Raumes*, Berlin, (1942).

Received February 23, 1976.

UNIVERSITY OF PENNSYLVANIA