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ON THE ROLE OF AN ABELIAN PHASE GROUP IN RELATIVIZED PROBLEMS IN TOPOLOGICAL DYNAMICS

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In this paper we are concerned with transformation groups having compact Hausdorff phase space. This paper is an outgrowth of an example that answers negatively the well-known question, must a proximally equicontinuous minimal set with abelian phase group be locally almost periodic; see Examples 3.1 and 3.3.

In §2 we associate with each countable or finite group T a function F from $X \times S$ onto T such that $F(x, ss') = F(x, s)F(xs, s')$, where $X = \Pi_1^\infty \{0, 1\}$ and S is the direct sum $\Pi_1^\infty \{0, 1\}$ given the discrete topology. And we associate with each transformation group (Y, T) the extension $(X \times Y, S)$ of (X, S) defined by $(x, y)s = (xs, yF(x, s))$. If (Y, T) is minimal, then $(X \times Y, S)$ is minimal; and if in addition, $\varphi: (Y, T) \to (W, T)$ is a homomorphism, then $\psi = \text{identity} \times \varphi: (X \times Y, S) \to (X \times W, S)$ is a homomorphism and shares many of the dynamical and topological properties of φ. For example, ψ is proximal iff φ is and ψ is distal iff φ is. This indicates that for many relativized problems one may assume that the phase group is abelian (or indeed $\Pi\{0, 1\}$) without loss of generality. There are a great many properties that φ and ψ may share, we study some of those properties that are of particular importance in the structure of minimal sets and homomorphisms in general.

The association we use in §2 is developed in a more general setting in §1. This general setting allows us to give an example of a proximally equicontinuous minimal set with abelian phase group and connected phase space that is not locally almost periodic, see Example 3.3.

I would like to thank T. S. Wu for his helpful suggestions.

This paper was announced in **[4]**, where a slightly different point of view is taken.

DEFINITIONS AND NOTATION. Suppose $\varphi: (X, T) \to (Y, T)$ is a homomorphism of minimal sets. Define $R(\varphi) = \{(x, x'): \varphi(x) = \varphi(x')\}$. The relativized regionally proximal relation is $Q(\varphi) = \{(x, x')$: for some nets $(x_n, x'_n) \in R(\varphi)$ and t_n in T, $\lim(x_n, x'_n)t_n = (x, x)$ and $\lim(x_n, x'_n) = (x, x')\}$; the relativized proximal relation is $P(\varphi) = R(\varphi) \cap P_X$, where $P_X = P = \{(x, x')$: for some net t_n, $\lim xt_n = \lim x't_n\}$ is the proximal

relation on X; the regionally proximal relation on X, $Q_X = Q$ equals $Q(\varphi)$ when Y is a singleton. The relativized distal structure relation, $S_d(\varphi)$, and the relativized equicontinuous structure relation, $S_e(\varphi) = S(\varphi)$, are the smallest closed invariant equivalence relations containing $P(\varphi)$ and $Q(\varphi)$ respectively. We say φ is proximal iff $R(\varphi) \subset P_X$. φ is distal iff $R(\varphi) \cap P_X = P(\varphi) = \Delta$, the diagonal of X. A closed invariant equivalence relation, R, on X is called proximal or distal if the induced homomorphism $(X, T) \to (X/R, T)$ is; so R is a proximal equivalence relation iff $R \subset P_X$, and R is a distal equivalence relation iff $R \cap P_X = \Delta$.

1. The basic idea for the construction in this section is as follows. Given homeomorphisms α, β of a compact Hausdorff space Y, take two copies of Y, (i.e. $\{0, 1\} \times Y$), and define homeomorphism θ_α and θ_β on $\{0, 1\} \times Y$ by $(0, y)\theta_\alpha = (1, y\alpha)$, $(1, y)\theta_\alpha = (0, y\alpha^{-1})$, $(0, y)\theta_\beta = (0, y\beta)$, and $(1, y)\theta_\beta = (1, y\alpha^{-1}\beta\alpha)$. One can check that $\theta_\beta\theta_\alpha = \theta_\alpha\theta_\beta$.

One could take the more general approach of assuming θ_α was defined by $(0, y)\theta_\alpha = (1, y\alpha)$, $(1, y)\theta_\alpha = (0, y\alpha')$ for some homeomorphism α' of Y. Now with θ_β defined so that $\theta_\beta(0, y) = (0, y\beta)$, in order that $\theta_\alpha\theta_\beta = \theta_\beta\theta_\alpha$ one must define,

$$\begin{aligned}(1, y)\theta_\beta &= (1, y)\theta_\alpha^{-1}\theta_\beta\theta_\alpha = (0, y\alpha^{-1})\theta_\beta\theta_\alpha \\ &= (0, y\alpha^{-1}\beta)\theta_\alpha = (1, y\alpha^{-1}\beta\alpha).\end{aligned}$$

Then $(1, y)\theta_\beta\theta_\alpha^{-1} = (1, y)\theta_\alpha^{-1}\theta_\beta$. Also

$$(0, y)\theta_\beta\theta_\alpha^{-1} = (0, y\beta)\theta_\alpha^{-1} = (1, y\beta\alpha'^{-1})$$

while $(0, y)\theta_\alpha^{-1}\theta_\beta = (1, y\alpha'^{-1})\theta_\beta = (1, y\alpha'^{-1}\alpha^{-1}\beta\alpha)$. So $\theta_\alpha\theta_\beta = \theta_\beta\theta_\alpha$ iff $\beta\alpha'^{-1} = \alpha'^{-1}\alpha^{-1}\beta\alpha$ iff $\beta\alpha\alpha' = \alpha\alpha'\beta$. So $\alpha\alpha' = \gamma$ in a homeomorphism of Y that commutes with β, that is, $\alpha' = \alpha^{-1}\gamma$ differs from α^{-1} by a homeomorphism that commutes with β. Conversely if γ is a homeomorphism of Y that commutes with β, define $(0, y)\theta_\gamma = (0, y\gamma)$, $(1, y)\theta_\gamma = (1, y)$, define $(0, y)\theta_\alpha^* = (1, y\alpha)$, $(1, y)\theta_\alpha^* = (1, y\alpha^{-1})$, and define $(0, y)\theta_\beta = (0, y\beta)$, $(1, y)\theta_\beta = (1, y\alpha^{-1}\beta\alpha)$, then $\theta_\beta\theta_\alpha^* = \theta_\alpha^*\theta_\beta$, $\theta_\alpha = \theta_\alpha^*\theta_\gamma$ and $\theta_\beta\theta_\alpha = \theta_\beta\theta_\alpha^*\theta_\gamma = \theta_\alpha^*\theta_\beta\theta_\gamma = \theta_\alpha^*\theta_\gamma\theta_\beta = \theta_\alpha\theta_\beta$. Thus the general, θ_α, is easily derived from the more special, θ_α^*; we will not consider this particular generalization in what follows.

To generalize from two homeomorphisms to a countable number, consider a countable set $\{\alpha_n\}_1^\infty$ of homeomorphisms of Y and let $X = \Pi_1^\infty\{0, 1\}$. Define $\theta_1: X \times Y \to X \times Y$ by $((0, l_2, \cdots), y)\theta_1 = ((1, l_2, l_3, \cdots), y\alpha_1)$ and $((1, l_2, \cdots), y)\theta_1 = ((0, l_2, \cdots), y\alpha_1^{-1})$. Then define θ_2 by

$$((0,0,l_3,\cdots),y)\theta_2 = ((0,1,l_3,\cdots),y\alpha_2)$$
$$((0,1,l_3,\cdots),y)\theta_2 = ((0,0,l_3,\cdots),y\alpha_2^{-1})$$
$$((1,0,l_3,\cdots),y)\theta_2 = ((1,1,l_3,\cdots),y\alpha_1^{-1}\alpha_2\alpha_1)$$
$$((1,1,l_3,\cdots),y)\theta_2 = ((1,0,l_3,\cdots),y\alpha_1^{-1}\alpha_2^{-1}\alpha_1).$$

One would define θ_3 similarly, for example

$$((0,0,0,l_4,\cdots),y)\theta_3 = ((0,0,1,l_4,\cdots),y\alpha_3)$$
$$((0,0,1,l_4,\cdots),y)\theta_3 = ((0,0,0,l_4,\cdots),y\alpha_3^{-1})$$
$$((1,0,0,l_4,\cdots),y)\theta_3 = ((1,0,1,l_4,\cdots),y\alpha_1^{-1}\alpha_3\alpha_1)$$
$$((1,1,0,l_4,\cdots),y)\theta_3 = ((1,1,1,l_4,\cdots),y\alpha_1^{-1}\alpha_2^{-1}\alpha_3\alpha_2\alpha_1).$$

For each n, θ_n could be defined similarly and one could check that $\theta_n\theta_m = \theta_m\theta_n$.

To generalize it one step further, for $n = 1,2,\cdots$, let G_n be a compact Hausdorff topological group and for each n let $(G_n \times Y, G_n)$ be a transformation group such that $(g,y)h = (gh, \psi_n(g,y,h))$ where $g,h \in G_n$, $y \in Y$, and ψ_n is the projection onto Y of $(g,y)h$. (To insure the continuity of the action of each s in S on X defined below, the phase group G_n must retain its topology in the above.) Let e denote the identity of G_n for all n, let $X = \prod_{n=1}^{\infty} G_n$, and let $x_n^h \in X$ be defined by $x_n^h(n) = h$ and $x_n^h(i) = e$ if $i \neq n$. Extend the action of G_n on $G_n \times Y$ to $X \times Y$ by $(x,y)h = (xx_n^h, \psi_n(g_n,y,h))$ where g_n is the nth coordinate of x and $h \in G_n$.

Now define $(x,y)\theta_n^h = (x,y)g_1^{-1}g_2^{-1}\cdots g_{n-1}^{-1}hg_{n-1}\cdots g_1$ where $x = (g_1,g_2,\cdots)$ and $h \in G_n$. Note the projection of $(x,y)\theta_n^h$ onto X is xx_n^h.

If we let $G_n = Z_2$ and the action of $(G_n \times Y, G_n)$ is defined by $(0,y)0 = (0,y)$, $(1,y) = (1,y)$, $(0,y)1 = (1,y\alpha)$, and $(1,y)1 = (0,y\alpha^{-1})$, then we have the more specific construction mentioned above for a countable number of homeomorphisms with $\theta_n = \theta_n^1$.

We now show that $\theta_n^h\theta_m^f = \theta_m^f\theta_n^h$ where $f \in G_m$, $h \in G_n$, and $m \neq n$. Let $m > n$, then

$$\begin{aligned}
((x,y)\theta_n^h)\theta_m^f &= ((x,y)\theta_n^h)g_1^{-1}\cdots g_{n-1}^{-1}(g_nh)^{-1}\cdots g_{m-1}^{-1}fg_{m-1}\cdots(g_nh)g_{n-1}\cdots g_1\\
&= (x,y)g_1^{-1}\cdots g_{n-1}^{-1}hg_{n-1}\cdots g_1g_1^{-1}\cdots g_{n-1}^{-1}(g_nh)^{-1}\\
&\quad\cdots g_{m-1}^{-1}fg_{m-1}\cdots(g_nh)g_{n-1}\cdots g_1\\
&= (x,y)g_1^{-1}\cdots g_{n-1}^{-1}g_n^{-1}\cdots g_{m-1}^{-1}fg_{m-1}\cdots(g_nh)g_{n-1}\cdots g_1\\
&= (x,y)g_1^{-1}\cdots g_{m-1}^{-1}fg_{m-1}\cdots g_ng_{n-1}\cdots g_1g_1^{-1}\cdots g_{n-1}^{-1}hg_{n-1}\cdots g_1\\
&= (x,y)\theta_m^f\theta_n^h.
\end{aligned}$$

Now clearly θ_n^h is a homeomorphism of $X \times Y$, $\theta_n^h \theta_n^f = \theta_n^{hf}$, θ_n^e is the identity, and $(\theta_n^h)^{-1} = \theta_n^{h^{-1}}$. The group generated by the $\{\theta_n^h\}$ is group isomorphic to the direct sum $\Pi G_n = \{x \in X: x = (g_1, g_2, \cdots),\ g_n = e$ except for finitely many n's$\}$. Let $S = \Pi G_n$ be given the discrete topology, then $(X \times Y, S)$ is a transformation group, where $s \in S$ acts on (x, y) by $(x, y)s = (x, y)\theta_{n_1}^{g_1} \cdots \theta_{n_k}^{g_k}$ where $s = x_{n_1}^{g_1} \cdots x_{n_k}^{g_k}$.

Note the projection Π of $X \times Y$ onto X is an open homomorphism of $(X \times Y, S)$ onto (X, S); (X, S) is an equicontinuous minimal set; and if G_n is abelian for all n, then S is abelian. Also for fixed x and s, $F(x, s)$: $Y \to Y$ defined by the equality $(xs, yF(x, s)) = (x, y)s$ is a homeomorphism of Y and for $s, s' \in S$, $F(x, s)F(xs, s') = F(x, ss')$ as it must for $(X \times Y, S)$ to be a transformation group.

2. From 2.2 onward, we are considering minimal transformation groups only.

Let (Y, T) be a transformation group with T countable or finite. Let $\{t_n\}_{n \in N}$ be an enumeration of the elements of T with each t in T repeated an infinite number of times, i.e. $\{n: t_n = t\}$ is infinite for each t in T. Now let $G_n = Z_2 = \{0, 1\}$, $1 + 1 = 0$, and let the action of G_n on $G_n \times Y$ be $(0, y)0 = (0, y)$, $(1, y)0 = (1, y)$, $(0, y)1 = (1, yt_n)$, $(1, y)1 = (0, yt_n^{-1})$. Define $(X \times Y, S)$ as in §1 and note that the action is $(x, y)s = (xs, yF(x, s))$ and $F(x, s) \in T$. This is clear since if $x = (x(1), x(2), \cdots)$ and x_n^1 is the element of S defined by $x_n^1(i) = 0$ if $i \neq n$, $x_n^1(n) = 1$, then $(x, y)x_n^1 = (xx_n^1, yt_1^{-x(1)} \cdots t_{n-1}^{-x(n-1)} t_n^* t_{n-1}^{x(n-1)} \cdots t_1^{x(1)})$ where $t_n^* = t_n$ if $x(n) = 0$, $t_n^* = t_n^{-1}$ if $x(n) = 1$. (Clearly F is independent of Y.)

Now let (W, T) be a transformation group and form $(X \times W, S)$ in the same manner as $(X \times Y, S)$ was formed (i.e. take $\{t_n\}$ to be the same enumeration — take the same F). If $\varphi: (Y, T) \to (W, T)$ is a homomorphism, then it is easy to show that $\psi: (X \times Y, S) \to (X \times W, S)$ defined by $\psi(x, y) = (x, \varphi(Y))$ is a homomorphism. Note for $x \in X$, $(y, y') \in R(\varphi)$ iff $((x, y), (x, y')) \in R(\psi)$. We will show that there is a very close relationship between φ and ψ.

Suppose $x \in S$ and $x(n) = 0$ for $n > M$, then for $n > M$ the action of $x_n^1 \in S$ on (x, y) is $(x, y)x_n^1 = (xx_n^1, y\hat{t}^{-1}t_n\hat{t})$ where $\hat{t} = t_M^{x(M)} \cdots t_2^{x(2)} t_1^{x(1)}$. Suppose in addition that $t \in T$, and consider $\hat{t}t\hat{t}^{-1}$, since $\hat{t}t\hat{t}^{-1} = t_n$ for some increasing sequence of n's in N with $n > M$, it is clear that for $y \in Y$, $(x, yt) \in \mathrm{cls}[(x, y)S]$. Thus we have the following lemma.

LEMMA 2.1. *Let $x \in S$, $t \in T$, then for some net s_λ in S, and for all y in Y, $(x, yt) = \lim(x, y)s_\lambda$.*

Note from this it is clear that $(X \times Y, S)$ is minimal, if (Y, T) is. For the remainder of this section we are considering only minimal sets.

PROPOSITION 2.2.

(I) *For* x, $x' \in X$, *if* $((x, y), (x'y')) \in P_{(X \times Y, S)}$, *then* $x = x'$ *and* $(y, y') \in P_{(Y, T)}$.

(ii) *For* $x \in S$, *if* $(y, y') \in P_{(Y, T)}$, *then* $((x, y), (x, y')) \in P_{(X \times Y, S)}$.

Proof. If $((x, y), (x', y')) \in P_{(X \times Y, S)}$, then $(x, x') = \Pi \times \Pi((x, y), (x', y')) \in P_{(X, S)} = \Delta$ and thus $x = x'$. And since action by s in S has the effect on the Y-coordinate of an action by some t in T, we have that $(y, y') \in P_{(Y, T)}$.

The second statement for $x \in S$ follows from Lemma 2.1.

COROLLARY 2.3. *φ is a distal homomorphism iff ψ is.*

PROPOSITION 2.4. *φ is a proximal homomorphism iff ψ is.*

Proof. Suppose $R(\varphi) = P(\varphi)$. If $((x, y), (x', y')) \in R(\psi)$, then $x = x'$ and so we may take a net s_λ with $((x, y), (x', y'))s_\lambda \to ((e, \hat{y}), (e, \hat{y}')) \in R(\psi)$ for some $\hat{y}, \hat{y}' \in Y$. Now $(\hat{y}, \hat{y}') \in R(\varphi) = P(\varphi)$ and $e \in S$, so $((e, \hat{y}), (e\hat{y}')) \in P(\psi)$. This implies $((x, y), (x', y')) \in P(\psi)$, $(x = x')$, and thus $R(\psi) = P(\psi)$ and ψ is a proximal homomorphism.

Now suppose $R(\psi) = P(\psi)$ and $(y, y') \in R(\varphi)$. Then $((e, y), (e, y')) \in R(\psi) = P(\psi)$. Then by 2.2, $(y, y') \in P(\varphi)$.

PROPOSITION 2.5. *For each* $x \in X$, $(y, y') \in Q(\varphi)$ *iff* $((x, y), (x, y')) \in Q(\psi)$.

Proof. If $(y, y') \in Q(\varphi)$, then for any neighborhoods U of x, V of y, and V' of y' there exists $x' \in S \cap U$, $(\hat{y}, \hat{y}') \in V \times V' \cap R(\varphi)$, and $t \in T$ such that $\hat{y}t \in V$ and $\hat{y}'t \in V$. Now by 2.1 there exists $s \in S$ with $((x', \hat{y}), (x', \hat{y}'))s \in (U \times V) \times (U \times V)$ while $((x', \hat{y}), (x', \hat{y}')) \in (U \times V) \times (U \times V') \cap R(\psi)$, thus $((x, y), (x, y')) \in Q(\psi)$.

If $((x, y), (x, y')) \in Q(\psi)$, then since an action by an s in S has the effect on the Y-coordinate of an action by some t in T, it is clear that $(y, y') \in Q(\varphi)$.

PROPOSITION 2.6.

(i) *For x in X,* $(y, y') \in S_d(\varphi)$ *iff* $((x, y), (x, y')) \in S_d(\psi)$.

(ii) *For x in X,* $(y, y') \in S_e(\varphi)$ *iff* $((x, y), (x, y')) \in S_e(\psi)$.

Proof.

(i) $S_d(\varphi)$ is generated by letting $C_1(\varphi) = \operatorname{cls} P(\varphi)$, $E_1(\varphi) = E(C_1(\varphi))$, where $E(C)$ is the smallest equivalence relation containing C, $C_2(\varphi) = \operatorname{cls} E_1$, $E_2(\varphi) = E(C_2(\varphi))$, and in general for any ordinal λ, $C_\lambda(\varphi) = \operatorname{cls}(\cup\{E_\mu(\varphi) \colon \mu < \lambda\})$, $E_\lambda(\varphi) = E(C_\lambda(\varphi))$. This gives a chain of sets which ends when $C_\lambda(\varphi) = E_\lambda(\varphi) = S_d(\varphi)$.

First note that if $((x, y), (x', y')) \in S_d(\psi)$, then $x = x'$. By 2.2, for all x

in X, $(y, y') \in \operatorname{cls} P(\varphi)$ iff $((x, y), (x, y')) \in \operatorname{cls} P(\psi)$; that is for all x in X, $(y, y') \in C_1(\phi)$ iff $((x, y), (x, y')) \in C_1(\psi)$. Suppose that for $\mu < \lambda$ we have, for all x in X, $(y, y') \in C_\mu(\phi)$ iff $((x, y), (x, y')) \in C_\mu(\psi)$. It follows easily that for $\mu < \lambda$ and for all x in X, $(y, y') \in E_\mu(\phi)$ iff $((x, y), (x, y')) \in E_\mu(\psi)$ and then for all x in X, $(y, y') \in C_\lambda(\phi)$ iff $((x, y), (x, y')) \in C_\lambda(\psi)$. Thus (i) follows.

(ii) Is proved in a similar manner.

COROLLARY 2.7. *$S(\varphi) = R(\varphi)$ iff $S(\psi) = R(\psi)$.*

PROPOSITION 2.8. *For $x \in X$, (y, y') is an almost periodic point in $(Y \times Y, T)$ iff $((x, y), (x, y'))$ is an almost periodic point in $((X \times Y) \times (X \times Y), S)$.*

Proof. Suppose (y, y') is an almost periodic point in $R(\varphi)$. First we show that $((e, y), (e, y'))$ is an almost periodic point. If $((x, \hat{y}), (x, \hat{y}')) \in \operatorname{cls}(((e, y), (e, y'))S)$, let s_λ be a net in S with $((e, \bar{y}), (e, \bar{y}')) = \lim((x, \hat{y}), (x, \hat{y}'))s_\lambda$ for some $\bar{y}, \bar{y}'$ in Y, then $(\bar{y}, \bar{y}') \in \operatorname{cls}((y, y')T)$, since S acts on the Y-coordinate by elements of T. So $(y, y') \in \operatorname{cls}((\bar{y}, \bar{y}')T)$. Let U, V, V' be neighborhoods of e, y, y' respectively. Then by 2.1 for some $s \in S$, $((e, \bar{y}), (e, \bar{y}'))s \in (U \times V) \times (U \times V')$ and so for some λ, $((x, \hat{y}), (x, \hat{y}'))s_\lambda s \in (U \times V) \times (U \times V')$. So $((e, y), (e, y')) \in \operatorname{cls}(((x, \hat{y}), (x, \hat{y}'))S)$ and is therefore an almost periodic point.

Now for $d \in S \subseteq X$ and any $y^* \in Y$, $(e, y^*)d = (d, y^*t)$ where $t \in T$ depends on d. By 2.1, $((e, yt^{-1}), (e, y't^{-1})) \in \operatorname{cls}(((e, y), (e, y'))S)$ and so $((d, y), (d, y')) = ((e, y't^{-1}), (e, y't^{-1}))d \in \operatorname{cls}(((e, y), (e, y'))S)$. And since S is dense in X, $((x, y), (x, y')) \in \operatorname{cls}(((e, y), (e, y'))S)$ for $x \in X$ and so is an almost periodic point.

Now suppose $((x, y), (x, y'))$ is an almost periodic point in $R(\psi)$. We will first show $((e, y), (e, y')) \in \operatorname{cls}(((x, y), (x, y'))S)$ and so is an almost periodic point in $R(\psi)$.

For some $\hat{y}, \hat{y}'$ in Y, $((e, \hat{y}), (e, \hat{y}')) \in \operatorname{cls}(((x, y), (x, y'))S)$. Then $((x, y), (x, y')) \in \operatorname{cls}(((e, \hat{y}), (e, \hat{y}'))S)$. Now let U, V, V' be neighborhoods of e, y, y' respectively. Then for some $s \in S$, $((e, \hat{y}), (e, \hat{y}'))s \in (X \times V) \times (X \times V')$ and $((e, \hat{y}), (e, \hat{y}'))s = ((s, \hat{y}t^*), (s, \hat{y}'t^*))$ for some t^* in T. For some integer M, $x_n^1 \in U$ for $n > M$, so take $n > M$ with $t_n = t^*$, then $((e, \hat{y}), (e, \hat{y}'))x_n^1 \in (U \times V) \times (U \times V')$. So $((e, y), (e, y')) \in \operatorname{cls}(((x, y), (x, y'))S)$ and so is an almost periodic point.

Now suppose $((e, y), (e, y'))$ is an almost periodic point and $(\hat{y}, \hat{y}') \in \operatorname{cls}((y, y')T)$. Then by 2.1, $((e, \hat{y}), (e, \hat{y}')) \in \operatorname{cls}(((e, y), (e, y'))S)$. So $((e, y), (e, y')) \in \operatorname{cls}(((e, \hat{y}), (e, \hat{y}'))S)$ and therefore $(y, y') \in \operatorname{cls}((\hat{y}, \hat{y}')T)$. Thus (y, y') is an almost periodic point.

COROLLARY 2.9. *The almost periodic points in $(R(\varphi), T)$ are dense in $R(\varphi)$ iff the almost periodic points in $(R(\psi), S)$ are dense in $R(\psi)$.*

We note that the restriction in 2.2 (ii) that x be an element of S is necessary. For example let (Y, T) be a minimal set with T = integers and for some $(y, y') \in P_{(Y,T)}$ and for some index α on Y, the set of negative integers is contained in the set $\{t\colon (y, y')t \notin \alpha\}$ (see **[6]** for an example of such a space). Now define x by

$$x(n) = \begin{cases} 0 & \text{if} \quad t_n \leqq 0 \\ 1 & \text{if} \quad t_n > 0. \end{cases}$$

Then $((x, y), (x, y')) \notin P_{(X \times Y, S)}$ since the effect on the Y-coordinate is only that of negative integers. Note that in view of 2.4, the smallest invariant closed equivalence relation on (Y, T) containing (y, y') is not proximal (i.e. is not contained in $P_{(Y,T)}$).

Up to now we have looked at a homomorphism $\varphi\colon (Y, T) \to (W, T)$ and the corresponding homomorphism $\psi = id \times \varphi\colon (X \times Y, S) \to (X \times W, S)$. Now we will consider a minimal set (Z, S) such that the diagram

$$\begin{tikzcd} (X \times Y, S) \arrow[rr, "\psi"] \arrow[dr, "\alpha"'] & & (X \times W, S) \\ & (Z, S) \arrow[ur, "\beta"'] & \end{tikzcd}$$

commutes. Now $R(\varphi) = \{(y, y')\colon \varphi(y) = \varphi(y')\}$, $R(\psi) = \{((x, y), (x, y'))\colon x \in X, (y, y') \in R(\varphi)\}$, and $R(\alpha) \subseteq R(\psi)$. This situation is treated in the following proposition, note (W, T) is isomorphic to $(Y/R(\varphi), T)$ and $(X \times W, S)$ is isomorphic to $(X \times Y/R(\psi), S)$.

PROPOSITION 2.10. *Suppose R is a closed invariant equivalence relation contained in $R(\psi)$. Then the set $R' = \{(y, y')\colon ((e, y), (e, y')) \in R\}$ is a closed invariant equivalence relation contained in $R(\varphi)$ and the associated equivalence relation $R^* = \{((x, y), (x, y'))\colon ((e, y), (e, y')) \in R$ (i.e. $(y, y') \in R')\}$ equals $\{((x, y), (x, y'))\colon ((x', y), (x', y')) \in R$ for every x' in $X\}$ and the natural homomorphism of $X \times Y/R^* \to X \times Y/R$ is such that the fiber over $(e, y)/R$ is a singleton, for y in Y. Moreover for s in S, $R' = \{(y, y')\colon ((s, y), (s, y')) \in R\}$ and $R^* = \{((x, y), (x, y'))\colon ((s, y), (s, y')) \in R\}$.*

Proof. Fix s in S, we first show that

$$\{((x, y), (x, y'))\colon ((s, y), (s, y')) \in R\} \\ = \{((x, y), (x, y'))\colon ((x', y), (x', y')) \in R \text{ for every } x' \text{ in } X\}.$$

(Note both of these sets are closed.) Let $((s, y), (s, y')) \in R$. Then for $s' \in S$, $((s, y), (s, y'))s' = ((ss', yt), (ss', y't)) \in RS = R$ for some t in T and by 2.1

$$((ss', y), (ss', y')) = ((ss', ytt^{-1}), (s, s', ytt^{-1})) \in \operatorname{cls} R = R.$$

Since $S = sS$ is dense in X, $((x', y), (x', y')) \in R$ for x' in X. So they are equal. Also for fixed s in S, the set

$$\{(y, y'): ((s, y), (s, y')) \in R\} = \{(y, y'): ((x, y), (x, y')) \in R \text{ for every } x \text{ in } X\}.$$

To show that R' is invariant, let $t \in T$ and suppose $t_n = t^{-1}$. If $(y, y') \in R'$, then $((e, y), (e, y')) \in R$, so $((x_n^1, y), (x_n^1, y')) \in R$ and $((x_n^1, y), (x_n^1, y'))x_n^1 = ((e, yt), (e, y't)) \in R$; thus $(yt, y't) \in R'$ and R' is invariant. Clearly then R' is a closed invariant equivalence relation.

Clearly R^* is a closed invariant equivalence relation on $X \times Y$ and $R^* \subseteq R$. Now for s in S, $((s, y), (s, y')) \in R^*$ iff $((s, y), (s, y')) \in R$, so the homomorphism $X \times Y/R^* \to X \times Y/R$ has the fiber over $(s, y)/R$ equal to $\{(s, y)/R^*\}$.

COROLLARY 2.11. *If R is a distal* (closed invariant) *equivalence relation, then $R = R^*$.*

Now suppose $R_1 \subseteq R_2 \subseteq R(\psi)$ and consider the diagram

$$\begin{tikzcd}
X \times Y \arrow[r] \arrow[dr] & X \times Y/R_1 \arrow[r, "\delta"] & X \times Y/R_2 \\
& X \times Y/R_1^* \arrow[u, "\alpha"] \arrow[r, "\delta^*"] & X \times Y/R_2^* \arrow[u, "\beta"']
\end{tikzcd}$$

where $\alpha, \beta, \delta, \delta^*$ are the homomorphisms induced by the equivalence relations. Note that if ψ is a distal homomorphism, then R_1 and R_2 are distal equivalence relations and so $R_1 = R_1^*$, $R_2 = R_2^*$ and $\delta = \delta^*$. Also clearly δ is a proximal homomorphism iff δ^* is. Considering S as a subset of X and any closed invariant equivalence relation $R \subseteq R(\psi)$, we see that the restriction of the induced map to $(S \times Y/R^*, S) \to (S \times Y/R, S)$ is an isomorphism and for s in S the map from $\{s\} \times Y/R \to Y/R'$ defined by $(s, y)/R \to y/R'$ is a homeomorphism of topological spaces.

Now suppose δ^* is open and let δ' be the induced homomorphism from $(Y/R_1', T)$ onto $(Y/R_2', T)$, then δ' is open and so $id \times \delta': X \times (Y/R_1') \to X \times (Y/R_2')$ is open, where id is the identity on X. We may identify δ^* with $id \times \delta'$, since for $R \subseteq R(\psi)$ the map ψ_R of $X \times (Y/R')$ onto $(X \times Y)/R$ defined by $(x, y/R') \mapsto (x, y)/R$ is an isomorphism. Then

δ^* is open. This illustrates one approach to studying the topological properties of δ^* in comparison with δ. Note that for the purpose of studying the relationship between δ and δ^* we may assume that $(X \times Y, S)$ is isomorphic to $(X \times Y/R_1^*, S)$, (since ψ_R is an isomorphism). We now compare some of the dynamical properties of δ and δ^*.

PROPOSITION 2.12. *If δ is distal, then δ^* is distal.*

Proof. Note δ is distal iff $R_2 \cap P \subseteq R_1$.

Now suppose $((x, y), (x', y)) \in R_2^* \cap P$. Then by definition of R_2^* and by 2.2, $((e, y), (e, y')) \in R_2 \cap P \subseteq R_1$ and so $((x, y), (x, y')) \in R_1^*$. Then $R_2^* \cap P \subseteq R_1^*$ and δ^* is distal.

PROPOSITION 2.13. *If the almost periodic points in $(R(\delta), S)$ are dense, then the almost periodic points in $(R(\delta^*), S)$ are dense.*

Proof. Note $R(\delta \circ \alpha) = R_2$ and $R(\delta^*) = R_2^*$ because of our identification of $(X \times Y, S)$ with $(X \times Y/R_1^*, S)$. Let $((e, y), (e, y')) \in R(\delta \circ \alpha)$, so $(\alpha(e, y), \alpha(e, y')) \in R(\delta)$. Now $(\alpha \times \alpha)^{-1}(\alpha \times \alpha((e, y), (e, y'))) = \{((e, y), (e, y'))\}$ and since $\alpha \times \alpha$ is a closed map we have for any open neighborhood U of $((e, y), (e, y'))$ there exists a neighborhood V^* of $\alpha \times \alpha((e, y), (e, y'))$ with $V = (\alpha \times \alpha)^{-1} V^* \subseteq U$. Now there exists an almost periodic point in $V^* \cap R(\delta)$ and so there exists an almost periodic point, $((x, \bar{y}), (x, \bar{y}'))$, in $V \cap R(\delta \circ \alpha) \subset U \cap R(\delta \circ \alpha) = U \cap R_2$. As shown in the proof of 2.8, $((e, \bar{y}), (e, \bar{y}')) \in \mathrm{cls}(((x, \bar{y}), (x, \bar{y}'))S)$ and so is an element of R_2. Thus $((x, y), (x, y')) \in R_2^* = R(\delta^*)$ and the almost periodic points are dense.

In working with the results of this section one should keep in mind that even if (Y, T) and (W, T) are not isomorphic, $(X \times Y, S)$ and $(X \times W, S)$ may be isomorphic. For example suppose α is a homeomorphism of $Y = W$ and for $T =$ integers, (Y, T) is defined by $yn = \alpha^n(y)$ and (W, T) by $wn = \alpha^{-n}(w)$. Then the map $(x, y) \mapsto (xx^*, y)$ is an isomorphism of $(X \times Y, S)$ onto $(X \times W, S)$ where $x^* \in X$ defined by $x^*(n) = 1$ for all n. See **[3]** for an example in which (Y, T) and (W, T) are not isomorphic.

Now suppose $(X \times Y, S)$, $(X \times W, S)$ are minimal sets, (X and S as usual) the first projections π of $(X \times Y, S)$ onto (X, S) and of $(X \times W, S)$ onto (X, S) are homomorphisms, and $\alpha: (X \times Y, S) \to (X \times W, S)$ is a homomorphism. Then there is an isomorphism $\alpha^*: (X, S) \to (X, S)$ such that the diagram

$$\begin{array}{ccc} X \times Y & \xrightarrow{\alpha} & X \times W \\ \pi \downarrow & & \downarrow \pi \\ X & \xrightarrow{\alpha^*} & X \end{array}$$

commutes, and $\alpha^*(x)$ equals the product of $\alpha^*(e)$ and x, where e is the identity of X. We define α^* as $\alpha^*(x) = \pi \circ \alpha \circ \pi^{-1}(x)$. To show α^* is well defined consider (x, y) and (x, y') in $\pi^{-1}(x)$ and let s_λ be a net with $(x, y)s_\lambda \to (x, y')$. Then $x s_\lambda = \pi((x, y)s_\lambda) \to \pi(x, y') = x$ and since S is a subgroup of X acting by right multiplication, $x' s_\lambda \to x'$ for every x' in X. Then

$$\begin{aligned}\pi(\alpha(x, y)) &= \lim[(\pi \circ \alpha(x, y))s_\lambda] \\ &= \pi \circ \alpha(\lim(x, y)s_\lambda) = \pi \circ \alpha(x, y').\end{aligned}$$

So α^* is well-defined and is clearly a homomorphism. Now for $x \in X$, $x = \lim s_\mu$ for some net in S, so $\alpha^*(x) = \alpha^*(\lim s_\mu) = \lim \alpha^*(s_\mu) = \lim(\alpha^*(e))s_\mu = \alpha^*(e)\, x$, where e is the identity of X. That is the effect of α^* on x is simply to multiply x by $\alpha^*(e)$.

Note however this difficulty might be overcome by working with pointed transformation groups (y_0, Y, T), $y_0 \in Y$, and associating it with $((e, y_0), X \times Y, S)$; in this case α^* is the identity map on X and the map α': $Y \to W$ defined by $\alpha'(y) = \pi_2\alpha(e, y)$ where π_2 is the projection onto the second coordinate, is a homomorphism of (Y, T) onto (W, T) since

$$\begin{aligned}(e, \alpha'(yt)) &= \alpha(e, yt) = \alpha(\lim(e, y)x'_n) \\ &= \lim((\alpha(e, y))x'_n) = (e, (\pi_2 \circ \alpha(e, y))t) \\ &= (e, \alpha'(y)t),\end{aligned}$$

where x'_n is a sequence in S converging to e and $t_n = t$ for all x'_n.

We have showed the close relationship between the homomorphism $\varphi: (Y, T) \to (W, T)$ and the homomorphism $\psi: (X \times Y, S) \to (W \times W, T)$ for some basic properties of homomorphisms. Below we mention some other properties that might be investigated. First we note that if (Y, T) is a regular minimal set, then $\theta: (X \times Y, S) \to (X, S)$ is a regular homomorphism (see **[8]** and **[9]**). Also if φ is a regular homomorphism, so is ψ. One might also define a τ-topology on (Y, T) and on the fibers of θ and define a group, $g(w_0)$, for factors (W, T) of (Y, T) as $\{y \in Yu: \varphi(y) = w_0\}$, where u is an idempotent in the semigroup Y, such a group may correspond to a similarly defined group of $(X \times W, S)$ as a factor of $(X \times Y, S)$, **[2]**. One may also look at p-universal minimal set relative to a fixed minimal set and a property p or at questions of relative disjointness **[9]**.

Section 3.

EXAMPLE 3.1. Proposition 2.4 permits us to give an example to a well known question. Let W be a singleton and Y be the circle acted on

by two homeomorphisms, one an irrational rotation, α, so (Y, T) will be minimal, and the other, β, the map $y \to e^{2\pi i r^2}$, where $y = e^{2\pi i r}$, so (Y, T) will be proximal (i.e. $\varphi: (Y, T) \to (W, T)$ is proximal), T is the subgroup of the group of homeomorphisms of Y generated by the two homeomorphisms above. Then $\psi: (X \times Y, S) \to (X, S) = (X \times W, S)$ is proximal. So $(X \times Y, S)$ is proximally equicontinuous, S is abelian, but $(X \times Y, S)$ is not locally almost periodic since it contains no distal points **[5]**.

EXAMPLE 3.2. In this example (Y, T) is equicontinuous but $(X \times Y, S)$, as defined in §2, is not equicontinuous.

Let $Y = Z_3$, $T = Z$ and the action be defined by $(z, t) \to z + t \bmod 3$. Let the enumeration of

$$T = \{t_n\}_{n=1}^{\infty} \quad \text{be} \quad t_n = \begin{cases} 0 & \text{if} \quad n \equiv 0 \bmod 3 \\ 1 & \text{if} \quad n \equiv 1 \bmod 3 \\ 2 & \text{if} \quad n \equiv 2 \bmod 3 \end{cases}$$

With $(X \times Y, S)$ as in §2, note

$$(e, 0)x^1_{3n+1} \to (e, 1)$$

and

$$(e, 1)(x^1_{3n+1})^{-1} = (e, 1)x^1_{3n+1} \to (e, 2);$$

we see $(e, 0)$ and $(e, 1)x^1_{3n+1}$ are moved near $(e, 1)$ by $x^1_{3n+1} \in S$, and so $(e, 0)$ and $(e, 2)$ are regionally proximal.

Note that the fibers of the projection $\pi: (X \times Y, S) \to (X, S)$ are finite, so this example shows that a finite-to-one extension of an equicontinuous minimal set may be distal without being equicontinuous even when the phase group is abelian. For the details involved in this question and for previous examples see **[8]** and **[6]**. In **[11]** Wu has shown that if a minimal set (Y, T) is a finite-to-one extension of an equicontinuous minimal set with connected phase space, then (Y, T) is equicontinuous (he proves this under some additional restrictions which are unnecessary).

In light of the role that connectedness plays in the above, the question occurs whether a proximally equicontinuous minimal set with abelian phase group and connected phase space would have to be locally almost periodic. To answer this question and to illustrate further the ideas of §1, we provide the following example.

EXAMPLE 3.3. Assuming the notation of §1 let $G_n = Y =$ unit circle in the complex plane and let α, β be as in Example 3.1. Define an action of G_n on $G_n \times Y$ by $(g', y)g = (g'g, yt_n g t_n^{-1})$, where

$$t_n = \begin{cases} \alpha & \text{if} \quad n \quad \text{is even} \\ \beta & \text{if} \quad n \quad \text{is odd} \end{cases}$$

Form $(X \times Y, S)$ as in §1 and note that by taking $g \in G_n = Y$ such that $\alpha(y) = gy$ (multiplication) for every y, we see $\lim[(e, y)x_{2n}^g] = (e, \alpha(y))$ and so $(e, \alpha^n(y)) \in \mathrm{cls}((e, y)S)$ for every n; thus $(X \times Y, S)$ is minimal since the projection π of $(X \times Y, S)$ onto (X, S) is open. Now to show that π is a proximal homomorphism, we need only show that for y, y' in Y, (e, y) and (e, y') are proximal. Fix $y, y' \in Y$ with $r' > r$ where $y = e^{2\pi i r}$, $y' = e^{2\pi i r'}$; i.e. y' is on the arc going clockwise from y to 1.

Now β^{-1} fixes 1 and moves all other points counter-clockwise while β fixes 1 and moves all other points clockwise. Take g such that $g\beta(y) = 1$, then β moves y clockwise toward 1 with y' in between; g rotates $\beta(y)$ and 1 to $g\beta(y) = 1$ and g with $g\beta(y')$ on the counter-clockwise arc from g to 1; and β^{-1} moves g counter-clockwise toward $g\beta(y)$ with $g\beta(y')$ in between; in effect, $\beta^{-1}(g(\beta(y)))$ and $\beta^{-1}(g(\beta(y')))$ are closer than $\beta(y)$ and $\beta(y')$. Now $(e, y)x_{2n+1}^g \to (e, \beta^{-1}(g(\beta(y))))$ and $(e, y')x_{2n+1}^g \to (e, \beta^{-1}(g(\beta(y'))))$. Since the y and y' were fixed but arbitrary (i.e. g depends on y and y'), it is clear that (e, y) and (e, y') are proximal.

Added in proof. B. Weiss and H. Furstenberg now have an example of a minimal set with X metric and T the integers that is proximally equicontinuous but not locally almost periodic (personal communication).

REFERENCES

1. Joseph Auslander, B. Horelick, *Regular minimal sets, II, the proximally equicontinuous case*, Compositio Math., **22** (1970), 203–214.

2. R. Ellis, *Lectures on topological dynamics*, Math. Lecture Note Series, Benjamin, N.Y. 1969.

3. W. H. Gottschalk, *An irreversible minimal set, Ergodic Theory*, Proc. of an international symposium (New Orleans, 1961). Academic Press, New York, 1963, 121–134.

4. D. McMahon, *Relativized problems with abelian phase group in topological dynamics*, Proc. Nat Acad. of Sci., U.S.A. **73** No. 4, (1976).

5. ———, T. S. Wu, *On weak mixing and local almost periodicity*, Duke Math. J., **39** (1972), 333–343.

6. ———, *On the connectedness of homomorphisms in topological dynamics*, submitted.

7. K. Petersen, L. Shapiro, *Induced flows*, Trans. Amer. Math. Soc., **177** (1973), 375–390.

8. R. Sacker, G. Sell, *Finite extensions of minimal transformation groups*, Trans. Amer. Math. Soc., **190** (1974), 325–334.

9. P. Schoenfeld, *Regular homomorphisms of minimal sets*, Thesis, (University of Maryland, 1974).

10. T. S. Wu, *Notes on topological dynamics, I. Relative disjointness, relative regularity and homomorphisms*, Bull. Inst. Math. Acad. Sinica, **2** (1974), 343–356.

11. ———, *Notes on topological dynamics, II, Distal extensions with discrete fibers and prime flows*, June 1975 Bull. Inst. Math. Acad. Sinica.

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