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THE SOLUTION OF A STIELTJES-VOLTERRA INTEGRAL EQUATION FOR RINGS

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For a triple (h, K, g) of functions and an interval $[a, x]$, the author defines a subdivision-refinement-type limit $V(a, x; h, K, dg)$ of the set $\{V(D, h, K, \Delta g)\}$ of determinants, where each subdivision $D = \{x_i\}_0^n$ of $[a, x]$ defines an $n \times n$ determinant of the set and each determinant has the form

$$\begin{vmatrix} h_1 + h_0K_{10}\Delta g_1 & -1 & 0 & 0 \\ h_2 + h_0K_{20}\Delta g_1 & K_{21}\Delta g_2 & -1 & 0 \\ h_3 + h_0K_{30}\Delta g_1 & K_{31}\Delta g_2 & K_{32}\Delta g_3 & -1 \\ h_4 + h_0K_{40}\Delta g_1 & K_{41}\Delta g_2 & K_{42}\Delta g_3 & K_{43}\Delta g_4 \end{vmatrix}$$

The following theorem is proved. If f, g, h and K are functions to a ring and g has bounded variation on $[a, b]$, then $(f, K, g) \in OA^*$ and $f(x) = h(x) + (L)\int_a^x f(t)K(x, t)dg(t)$ on $[a, b]$ iff $(h, K, g) \in OM^*$ and $f(x) = V(a, x; h, K, dg)$ on $[a, b]$. The OA^* and OM^* sets are defined and sufficient conditions are proved for $(f, K, g) \in OA^*$ and $(h, K, g) \in OM^*$, and for the existence of the limit $V(a, x; h, K, dg)$, and for $V(a, x; h, K, dg) = h(x) - (L)\int_a^x h(t)dV(t, x; 1, K, dg)$.

Although the Volterra equation $f(x) = h(x) + \int_a^x f(t)K(x, t)dt$ has been studied in depth by many persons, it seems that only Hinton **[3]**, Reneke **[4]** **[5]** and Bitzer **[1]** **[2]** have published papers on the Volterra integral equation in which the integral is a subdivision-refinement-type Stieltjes integral. In this paper the solution of the Volterra equation and the development of the related properties do not depend on a Picard expansion or on the above quoted references. So far as the author has been able to determine, this subdivision-refinement definition of the solution $V(a, x; h, K, dg)$ of the Volterra equation has not been published previously.

Definitions and notations. The symbol R denotes the set of real numbers and N is a ring which has a multiplicative identity element 1 and a norm $|\cdot|$ with respect to which N is complete and $|1| = 1$; f, g and h are functions from R to N and K is a function from RXR to N. Also, $dg \in OB^0$ on $[a, b]$ means g has bounded variation on $[a, b]$. All integrals are of the subdivision-refinement-type limits; the approximating sum for $(L)\int E(t)dg(t)$ is $\Sigma E(t_{i-1})[g(t_i)-g(t_{i-1})]$ and for $(R)\int E(t)dg(t)$ is $\Sigma E(t_i)[g(t_i)-g(t_{i-1})]$. If no misunderstanding is likely, the symbols K_{ij}, f_i and Δg_i will be used for $K(x_i, x_j)$, $f(x_i)$ and $g(x_i)-g(x_{i-1})$, respectively.

If $\{a_{ij}\}_{i,j=1}^n$ is a sequence of elements of N and p and q are integers such that $1 \leqq p \leqq q \leqq n$, then the symbol $|a_{ij}|_p^q$ denotes the determinant

$$\begin{vmatrix} a_{pp} & \cdots & a_{pq} \\ \cdots & \cdots & \cdots \\ a_{qp} & \cdots & a_{qq} \end{vmatrix}$$

and is defined by the sum of the $(q-p+1)!$ products obtained as follows: (1) each term of the sum is a product, or the negative of a product, which contains one and only one element from each row and each column of $|a_{ij}|_p^q$; (2) the factors of each term are ordered so that the second subscripts appear in the order $p, p+1, \cdots, q$; and (3) the product or the negative of a product is used as a term according as the number of inversions of the first subscripts is even or odd. Note that the usual theorems pertaining to determinants will hold, except where multiplicative commutativity is needed in the proofs. Also, if $A = |a_{ij}|_1^n$, then $|A|$ denotes the norm of A and, if $1 \leqq p \leqq n$, A_p, $^*A_p$ and $^*A_{pk}$ denote the determinants defined as follows: $A_p = |a_{ij}|_1^p$, $^*A_p = |a_{ij}|_p^n$, $A_0 = 1$, $^*A_{n+1} = 1$, and if $1 \leqq k \leqq p$, then $^*A_{pk}$ is the determinant obtained by replacing the first column of $|a_{ij}|_p^n$ with the column $\{a_{ik}\}_{i=p}^n$ of elements of $\{a_{ij}\}_{i,j=1}^n$.

$A = |a_{ij}|_1^n$ is a Volterra determinant means $\{a_{ij}\}_{i,j=1}^n$ is a sequence such that $a_{ij} = -1$ for $j = i+1$ and $a_{ij} = 0$ for $j > i+1$. $A = |a_{ij}|_1^n$ is a delta determinant defined by the sequences $\{c_{ij}\}_{i,j=1}^n$ and $\{d_j\}_{j=0}^n$ means A is a Volterra determinant and $a_{ij} = c_{ij}(d_j - d_{j-1})$ for $1 \leqq j \leqq i \leqq n$.

If $D = \{x_i\}_0^n$ is a subdivision of a number interval $[a, b]$, then $V(D, h, K, \Delta g)$ denotes the $n \times n$ Volterra determinant $|a_{ij}|_1^n$ such that $a_{i1} = h(x_i) + h(x_0)K(x_i, x_0)[g(x_1) - g(x_0)]$ for $i = 1, 2, \cdots, n$ and $a_{ij} = K(x_i, x_{j-1})[g(x_j) - g(x_{j-1})]$ for $1 < j \leqq i \leqq n$. If no misunderstanding is likely, $V(D)$ will be used to denote $V(D, h, K, \Delta g)$.

The limit $V(a, b; h, K, dg)$ exists means there is an element J of N such that if $\epsilon > 0$ then there is a subdivision D of $[a, b]$ such that if D' is a refinement of D then $|J - V(D', h, K, \Delta g)| < \epsilon$. The symbol $V(a, b; h, K, dg)$ will be used to denote this limit J.

If $m > 1$, the number M is an m-bound for $V(\ ,\ ,\ , \Delta g)$ on $[a, b]$ means $M \geqq m$ and, if $|h| < m$ on $[a, b]$ and $|K| < m$ on $[a, b] \times [a, b]$ and D is a subdivision of a subinterval of $[a, b]$ and $A = |a_{ij}|_1^n = V(D, h, K, \Delta g)$, then $|A| < M$ and each of $|A_p|, |^*A_p|$ and $|^*A_{pj}|$ is less than M for $1 \leqq p \leqq n$ and $1 \leqq j \leqq p$.

The triple $(f, K, g) \in OA^*$ on $[a, b]$ means that $(L)\int_a^x f(t)K(x, t)dg(t)$ exists for $x \in [a, b]$ and if $\epsilon > 0$ then there is a subdivision D of $[a, b]$ such that, if $\{t_i\}_0^n$ is a refinement of D and $0 < p \leqq n$ and $x = t_p$, then

$$\left|(L)\int_a^x f(t)K(x, t)dg(t) - \Sigma_1^p f(t_{i-1})K(x, t_{i-1})[g(t_i) - g(t_{i-1})]\right| < \epsilon.$$

The triple $(h, K, g) \in OM^*$ means $V(a, x; h, K, dg)$ exists for $x \in [a, b]$ and if $\epsilon > 0$ then there is a subdivision D of $[a, b]$ such that, if $\{x_i\}_0^n$ is a refinement of D and $0 < p \leqq n$ and $H = \{x_i\}_0^p$, then

$$|V(a, x_p; h, K, dg) - V(H, h, K, \Delta g)\Big| < \epsilon.$$

The triple $(1, K, g) \in OM^{**}$ on $[a, b]$ means $V(x, b; 1, K, dg)$ exists for $x \in [a, b]$ and if $\epsilon > 0$ then there is a subdivision D of $[a, b]$ such that, if $\{x_i\}_0^n$ is a refinement of D and $0 \leqq p < n$ and $H = \{x_i\}_p^n$, then

$$|V(x_p, b; 1, K, dg) - V(H, 1, K, \Delta g)| < \epsilon,$$

where 1 denotes the identity function.

In the following three definitions, $G(x, y) = \int_x^y |dg|$.

$\int_a^b \int_a^b |dK||dg||dg| = 0$ means if $\epsilon > 0$ then there is a subdivision D of $[a, b]$ such that, if $\{x_i\}_0^n$ is a refinement of D, then

$$\Sigma_{i=1}^n \Sigma_{j=1}^n M_{ij} G(x_{i-1}, x_i) G(x_{j-1}, x_j) < \epsilon,$$

where, for each i and j, M_{ij} is the lub of $|K(x_{i-1}, x_{j-1}) - K(x, y)|$ for $x_{i-1} \leqq x < x_i$ and $x_{j-1} \leqq y < x_j$.

If $a \leqq p \leqq b$, $\int_a^b |dK(p, x)||dg(x)| = 0$ means if $\epsilon > 0$ then there is a subdivision D of $[a, b]$ such that, if $\{x_i\}_0^n$ is a refinement of D, then $\Sigma_1^n M_i G(x_{i-1}, x_i) < \epsilon$, where, for each i, M_i is the lub of $|K(p, x_{i-1}) - K(p, x)|$ for $x_{i-1} \leqq x < x_i$.

$\int_a^b |dK(\ , x)||dg(x)| = 0$ uniformly on $[a, b]$ means if $\epsilon > 0$ then there is a subdivision D of $[a, b]$ such that, if $\{x_i\}_0^n$ is a refinement of D and $a \leqq p \leqq b$, then $\Sigma_1^n M_i G(x_{i-1}, x_i) < \epsilon$, where, for each i, M_i is the lub of $|K(p, x_{i-1}) - K(p, x)|$ for $x_{i-1} \leqq x < x_i$.

The set S of functions is bounded uniformly on $[a, b]$ means there is a number M such that, if $f \in S$ and $x \in [a, b]$, then $|f(x)| < M$. The set S

of functions is quasicontinuous uniformly on $[a, b]$ means S is bounded uniformly on $[a, b]$ and if $\epsilon > 0$ then there is a subdivision $D = \{x_i\}_0^n$ of $[a, b]$ such that, if $f \in S$ and $0 < i \leqq n$ and $x_{i-1} < r < t < x_i$, then $|f(r) - f(t)| < \epsilon$.

THEOREMS. In Theorems 1–5 we develop properties of the Volterra determinant. Theorem 6 gives the solution to the Stieltjes-Volterra integral equation.

THEOREM 1. *If* $A = |a_{ij}|_1^n$ *is a Volterra determinant and* $0 < p \leqq n$, *then*

(a) $A = a_{11}{}^*A_2 + {}^*A_{21}$;

(b) $A = \Sigma_{i=1}^n a_{i1}{}^*A_{i+1} = \Sigma_{j=1}^n A_{j-1}a_{nj}$;

(c) *if* $0 < j \leqq p$, *then* ${}^*A_{pj} = a_{pj}{}^*A_{p+1} + {}^*A_{p+1,j} = \Sigma_{i=p}^n a_{ij}{}^*A_{i+1}$;

(d) $A = \Sigma_{j=1}^p A_{j-1}{}^*A_{pj} = \Sigma_{j=1}^p A_{j-1}(\Sigma_{i=p}^n a_{ij}{}^*A_{i+1})$;

(e) *if* $p < n$ *and* $B = |b_{ij}|_1^n$ *and* $b_{p,p+1} = 0$ *and* $b_{ij} = a_{ij}$ *otherwise, then* $B = \Sigma_{j=1}^p A_{j-1}a_{pj}{}^*A_{p+1}$; *and*

(f) *if* $a_{ij} = 0$ *whenever* $1 < j \leqq p$ *and* $j \leqq i \leqq n$, *then* $A = {}^*A_{p1}$.

Each item in Theorem 1 can be proved using the definition of a determinant or by mathematical induction. Note that $A_0 = 1$ and ${}^*A_{n+1} = 1$.

THEOREM 2. *If A is a delta determinant defined by the sequences* $\{c_{ij}\}_1^n$ *and* $\{g_i\}_0^n$ *and* $|c_{ij}| \leqq m$ *for* $i, j = 1, 2, \cdots, n$, *then*

$$|A| \leqq m|g_1 - g_0| \Pi_2^n(1 + m|g_i - g_{i-1}|).$$

Proof. (by induction) If A is a 2×2 delta determinant defined by $\{c_{ij}\}_1^2$ and $\{g_i\}_0^2$, then

$$\begin{aligned}|A| &= |c_{11}\Delta g_1 c_{22}\Delta g_2 + c_{21}\Delta g_1| \\ &\leqq |\Delta g_1|(|c_{11}||c_{22}||\Delta g_2| + |c_{21}|) \leqq m|\Delta g_1|(1 + m|\Delta g_2|).\end{aligned}$$

Suppose that the theorem is true for $n = p$ and A is a $(p+1) \times (p+1)$ delta determinant defined by $\{c_{ij}\}_1^{p+1}$ and $\{g_i\}_0^{p+1}$; then $A = c_{11}\Delta g_1{}^*A_2 + {}^*A_{21}$ (Th. 1a). Since ${}^*A_2$ and ${}^*A_{21}$ are $p \times p$ delta determinants which have Δg_2 and Δg_1 as factors of each element of the first columns, respectively, then

$$\begin{aligned}|A| &\leqq |c_{11}||\Delta g_1||{}^*A_2| + |{}^*A_{21}| \\ &\leqq |c_{11}||\Delta g_1|[m|\Delta g_2|\Pi_3^{p+1}(1 + m|\Delta g_i|)] \\ &\quad + m|\Delta g_1|\Pi_3^{p+1}(1 + m|\Delta g_i|) \\ &\leqq m|\Delta g_1|\Pi_2^{p+1}(1 + m|\Delta g_i|).\end{aligned}$$

THEOREM 3. *If $m > 1$ and $dg \in OB^0$ on $[a, b]$, then there is a number M such that M is an m-bound for $V(\ ,\ ,\ ,\Delta g)$ on $[a, b]$.*

Proof. Suppose that $m > 1$ and k is a bound for $|g|$ on $[a, b]$. Let $M = P + Q$, where $P = m^2(1+2k)$ and $Q = Pm\int_a^b |dg| \exp \int_a^b m\,|dg|$. Let $D = \{x_i\}_{i=0}^n$ be a subdivision of a subinterval of $[a, b]$ and let K and h be functions such that m bounds $|K|$ on $[a, b] \times [a, b]$ and $|h|$ on $[a, b]$. Let $A = |a_{ij}|_1^n = V(D, h, K, \Delta g)$; then ${}^*A_{i+1}$ is a delta determinant for $i = 1, 2, \cdots, n-1$. Hence, for $i = 1, 2, \cdots, n$ and $1 < j \leqq i$, $P = m^2 + 2m^2k > |K_{i,j-1}||\Delta g_j| \geqq |a_{ij}|$ and $P = m^2 + 2m^2k > |h_i| + |h_0||K_{i0}||\Delta g_1| \geqq |a_{i1}|$; hence, if $0 < i \leqq n$ and $0 < j \leqq n$, then $|a_{ij}| < P$. Therefore,

$$
\begin{aligned}
|A| &\leqq \Sigma_{i=1}^{n-1} |a_{i1}||{}^*A_{i+1}| + |a_{n1}| && \text{(Th. 1b)}\\
&< \Sigma_1^{n-1} P|{}^*A_{i+1}| + P\\
&\leqq \Sigma_1^{n-1} Pm\,|\Delta g_{i+1}|\, \Pi_{j=i+2}^n (1 + m\,|\Delta g_j|) + P && \text{(Th. 2)}\\
&\leqq Pm\int_a^b |dg| \exp \int_a^b m\,|dg| + P = Q + P = M.
\end{aligned}
$$

Similarly, $|A_p| < M$, $|{}^*A_p| < M$ and $|{}^*A_{pj}| < M$ for $1 \leqq p \leqq n$ and $j = 1, 2, \cdots, p$.

THEOREM 4. *If $\{f_i\}_1^n$ and $\{a_{ij}\}_1^n$ are sequences of elements of N and $A = |a_{ij}|_1^n$ is an $n \times n$ Volterra determinant, then the following statements are equivalent.*

(1) *$f_1 = a_{11}$ and $f_i = a_{i1} + \Sigma_{j=2}^i f_{j-1} a_{ij}$ for $1 < i \leqq n$; and*

(2) *$f_i = A_i$ for $0 < i \leqq n$.*

Proof. If $0 < i \leqq n$, it follows from Theorem 1b that $A_i = a_{i1} + \Sigma_{j=2}^i A_{j-1} a_{ij}$; therefore, $1 \rightarrow 2$ by induction and $2 \rightarrow 1$ by induction.

THEOREM 5. *If g is a function and m is a number such that $m > 1$ and $dg \in OB^0$ on $[a, b]$, then there is a number Q such that, if m bounds the functions H and K on $[a, b] \times [a, b]$ and h and k on $[a, b]$ and $D = \{x_i\}_0^n$ is a subdivision of a subinterval of $[a, b]$, then*

$$|A - B| \leqq Q\Sigma_{p=1}^n \Sigma_{j=1}^p |a_{pj} - b_{pj}||g(x_{p+1}) - g(x_p)|,$$

where $A = |a_{ij}|_1^n = V(D, h, H, \Delta g)$, $B = |b_{ij}|_1^n = V(D, k, K, \Delta g)$ and $|\Delta g_{n+1}| = 1$.

Proof. Let g be a function and m be a number such that $dg \in OB^0$ on $[a, b]$ and $m > 1$. It follows from Theorem 3 that there is a number M

which is an m-bound for $V(\ ,\ ,\ ,\Delta g)$ on $[a,b]$. Let $Q = Mm \exp\left(m \int_a^b |dg|\right)$. Let H, K, h and k be functions which are bounded by m on $[a,b]$.

First we will consider a special case. Suppose that $D = \{x_i\}_0^n$ is a subdivision of a subinterval of $[a,b]$, $A = |a_{ij}|_1^n = V(D,h,H,\Delta g)$, $B = |b_{ij}|_1^n = V(D,k,K,\Delta g)$, $1 \leqq p \leqq n$, and $a_{ij} = b_{ij}$ for $i \neq p$; then $A - B = |a_{ij}|_1^n - |b_{ij}|_1^n$ is an $n \times n$ determinant $C = |c_{ij}|_1^n$ such that $c_{ij} = a_{ij}$ for $i \neq p$, $c_{pj} = a_{pj} - b_{pj}$ for $1 \leqq j \leqq p$ and $c_{p,p+1} = 0$ for $p < n$.

If $p = n$, then

$$\begin{aligned} |A - B| = |C| &= |\Sigma_{j=1}^n C_{j-1} c_{nj}| & \text{(Th. 1b)} \\ &\leqq \Sigma_{j=1}^n M |a_{nj} - b_{nj}| \leqq Q \Sigma_{j=1}^n |a_{nj} - b_{nj}| |\Delta g_{n+1}|. \end{aligned}$$

If $1 \leqq p < n$, then

$$\begin{aligned} |A - B| = |C| &= |\Sigma_{j=1}^p C_{j-1} c_{pj} {}^*C_{p+1}| & \text{(Th. 1e)} \\ &\leqq \Sigma_{j=1}^p M |a_{pj} - b_{pj}| |{}^*C_{p+1}| \\ &\leqq \Sigma_{j=1}^p M |a_{pj} - b_{pj}| m |\Delta g_{p+1}| \exp \int_a^b m |dg| & \text{(Th. 2)} \\ &\leqq Q \Sigma_{j=1}^p |a_{pj} - b_{pj}| |\Delta g_{p+1}|. \end{aligned}$$

We will now prove the general case. Suppose that $D = \{x_i\}_0^n$ is a subdivision of $[a,b]$ and that A and B are the determinants $A = |a_{ij}|_1^n = V(D,h,H,\Delta g)$ and $B = |b_{ij}|_1^n = V(D,k,K,\Delta g)$. There exists a sequence $\{R_p\}_0^n$ of $n \times n$ determinants such that $A = R_0$, $B = R_n$, and $A - B = \Sigma_{p=1}^n (R_{p-1} - R_p)$ and such that, if $a < p \leqq n$ and $R_{p-1} = |u_{ij}|_1^n$ and $R_p = |v_{ij}|_1^n$, then $u_{ij} = v_{ij}$ for $i \neq p$. For each integer p, $0 < p \leqq n$, $R_{p-1} - R_p$ is the difference of two determinants as defined in the special case above; therefore,

$$\begin{aligned} |A - B| &= |\Sigma_{p=1}^n (R_{p-1} - R_p)| \leqq \Sigma_{p=1}^n |R_{p-1} - R_p| \\ &\leqq \Sigma_{p=1}^n Q \Sigma_{j=1}^p |a_{pj} - b_{pj}| |\Delta g_{p+1}|. \end{aligned}$$

THEOREM 6. *Given. K is a bounded function from $R \times R$ to N and f, h and g are functions from R to N and $dg \in OB^0$ on $[a,b]$. Conclusion. The following statements are equivalent:*

(1) *$(f,K,g) \in OA^*$ on $[a,b]$ and, if $x \in [a,b]$, then*

$$f(x) = h(x) + (L) \int_a^x f(t) K(x,t) dg(t);$$

(2) $(h, K, g) \in OM^*$ *on* $[a, b]$ *and, if* $x \in [a, b]$, *then* $f(x) = V(a, x; h, K, dg)$.

Proof of $1 \to 2$. Suppose that m is a bound for K and $\epsilon > 0$. Since $(f, K, g) \in OA^*$ on $[a, b]$, there exists a subdivision H of $[a, b]$ such that, if $H' = \{x_i\}_0^n$ is a refinement of H and $0 < i \leqq n$ and $x = x_i$, then

$$\left| (L) \int_a^x f(t)K(x, t)dg(t) - \Sigma_{j=1}^i f(x_{j-1})K(x, x_{j-1})\Delta g_j \right| < \epsilon/M,$$

where $M = 4\left[m \int_a^b |dg| \exp \int_a^b m\,|dg| + 1 \right]$. Let $x \in (a, b]$ and let $H' = \{x_i\}_0^n$ be any refinement of H such that $x \in H'$; let $x = x_p$ and $D = \{x_i\}_0^p$, where $0 < p \leqq n$. For each integer i such that $0 < i \leqq p$ there exists an element $\epsilon_i \in N$ such that

$$\begin{aligned} f(x_i) &= h(x_i) + (L) \int_a^{x_i} f(t)K(x_i, t)dg \\ &= h(x_i) + \Sigma_{j=1}^i f(x_{j-1})K(x_i, x_{j-1})[g(x_j) - g(x_{j-1})] + \epsilon_i \\ &= (h_i + \epsilon_i + f_0 K_{i,0}\Delta g_1) + \Sigma_{j=2}^i f_{j-1}K_{i,j-1}\Delta g_j, \end{aligned}$$

where $\Sigma_{j=2}^1(\ ) = 0$. Let ϵ be a function such that $\epsilon(a) = 0$ and $\epsilon(x_i) = \epsilon_i$, for $i = 1, 2, \cdots, p$. Let $V(D, h + \epsilon, K, \Delta g) = |v_{ij}|_1^p$; then $|v_{ij}|_1^p$ is a Volterra determinant such that $v_{i1} = h_i + \epsilon_i + h_0 K_{i,0}\Delta g_1$ for $i = 1, 2, \cdots, p$, and $v_{ij} = K_{i,j-1}\Delta g_j$ for $1 < j \leqq i \leqq p$. Hence, $f_1 = v_{11}$ and $f_i = v_{i1} + \Sigma_{j=2}^i f_{j-1}v_{ij}$ for $1 < i \leqq p$. Therefore,

$$\begin{aligned} f(x) = f(x_p) &= V(D, h + \epsilon, K, \Delta g) && \text{(Th. 4)} \\ &= V(D, h, K, \Delta g) + V(D, \epsilon, K, \Delta g). \end{aligned}$$

Let $A = |a_{ij}|_1^p = V(D, \epsilon, K, \Delta g)$, then ${}^*A_{i+1}$ is a delta determinant for $i = 1, 2, \cdots, p-1$, and

$$\begin{aligned} |A| &= |\Sigma_1^p \epsilon_i {}^*A_{i+1}| && \text{(Th. 1b)} \\ &\leqq \Sigma_1^{p-1} |\epsilon_i|\, m\, |\Delta g_{i+1}| \exp \int_a^b m\,|dg| + |\epsilon_p| < \epsilon, && \text{(Th. 2).} \end{aligned}$$

Therefore, $|f(x) - V(D, h, K, \Delta g)| = |A| < \epsilon$. Since x is an arbitrary element of $(a, b]$ and H' is an arbitrary refinement of H containing x, it follows that $V(a, x; h, K, dg) = f(x)$ for $x \in [a, b]$ and that $(h, K, g) \in OM^*$ on $[a, b]$.

Proof of $2 \to 1$. Suppose that $\epsilon > 0$. Since $(h, K, g) \in OM^*$ on

$[a, b]$, there exists a subdivision H of $[a, b]$ such that if $H' = \{x_i\}_0^n$ is a refinement of H and $0 < i \leqq n$, then

$$|f(x_i) - V(D_i, h, K, \Delta g)| < \epsilon/2\left(1 + m\int_a^b |dg|\right),$$

where m is a bound for K and $D_i = \{x_j\}_0^i$. Let $x \in (a, b]$ and let $H' = \{x_i\}_0^n$ be a refinement of H such that $x \in H'$. Let $x = x_p$ and $D = \{x_i\}_0^p$, where $0 < p \leqq n$. Then there is a sequence $\{\epsilon_i\}_1^p$ such that $f(x_i) - \epsilon_i = V(D_i, h, K, \Delta g)$ for $0 < i \leqq p$, where $D_i = \{x_j\}_0^i$. Let $A = |a_{ij}|_1^p = V(D, h, K, \Delta g)$; then $A_j = V(D_j, h, K, \Delta g)$ for $j = 1, 2, \cdots, p$, and

$$\begin{aligned} f(x) = f(x_p) &= V(a, x; h, K, dg) = V(D, h, K, \Delta g) + \epsilon_p \\ &= a_{p1} + \Sigma_{j=2}^p A_{j-1}a_{pj} + \epsilon_p \qquad \text{(Th. 1b)} \\ &= (h_p + h_0 K_{p,0}\Delta g_1) + \Sigma_{j=2}^p V(D_{j-1}, h, K, \Delta g)K_{p,j-1}\Delta g_j + \epsilon_p \\ &= (h_p + h_0 K_{p,0}\Delta g_1) + \Sigma_{j=2}^p (f_{j-1} - \epsilon_{j-1})K_{p,j-1}\Delta g_j + \epsilon_p. \end{aligned}$$

Since $h_p = h(x)$ and $h_0 = f_0 = f(a)$, then

$$\begin{aligned} |f(x) - h(x) - \Sigma_1^p f_{j-1}K_{n,j-1}\Delta g_j| &\leqq |\epsilon_p| + \Sigma_2^p|\epsilon_{j-1}K_{n,j-1}\Delta g_j| \\ &\leqq |\epsilon_p| + \left[\epsilon/2\Big/\left(1 + m\int_a^b |dg|\right)\right] < \epsilon. \end{aligned}$$

Since x is an arbitrary element of $(a, b]$ and H' is an arbitrary refinement of H containing x, then $f(x) - h(x) = (L)\int_a^x f(t)K(x, t)dg(t)$ for $x \in [a, b]$ and $(f, K, g) \in OA^*$ on $[a, b]$.

In the next three theorems, we prove a set of sufficient conditions for a function triple (h, K, g) to belong to each of OA^*, OM^* and OM^{**} and show that, with appropriate restrictions,

$$V(a, b; h, K, dg) = h(b) - (L)\int_a^b h(t)dV(t, b; 1, K, dg).$$

The following lemma is used in the proofs of these theorems.

LEMMA. *Given. f is a function from R to N and if $\epsilon > 0$ then there is a subdivision $D = \{x_i\}_0^n$ of $[a, b]$ such that, if $0 < i \leqq n$ and $x_{i-1} < x < y < x_i$, then $|f(x) - f(y)| < \epsilon$. Conclusion. The function f is quasicontinuous on $[a, b]$.*

THEOREM 7. *Given.* (1) *The functions f and K are bounded and*

$dg \in OB^0$ on $[a, b]$ and $F(x) = (L)\int_a^x f(t)K(x, t)dg(t)$ exists for $a \leqq x \leqq b$; and (2) if $\epsilon > 0$ then there is a subdivision $D = \{x_i\}_0^n$ of $[a, b]$ such that, if $0 < i \leqq n$ and $x_{i-1} < x < y < x_i$ and $\{t_i\}_0^m$ is a refinement of D such that $t_s \in \{t_i\}_0^m$ and $y = t_s$, then

$$|\Sigma_1^s f(t_{i-1})[K(x, t_{i-1}) - K(y, t_{i-1})][g(t_i) - g(t_{i-1})]| < \epsilon.$$

Conclusion. (1) The function F is quasicontinuous on $[a, b]$; and (2) $(f, K, g) \in OA^*$ *on* $[a, b]$.

Proof of Conclusion 1. Suppose that $\epsilon > 0$ and M is a bound for $|f||K|$. There is a subdivision $D = \{x_i\}_0^n$ of $[a, b]$ such that, if $0 < p \leqq n$ and x and $y \in (x_{p-1}, x_p)$, then $\int_x^y |dg| < \epsilon/4M$ and, if $\{t_i\}_0^m$ is any refinement of D and $y = t_s$, then

$$|\Sigma_1^s f(t_{i-1})[K(x, t_{i-1}) - K(y, t_{i-1})][g(t_i) - g(t_{i-1})]| < \epsilon/4.$$

Let p be an integer and x and y be numbers such that $0 < p \leqq n$ and x and $y \in (x_{p-1}, x_p)$. There is a refinement $D' = \{t_i\}_0^m$ of D and integers r and s such that $x = t_r$, $y = t_s$ and such that $|A| < \epsilon/4$ and $|B| < \epsilon/4$, where

$$A = F(x) - \Sigma_1^r f(t_{i-1})K(x, t_{i-1})[g(t_i) - g(t_{i-1})], \quad \text{and}$$
$$B = F(y) - \Sigma_1^s f(t_{i-1})K(y, t_{i-1})[g(t_i) - g(t_{i-1})].$$

Hence,

$$|F(y) - F(x)| \leqq |A| + |B| + |C| + |E| < \epsilon,$$

where

$$C = \Sigma_1^r f(t_{i-1})[K(y, t_{i-1}) - K(x, t_{i-1})][g(t_i) - g(t_{i-1})], \quad \text{and}$$
$$E = \Sigma_{r+1}^s f(t_{i-1})K(y, t_{i-1})[g(t_i) - g(t_{i-1})],$$

and $|C| < \epsilon/4$ and $|E| < \epsilon/4$. Therefore, F is quasicontinuous on $[a, b]$.

Proof of Conclusion 2. Let $\epsilon > 0$ and M be a bound for $|f||K|$. Since F is quasicontinuous on $[a, b]$, then there is a subdivision $H_1 = \{z_i\}_0^m$ of $[a, b]$ which is a refinement of the subdivision D defined above and such that, if $0 < p \leqq m$ and $z_{p-1} < x < y < z_p$, then $|F(x) - F(y)| < \epsilon/4$. Let H_2 be an interpolating sequence for H_1 and let H be a refinement of $H_1 \cup H_2$ such that, if $H' = \{y_i\}_0^n$ is a refinement of H and $y_q \in H_1 \cup H_2$, then

$$|F(y_q) - \Sigma_1^q f(y_{i-1})K(y_q, y_{i-1})[g(y_i) - g(y_{i-1})]| < \epsilon/4.$$

We now show that this subdivision H satisfies the definition for OA^*. Let $H' = \{t_i\}_0^p$ be a refinement of H and let $x = t_r \in H'$. If $x \in H_1 \cup H_2$, then the OA^* inequality $|\cdot| < \epsilon$ is satisfied. Suppose that $x \notin H_1 \cup H_2$; then there exist $y = t_s \in H_2$ and $z_{j-1}, z_j \in H_1$ such that x and $y \in (z_{j-1}, z_j)$. For convenience we will assume that $x < y$. Hence,

$$|F(x) - \Sigma_1^r f(t_{i-1})K(x, t_{i-1})[g(t_i) - g(t_{i-1})]| \leqq |A| + |B| + |C| + |E| < \epsilon,$$

where $A = F(x) - F(y)$ and B, C and E are defined as in Conclusion 1 of this proof. If $x > y$, the steps would be similar. Therefore, $(f, K, g) \in OA^*$ on $[a, b]$.

THEOREM 8. *Given. The function K is bounded on $[a, b] \times [a, b]$ and on $[a, b]$ the functions h and g have bounded variation, the set*

$$\left\{F_q \mid q \in [a, b],\ F_q = \int_a^x |dK(\ , q)|\right\}$$

of functions is quasicontinuous uniformly and $F(x) = V(a, x; h, K, dg)$ exists. Conclusion. (1) *F is quasicontinuous on $[a, b]$; and* (2) *$(h, K, g) \in OM^*$ on $[a, b]$.*

Proof of Conclusion 1. Suppose that $0 < \epsilon < 1$ and $m > 1$ is a bound for h and K; then there is a number M which is an m-bound for $V(\ ,\ ,\ , \Delta g)$ on $[a, b]$ and a number $Q > 1$ which has the properties stated in Theorem 5. There is a subdivision $D = \{x_i\}_0^n$ of $[a, b]$ such that, if $0 < i \leqq n$ and $x_{i-1} < x < y < x_i$, then

$$\int_x^y |dK(t, q)| < \epsilon/8mM\left(\int_a^b |dg| + 1\right) \quad \text{for} \quad q \in [a, b],$$

$$\int_x^y |dg| < \epsilon/8QM, \quad \text{and} \quad \int_x^y |dh| < \epsilon/8M.$$

Suppose that $0 < i \leqq n$ and that $x_{i-1} < x < y < x_i$; then there is a refinement $\{z_i\}_0^r$ of D such that $x = z_p$ and $y = z_q$ and such that $|F(x) - V(P, h, K, \Delta g)| < \epsilon/8$ and $|F(y) - V(R, h, K, \Delta g)| < \epsilon/8$, where $P = \{z_i\}_0^p$ and $R = \{z_i\}_0^q$. Let $A = |a_{ij}|_1^q = V(R, h, K, \Delta g)$; then $V(P, h, K, \Delta g) = |a_{ij}|_1^p = A_p$.

Let $B = |b_{ij}|_1^q$ be the $q \times q$ determinant such that $b_{ij} = 0$ for $p < j \leqq i \leqq q$ and $b_{ij} = a_{ij}$ otherwise. It follows from Theorem 5 that

$$\begin{aligned}
|A - B| &\leqq Q\Sigma_{i=1}^{q}\Sigma_{j=1}^{i}|a_{ij} - b_{ij}||g(z_{i+1}) - g(z_i)| \\
&= Q[\Sigma_{i=p+1}^{q-1}\Sigma_{j=p+1}^{i}|a_{ij}||\Delta g_{i+1}| + \Sigma_{j=p+1}^{q}|a_{qj}|] \\
&= Q[\Sigma_{i=p+1}^{q-1}\Sigma_{j=p+1}^{i}|K_{i,j-1}\Delta g_j||\Delta g_{i+1}| \\
&\quad + \Sigma_{j=p+1}^{q}|K_{q,j-1}\Delta g_j|] \\
&\leqq Q\left[M\left(\int_{x_p}^{x_q}|dg|\right)^2 + M\int_{x_p}^{x_q}|dg|\right] < \epsilon/4.
\end{aligned}$$

Also, $B_{j-1} = A_{j-1}$ and $^*B_{pj} = a_{qj}$ (Th. 1f) for $j = 1, 2, \cdots, p$; therefore,

$$\begin{aligned}
|B - A_p| &= |\Sigma_{j=1}^{p} B_{j-1}{}^*B_{pj} - \Sigma_{j=1}^{p} A_{j-1}a_{pj}| \qquad \text{(Th. 1d,b)} \\
&\leqq \Sigma_{j=1}^{p}|A_{j-1}||^*B_{pj} - a_{pj}| \leqq M\Sigma_{j=1}^{p}|a_{qj} - a_{pj}| \\
&\leqq M[|h_q - h_p| + |h_0||K_{q0} - K_{p0}||\Delta g_1| + \Sigma_{j=2}^{p}|K_{q,j-1} \\
&\qquad - K_{p,j-1}||\Delta g_j|] \\
&\leqq M\left[\epsilon/8M + m\Sigma_{j=1}^{p}\left(\epsilon/8mM\left(\int_a^b|dg| + 1\right)\right)|\Delta g_j|\right] < \epsilon/4.
\end{aligned}$$

Hence,

$$\begin{aligned}
|F(y) - F(x)| &\leqq |F(y) - A| + |A - B| + |B - A_p| + |A_p - F(x)| \\
&< \epsilon/8 + \epsilon/4 + \epsilon/4 + \epsilon/8 < \epsilon.
\end{aligned}$$

The proof of Conclusion 2 is similar to the proof of Conclusion 2 of Theorem 7.

THEOREM 9. *Given. The function K is bounded on $[a, b] \times [a, b]$ and on $[a, b]$ g has bounded variation and $F(x) = V(a, b; 1, K, dg)$ exists, where* 1 *denotes the identity function. Conclusion.* (1) *F is quasicontinuous on $[a, b]$;* (2) *$(1, K, g) \in OM^{**}$ on $[a, b]$; and* (3) *if $dh \in OB^0$ on $[a, b]$, then $V(a, b; h, K, dg)$ exists and is $h(b) - (L)\int_a^b h(t)dF(t)$.*

Proof of Conclusion 1. Let $\epsilon > 0$, let M be a bound for $|K|$ and let Q be a number having the properties defined in Theorem 5. Since $dg \in OB^0$, there is a subdivision $D = \{x_i\}_0^m$ of $[a, b]$ such that, if $0 < i \leqq m$ and $x_{i-1} < x < y < x_i$, then $\int_x^y |dg| < \epsilon/6QM\left(1 + \int_a^b |dg|\right)$. Suppose that $0 < r \leqq m$ and $x_{r-1} < x < y < x_r$. Since $F(x) = V(x, b; 1, K, dg)$ and $F(y) = V(y, b; 1, K, dg)$ exist, then there exists a subdivision $R = \{z_i\}_0^n$ of $[x, b]$ and an integer p such that $0 < p < n$ and a subdivision $P = \{z_i\}_{p-1}^n$ of $[y, b]$ such that $x = z_0$, $y = z_{p-1}$, $|F(x) - V(R, 1, K, \Delta g)| < \epsilon/6$, and $|F(y) - V(P, 1, K, \Delta g)| < \epsilon/6$.

Let $A = V(R, 1, K, \Delta g) = |a_{ij}|_1^n$ and $C = V(P, 1, K, \Delta g) = |a_{ij}|_p^n = {}^*A_p$; let $B = |b_{ij}|_1^n$ be the $n \times n$ determinant such that $b_{i1} = a_{ip}$ for $p \leqq i \leqq n$, $b_{ij} = 0$ for $1 < j \leqq i \leqq p$, $b_{ij} = 0$ for $p < i \leqq n$ and $2 \leqq j \leqq p$, and $b_{ij} = a_{ij}$ otherwise. In the following manipulations, $|\Delta g_{n+1}| = 1$ and $|a_{ij}|$ denotes the norm of the element a_{ij}; hence,

$$\begin{aligned} |A - B| &\leqq Q\Sigma_{i=1}^n \Sigma_{j=1}^i |a_{ij} - b_{ij}| |\Delta g_{i+1}| \qquad \text{(Th. 5)} \\ &= Q[|a_{n1} - b_{n1}| + \Sigma_{j=2}^p |a_{nj}|] + Q\Sigma_{i=p}^{n-1} |a_{i1} - b_{i1}| |\Delta g_{i+1}| \\ &\quad + Q[\Sigma_{i=2}^p \Sigma_{j=2}^i |a_{ij}| |\Delta g_{i+1}| + \Sigma_{i=p+1}^{n-1} \Sigma_{j=2}^p |a_{ij}| |\Delta g_{i+1}|] \\ &= Q[|(1 + K_{n0}\Delta g_1) - (1 + K_{n,p-1}\Delta g_p)| + \Sigma_{j=2}^p |K_{n,j-1}\Delta g_j|] \\ &\quad + Q[\Sigma_{i=p}^{n-1} |(1 + K_{i0}\Delta g_1) - (1 + K_{i,p-1}\Delta g_p)| |\Delta g_{i+1}|] \\ &\quad + Q[\Sigma_{i=2}^p \Sigma_{j=2}^i |K_{i,j-1}\Delta g_j| + \Sigma_{i=p+1}^{n-1} \Sigma_{j=2}^p |K_{i,j-1}\Delta g_j| |\Delta g_{i+1}|] \\ &\leqq 2QM\int_x^y |dg| + QM(|\Delta g_1| + |\Delta g_p|)\int_a^b |dg| \\ &\qquad\qquad + QM\int_x^y |dg| \int_a^b |dg| \\ &< \epsilon/3 + \epsilon/6 + \epsilon/6 = 2\epsilon/3. \end{aligned}$$

It follows from Theorem 1f that $B = {}^*A_p = C$; hence,

$$\begin{aligned} |F(x) - F(y)| &\leqq |F(x) - A| + |A - B| + |B - C| + |C - F(y)| \\ &< \epsilon/6 + 2\epsilon/3 + 0 + \epsilon/6 = \epsilon. \end{aligned}$$

Therefore, F is quasicontinuous on $[a, b]$.

The proof of Conclusion 2 is similar to the proof of Conclusion 2 of Theorem 7.

Proof of Conclusion 3. Suppose that $\epsilon > 0$. Since $dh \in OB^0$ and F is quasicontinuous, then $(R)\int_a^b dhF$ exists. Since $(R)\int_a^b dhF$ and $V(a, b; 1, K, dg)$ exist and $(h, K, g) \in OM^{**}$, there exists a subdivision D of $[a, b]$ such that if $D' = \{x_i\}_0^n$ is a refinement of D and $0 < i \leqq n$, then

$$\left|(R)\int_a^b dhF - \Sigma_1^n \Delta h_i F_i\right| < \epsilon/3$$

and

$$|F(x_i) - V(D_i, 1, K, \Delta g)| < \epsilon/3\left(\int_a^b |dh| + 1\right)$$

for $i = 0, 1, 2, \cdots, n$, where $D_i = \{x_p\}_{p=i}^n$. Let $D' = \{x_i\}_0^n$ be a refinement of D and let $D_i = \{x_p\}_{p=i}^n$ for $i = 1, 2, \cdots, n$. Also, let $V(D, h, K, \Delta g) = A = |a_{ij}|_1^n$ and let $V^*(D, h, K, \Delta g) = B = |b_{ij}|_1^n$ be the $n \times n$ Volterra determinant such that (1) $b_{11} = a_{11}$, and (2) if $1 < i \leqq n$, then $b_{ij} = a_{ij} - a_{i-1,j}$ for $j = 1, 2, \cdots, n$. Note that $A = |a_{ij}|_1^n$ can be transformed into $B = |b_{ij}|_1^n$ by adding the negative of the elements of the $n - 1$st row of A to the nth row of A, the negative of the elements of the $n - 2$nd row to the $n - 1$st row, etc. Hence, $A = B$, and for $i = 1, 2, 3, \cdots, n$, the determinant ${}^*B_{i+1}$ can be transformed into $V(D_i, 1, K, \Delta g)$ by adding the elements of the first row of ${}^*B_{i+1}$ to the 2nd row, the elements of the new 2nd row to the 3rd row, etc. Hence, there exists an element α of N such that $|\alpha| < \epsilon$ and

$$
\begin{aligned}
V(D, h, K, \Delta g) &= |b_{ij}|_1^n = \Sigma_{i=1}^n b_{ij} {}^*B_{i+1} \qquad \text{(Th. 1b)} \\
&= (h_1 + h_0 K_{10} \Delta g_1)^* B_2 + \Sigma_{i=2}^n [\Delta h_i + h_0(K_{i0} - K_{i-1,0})\Delta g_1]^* B_{i+1} \\
&= h_0[(1 + K_{10}\Delta g_1)^* B_2 + \Sigma_{i=2}^n (K_{i0} - K_{i-1,0})\Delta g_1 {}^* B_{i+1}] + \Sigma_{i=1}^n \Delta h_i {}^* B_{i+1} \\
&= h_0 V(D, 1, K, \Delta g) + \Sigma_{i=1}^n \Delta h_i F(x_i) + \Sigma_{i=1}^n \Delta h_i [{}^*B_{i+1} - F(x_i)] \\
&= h_0 V(a, b; 1, K, dg) + (R)\int_a^b dhF + \alpha \\
&= h(a)F(a) + h(x)F(x)|_a^b - (L)\int_a^b hdF + \alpha \\
&= h(b) - (L)\int_a^b hdF + \alpha.
\end{aligned}
$$

Therefore, $V(a, b; h, K, dg)$ exists and

$$
\begin{aligned}
V(a, b; h, K, dg) &= h(a)V(a, b; 1, K, dg) + (R)\int_a^b dhF \\
&= h(b) - (L)\int_a^b hdV(t, b; 1, K, dg).
\end{aligned}
$$

In Theorem 11 we prove a set of sufficient conditions for the existence of the limit $V(a, b; h, K, dg)$. Theorem 10 is a lemma which is used in the proof of Theorem 11.

THEOREM 10. *Given. The symbols n, r and p represent positive integers, $p < n$, and $A = |a_{ij}|_1^n$ is an $n \times n$ Volterra determinant and $B = |b_{ij}|_1^{n+r}$ is an $(n + r) \times (n + r)$ Volterra determinant such that*

(1) *if $0 < j \leqq i \leqq p$, then $b_{ij} = a_{ij}$;*
(2) *if $p < i \leqq p + r$ and $0 < j \leqq p$, then $b_{ij} = a_{pj}$;*
(3) *if $p + r < i \leqq n + r$ and $0 < j \leqq p$, then $b_{ij} = a_{i-r,j}$;*
(4) *if $p + r < i \leqq n + r$, then $\Sigma_{j=p+1}^{p+r+1} b_{ij} = a_{i-r,p+1}$;*

(5) *if* $p+r+1<j\leqq i\leqq n+r$, *then* $b_{ij}=a_{i-r,j-r}$; *and*
(6) *if* $p<j\leqq i\leqq p+r$, *then* $b_{ij}=0$.

Conclusion. $A=B$, where A and B represent elements of N.

Proof. Note that $A_i=B_i$ for $i=1,2,\cdots,p$ and ${}^*A_i={}^*B_{i+r}$ for $i>p+1$. It follows from (6) and (4) above that ${}^*B_{i+1}={}^*B_{p+r+1,i+1}$ for $i=p,p+1,\cdots,p+r$ and $\Sigma_{i=p}^{p+r}{}^*B_{i+1}=\Sigma_{i=p}^{p+r}{}^*B_{p+r+1,i+1}={}^*A_{p+1}$. Hence,

$$\begin{aligned}B&=\Sigma_{j=1}^{p}B_{j-1}(\Sigma_{i=p}^{n+r}b_{ij}{}^*B_{i+1}) &&\text{(Th. 1d)}\\&=\Sigma_{j=1}^{p}B_{j-1}(\Sigma_{i=p}^{p+r}b_{ij}{}^*B_{i+1}+\Sigma_{i=p+r+1}^{n+r}b_{ij}{}^*B_{i+1})\\&=\Sigma_{j=1}^{p}A_{j-1}(\Sigma_{i=p}^{p+r}a_{pj}{}^*B_{i+1}+\Sigma_{i=p+1}^{n}a_{ij}{}^*A_{i+1}) &&(2,3,5)\\&=\Sigma_{j=1}^{p}A_{j-1}(a_{pj}{}^*A_{p+1}+\Sigma_{i=p+1}^{n}a_{ij}{}^*A_{i+1})\\&=\Sigma_{j=1}^{p}A_{j-1}\Sigma_{i=p}^{n}a_{ij}{}^*A_{i+1}=A, &&\text{(Th. 1d).}\end{aligned}$$

THEOREM 11. *Given.* $[a,b]$ *is a number interval,* K *is a bounded function from* $R\times R$ *to* N *and* h *and* g *are functions from* R *to* N *such that* dg *and* $dh\in OB^0$ *on* $[a,b]$, $\int_a^b\int_a^b|dK||dg||dg|=0$ *and* $\int_a^b|dK(b,t)||dg(t)|=0$. *Conclusion.* (1) $V(a,b;h,K,dg)$ *exists, and* (2) *if* $\int_a^b|dK(\ ,t)||dg(t)|=0$ *uniformly on* $[a,b]$, *then on* $[a,b]$ *the function* $f(x)=V(a,x;h,K,dg)$ *exists,* $(h,K,g)\in OM^*$ *and* f *is the solution of the equation*

$$f(x)=h(x)+(L)\int_a^x f(t)K(x,t)dg(t).$$

Proof. We will show that the limit $V(a,b;h,K,dg)$ exists by showing that the following Cauchy criterion condition is satisfied: if $\epsilon>0$ then there is a subdivision D of $[a,b]$ such that, if D' is a refinement of D, then $|V(D,h,K,\Delta g)-V(D',h,K,\Delta g)|<\epsilon$. Let $\epsilon>0$ and let M be a bound for $(1+|h|)(1+|K|)\left(1+\int_a^b|dg|\right)$ on $[a,b]$. It follows from Theorem 5 that there is a number Q such that, if U, W, u and w are functions bounded by M on $[a,b]$ and $D=\{x_i\}_0^n$ is a subdivision of $[a,b]$, then

$$|A-B|\leqq Q\Sigma_{p=1}^{n}\Sigma_{j=1}^{p}|a_{pj}-b_{pj}||g(x_{p+1})-g(x_p)|,$$

where $A=|a_{ij}|_1^n=V(D,u,U,\Delta g)$, $B=|b_{ij}|_1^n=V(D,w,W,\Delta g)$ and $|\Delta g_{n+1}|=1$.

Since dg and $dh \in OB^0$ and $\int_a^b \int_a^b |dK||dg||dg| = 0$ and $\int_a^b |dK(b,t)||dg(t)| = 0$, there is a subdivision $D = \{x_i\}_0^n$ of $[a,b]$ such that

$$(1) \qquad \Sigma_{i=1}^n \int_{x_{i-1}}^{x_i^-} |dg| \int_{x_{i-1}^+}^{x_i} |dg| < \epsilon/9MQ,$$

$$\Sigma_{i=1}^n \int_{x_{i-1}}^{x_i^-} |dh| \int_{x_{i-1}^+}^{x_i} |dg| < \epsilon/9Q, \quad \text{and} \quad \int_{a^+}^{x_2} |dg| < \epsilon/18MQ;$$

$$(2) \qquad \Sigma_{i=1}^n \Sigma_{j=1}^i M_{ij} \int_{x_{i-1}}^{x_i} |dg| \int_{x_{j-1}}^{x_j} |dg| < \epsilon/9MQ,$$

where for each i and j, M_{ij} is the lub of $|K(x_{i-1}, x_{j-1}) - K(x,y)|$ for $x_{i-1} \leqq x < x_i$ and $x_{j-1} \leqq y < x_j$; and

$$(3) \qquad \Sigma_1^n M_i \int_{x_{i-1}}^{x_i} |dg| < \epsilon/9Q,$$

where for each i, M_i is the lub of $|K(b, x_{i-1}) - K(b,x)|$ for $x_{i-1} \leqq x < x_i$.

Let $D' = \{z_i\}_0^m$ be a refinement of D, $A = |a_{ij}|_1^n = V(D, h, K, \Delta g)$ and $B = |b_{ij}|_1^m = V(D', h, K, \Delta g)$. Let $\{n_i\}_0^n$ be the sequence of integers such that $x_i = z_{n_i}$ for $i = 1, 2, \cdots, n$. We now define an $m \times m$ determinant $C = |c_{ij}|_1^m$ such that $C = A$ and $|B - C| < \epsilon$ and, hence, $|A - B| \leqq |A - C| + |C - B| < \epsilon$. In the following paragraphs, the symbols h_i, K_{ij} and Δg_i represent $h(z_i)$, $K(z_i, z_j)$ and $g(z_i) - g(z_{i-1})$, respectively.

Let P_1 be the set of integer pairs such that $i, j \in P_1$ iff $j = 1$ and $1 \leqq i < n_2$. Let $c_{ij} = a_{11}$ for $i, j \in P_1$; then

$$\Sigma_{i,j \in P_1} |b_{ij} - c_{ij}||\Delta g_{i+1}| \leqq \Sigma_{i=1}^{n_2 - 1} 2M|\Delta g_{i+1}| \leqq 2M \int_{a^+}^{x_2} |dg|$$
$$< 2M(\epsilon/18MQ) = \epsilon/9Q.$$

Let P_2 be the set of integer pairs such that $i, j \in P_2$ iff $j = 1$ and $n_2 \leqq i < m$. If $i, j \in P_2$ and $2 < p \leqq n$ and $n_{p-1} \leqq i < n_p$, then $c_{ij} = a_{p-1,1}$. Let $N_p = [n_{p-1}, n_p)$. Since $x_0 = z_0 = a$, then

$$\Sigma_{i,j \in P_2} |b_{ij} - c_{ij}||\Delta g_{i+1}| = \Sigma_{p=3}^n \Sigma_{i \in N_p} |h(z_i) + h(z_0)K(z_i, z_0)(\Delta g_1)$$
$$- h(x_{p-1}) - h(x_0)K(x_{p-1}, x_0)[g(x_1) - g(x_0)]||\Delta g_{i+1}|$$
$$= \Sigma_{p=3}^n \Sigma_{i \in N_p} |[h(z_i) - h(x_{p-1})] + h(x_0)\{K(z_i, z_0)[g(z_1) - g(z_0)]$$
$$- K(x_{p-1}, x_0)[g(x_1) - g(x_0)]\}||\Delta g_{i+1}|$$

$$\leqq \Sigma_{p=3}^{n} \Sigma_{i \in N_p} |h(z_i) - h(x_{p-1})| |\Delta g_{i+1}|$$
$$+ \Sigma_{p=3}^{n} \Sigma_{i \in N_p} |h(x_0)| |K(z_i, x_0) - K(x_{p-1}, x_0)| |g(x_1) - g(x_0)| |\Delta g_{i+1}|$$
$$+ |g(z_1) - g(x_1)| \Sigma_{p=3}^{n} \Sigma_{i \in N_p} |h(x_0)| |K(z_i, z_0)| |\Delta g_{i+1}|$$
$$\leqq \Sigma_{p=3}^{n} \int_{x_{p-1}}^{x_p^-} |dh| \int_{x_{p-1}^+}^{x_p} |dg|$$
$$+ |h(x_0)| \Sigma_{p=3}^{n} \Sigma_{i \in N_p} M_{p,0} |g(x_1) - g(x_0)| |\Delta g_{i+1}|$$
$$+ (\epsilon/18MQ) |h(x_0)| |K(z_i, z_0)| \left(\int_a^b |dg| + 1 \right)$$
$$< \epsilon/9Q + M(\epsilon/9MQ) + \epsilon/9Q = \epsilon/3Q,$$

where M_{p0} is the lub of $|K(x_{p-1}, x_0) - K(x, y)|$ for $x_{i-1} \leqq x < x_i$ and $x_{j-1} \leqq y < x_j$.

Let P_3 be the set of integer pairs such that $i, j \in P_3$ iff $1 < j \leqq n_1$ and $j \leqq i \leqq m$. Let $c_{ij} = 0$ for $i, j \in P_3$; then

$$\Sigma_{i,j \in P_3} |b_{ij} - c_{ij}| |\Delta g_{i+1}| = \Sigma_{j=2}^{n_1} \Sigma_{i=j}^{m} |K(z_i, z_{j-1}) \Delta g_j| |\Delta g_{i+1}|$$
$$\leqq \Sigma_{j=2}^{n_1} M |g(z_j) - g(z_{j-1})| \leqq M \int_{a^+}^{x_1} |dg|$$
$$< M(\epsilon/9MQ) = \epsilon/9Q.$$

Let P_4 be the set of integer pairs such that the pair $i, j \in P_4$ iff there is an integer p such that $1 < p \leqq n$ and $n_{p-1} < j \leqq i < n_p$. Let $c_{ij} = 0$ for $i, j \in P_4$; then

$$\Sigma_{i,j \in P_4} |b_{ij} - c_{ij}| |\Delta g_{i+1}| = \Sigma_{i,j \in P_4} |K(z_i, z_{j-1}) \Delta g_j| |\Delta g_{i+1}|$$
$$\leqq M \Sigma_{p=2}^{n} \Sigma_{i=n_{p-1}+1}^{n_p - 1} \Sigma_{j=n_{p-1}+1}^{i} |g(z_j) - g(z_{j-1})| |g(z_{i+1}) - g(z_i)|$$
$$\leqq M \Sigma_{p=2}^{n} \int_{x_{p-1}}^{x_p^-} |dg| \int_{x_{p-1}^+}^{x_p} |dg| < M(\epsilon/9MQ).$$

Let P_5 be the set of integer pairs such that $i, j \in P_5$ iff $i = m$ and also $j = 1$ or $n_1 < j \leqq m$. Let $c_{m1} = a_{n1} = h(x_n) + h(x_0) K(x_n, x_0)[g(x_1) - g(x_0)]$ and, if $1 < p \leqq n$ and $n_{p-1} < j \leqq n_p$, let $c_{mj} = K(x_n, x_{p-1})[g(z_j) - g(z_{j-1})]$. Since $z_m = x_n$ and $z_0 = x_0$, then

$$\Sigma_{i,j \in P_5} |b_{ij} - c_{ij}| = |b_{m1} - c_{m1}| + \Sigma_{p=2}^{n} \Sigma_{j=n_{p-1}+1}^{n_p} |b_{mj} - c_{mj}|$$
$$= |h(z_m) + h(z_0) K(z_m, z_0)[g(z_1) - g(z_0)]$$
$$- h(x_n) - h(x_0) K(x_n, x_0)[g(x_1) - g(x_0)]|$$
$$+ \Sigma_{p=2}^{n} \Sigma_{j=n_{p-1}+1}^{n_p} |K(z_m, z_{j-1})[g(z_j) - g(z_{j-1})]$$
$$- K(x_n, x_{p-1})[g(z_j) - g(z_{j-1})]|$$

$$\begin{aligned}
&\leqq |h(z_0)||K(z_m, z_0)||g(x_1)-g(z_1)| \\
&\quad + \Sigma_{p=2}^{n} \Sigma_{j=n_{p-1}+1}^{n_p} |K(x_n, z_{j-1}) - K(x_n, x_{p-1})||g(z_j)-g(z_{j-1})| \\
&\leqq M|g(x_1)-g(z_1)| + \Sigma_{p=2}^{n} \Sigma_{j=n_{p-1}+1}^{n_p} M_p |g(z_j)-g(z_{j-1})| \\
&\leqq M(\epsilon/18MQ) + \Sigma_{p=2}^{n} M_p \int_{x_{p-1}}^{x_p} |dg| < \epsilon/3Q,
\end{aligned}$$

where M_p is the lub of $|K(b, x_{p-1}) - K(b, z)|$ for $x_{p-1} \leqq z < x_p$.

Let P_6 be the set of integer pairs such that $i, j \in P_6$ iff there are integers p and q such that $2 \leqq q < p \leqq n$ and such that $n_{p-1} \leqq i < n_p$ and $n_{q-1} < j \leqq n_q$. If $i, j \in P_6$ and $n_p \leqq i < n_{p+1}$ and $n_{q-1} < j \leqq n_q$, let $c_{ij} = K(x_p, x_{q-1})[g(z_j) - g(z_{j-1})]$; then

$$\begin{aligned}
&\Sigma_{i,j\in P_6} |b_{ij} - c_{ij}||\Delta g_{i+1}| \\
&\quad = \Sigma_{P_6} |K(z_i, z_{j-1})\Delta g_j - K(x_{p-1}, x_{q-1})\Delta g_j||\Delta g_{i+1}| \\
&\quad \leqq \Sigma_{p=3}^{n} \Sigma_{q=2}^{p-1} \Sigma_{i=n_{p-1}}^{n_p-1} \Sigma_{j=n_{q-1}}^{n_q} |K_{i,j-1} - K(x_{p-1}, x_{q-1})||\Delta g_j||\Delta g_{i+1}| \\
&\quad \leqq \Sigma_{P_6} M_{pq} |\Delta g_j||\Delta g_{i+1}| \\
&\quad \leqq \Sigma_{p=3}^{n} \Sigma_{q=2}^{p-1} M_{pq} \int_{x_{p-1}}^{x_p} |dg| \int_{x_{q-1}}^{x_q} |dg| < \epsilon/9Q,
\end{aligned}$$

where M_{pq} is the lub of $|K(x_{p-1}, x_{q-1}) - K(x, y)|$ for $x_{p-1} \leqq x < x_p$ and $x_{q-1} \leqq y < x_q$.

The determinant $|c_{ij}|_1^m$ can be reduced to the determinant $|a_{ij}|_1^n$ by the following steps.

(1) If $n_1 > 1$, use Theorem 1f and obtain a determinant of lower order.

(2) For each integer p such that $2 < p \leqq n$ and $n_p > n_{p-1} + 1$, use Theorem 10 and the definition of the determinant $|c_{ij}|_1^m$ to obtain a determinant of lower order. Note that, if $1 < n_p \leqq i < n_{p+1}$, then

$$\Sigma_{j=n_{q-1}+1}^{n_q} c_{ij} = K(x_p, x_{q-1})[g(x_q) - g(x_{q-1})] = a_{pq}.$$

Hence,

$$\begin{aligned}
|B - A| &\leqq |B - C| + |C - A| = |B - C| \\
&\leqq Q \Sigma_{i=1}^{m} \Sigma_{j=1}^{i} |b_{ij} - c_{ij}||\Delta g_{i+1}| && \text{(Th. 5)} \\
&= Q(\Sigma_{r=1}^{6} \Sigma_{i,j\in P_r} |b_{ij} - c_{ij}||\Delta g_{i+1}|) < Q(\epsilon/Q) = \epsilon.
\end{aligned}$$

Therefore, if $\epsilon > 0$ then there is a subdivision D of $[a, b]$ such that if D' is a refinement of D, then $|V(D, h, K, \Delta g) - V(D', h, K, \Delta g)| < \epsilon$; hence, the limit $V(a, b; h, K, dg)$ exists.

Suppose that $\int_a^b |dK(\ ,t)||dg(t)| = 0$ uniformly on $[a, b]$. If $a < x \leqq b$, it follows from Conclusion 1 that $V(a, x; h, K, dg)$ exists. We now prove that $(h, K, g) \in OM^*$ on $[a, b]$. Let $\epsilon > 0$ and define a subdivision D of $[a, b]$ in the same manner as in Conclusion 1 except that $\int_a^b |dK(\ ,t)||dg(t)| = 0$ uniformly is used in defining D in place of $\int_a^b |dK(b, t)||dg(t)| = 0$.

If $\{x_i\}_0^n$ is a refinement of D and $0 < p \leqq n$, then a repetition of the steps in the proof of Conclusion 1 shows that, if Q' is a refinement of $Q = \{x_i\}_0^p$, then

$$|V(Q', h, K, \Delta g) - V(Q, h, K, \Delta g)| < \epsilon.$$

Since $V(a, x_p; h, K, dg)$ exists, there is a refinement Q' of Q such that $|V(Q', h, K, \Delta g) - V(a, x_p; h, K, dg)| < \epsilon$; hence,

$$\begin{aligned} &|V(a, x_p; h, K, dg) - V(Q, h, K, \Delta g)| \\ &\qquad \leqq |V(a, x_p; h, K, dg) - V(Q')| + |V(Q') - V(Q)| < 2\epsilon. \end{aligned}$$

Therefore, $(h, K, g) \in OM^*$ on $[a, b]$. It follows from Theorem 3 that f is bounded on $[a, b]$ and from Theorem 6 that f is the solution on $[a, b]$ of the equation $f(x) = h(x) + (L)\int_a^x f(t)K(x, t)dg(t)$.

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