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THE EXTREMAL STRUCTURE OF LOCALLY COMPACT CONVEX SETS

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Let X be a locally compact closed convex subset of a locally convex Hausdorff topological linear space E. Then every exposed point of X is strongly exposed. The definitions of denting (strongly extreme) ray and strongly exposed ray are given for convex subsets of E. If X does not contain a line, then every extreme ray is strongly extreme and every exposed ray is strongly exposed. An example is given to show that the hypothesis that X be locally compact is necessary in both cases.

By a locally convex space we mean a real Hausdorff locally convex topological linear space E. E^* will denote the topological dual of E. The set of extreme points of X will be denoted by ext X. The closed line segment between the points x and y in E will be denoted $[x, y]$. The following definition was given by M. Rieffel [**6**, p. 75] for subsets of a Banach space. I. Namioka also studied these points in [**4**].

DEFINITION 1. If X is a subset of a locally convex space, then $x \in X$ is called a denting (strongly extreme) point of X if for any nbhd U of x, $x \notin \text{cl-conv}(X \setminus U)$. The set of all denting points of X will be denoted by dent X.

Clearly, every denting point is an extreme point. It follows from the separation theorem for convex sets that x_0 is a denting point of X iff for each nbhd U of x_0 there exist $f \in E^*$ and $\alpha \in R$ such that $x_0 \in \{x : f(x) < \alpha\} \cap X \subseteq X \cap U$. An example is given in [**6**, p. 75] to show that not every extreme point is a denting point. However, this is not the case in a locally compact set. For completeness we state the following theorem due to J. Reif and V. Zizler [**5**, p. 64].

THEOREM 1. *Assume X is a locally compact closed convex set in a locally convex space E. Then any extreme point of X is a strongly extreme point of X with respect to the relative topology from E.*

A point p of a set X in a locally convex space E is an exposed point of X if there exists an $f \in E^*$ such that $f(x) > f(p)$ for each $x \in X \setminus \{p\}$. The following definition was given by J. Lindenstrauss [**3**, p. 140] for subsets of a Banach space.

DEFINITION 2. A point $x \in X$, where $X \subseteq E$, is called a strongly exposed point of X whenever (i) there exists an $f \in E^*$ such that $f(y) > f(x)$ for each $y \in X \backslash \{x\}$, and (ii) for any net $\{x_\alpha\} \subseteq X$, $f(x_\alpha) \to f(x)$ in R implies that $x_\alpha \to x$ in E. The set of all strongly exposed points of X is denoted by strexp X.

It is easy to see from the definition that every strongly exposed point is an exposed point. J. Lindenstrauss in [**3**, p. 145] gave an example of a set which has an exposed point that is not strongly exposed. However, this is not the case if the set is locally compact.

THEOREM 2. *Let X be a locally compact closed convex subset of a locally convex space E, then every exposed point of X is a strongly exposed point of X.*

Proof. Let U be a closed convex nbhd of x such that $U \cap X$ is compact and assume $f \in E^*$ such that $f(x) < f(y)$, for all $y \in X \backslash \{x\}$. Since x is an exposed point of X, x is an extreme point of X. By Theorem 1, x is a denting point of X. Thus, there exist $g \in E^*$ and $\alpha \in R$ such that $\{x: g(x) < \alpha\} \cap X \subseteq (\text{int } U) \cap X$.

If $\{x: g(x) \geqq \alpha\} \cap (X \cap U) = \emptyset$, then it follows immediately that $U \cap X \subseteq \{x: g(x) < \alpha\} \cap X \subseteq (\text{int } U) \cap X$. Therefore $U \cap X$ is a nonempty open and closed set in the connected set X. Hence, $U \cap X = X$ which implies X is compact. Let $\{x_\alpha\}$ be a net in X such that $f(x_\alpha) \to f(x)$ in R. Since X is compact, there is a subnet $\{x_\beta\}$ of $\{x_\alpha\}$ and a vector $y \in X$ such that $x_\beta \to y$. Thus, $f(x_\beta) \to f(y) = f(x)$ in R and so $y = x$. For any subnet $\{x_\gamma\} \subseteq \{x_\alpha\}$ there is similarly a subnet which converges to x, which proves that $x_\alpha \to x$ in E.

On the other hand, if $W = \{x: g(x) \geqq \alpha\} \cap (X \cap U) \neq \emptyset$, then W is a nonempty compact convex subset of X which does not contain x. Hence, there is a $w \in W$ such that $f(x) < f(w) = \inf f(W)$. Let $y \in X \backslash U$, then $[x, y] \subseteq X$. U is a closed convex nbhd of x; hence, there exists a $z \in \text{Bdry } U$ such that $z \in [x, y]$. Since $z \in \text{Bdry } U$, then $z \notin \text{int } U$ and $z \notin \{x: g(x) < \alpha\}$. Therefore, $z \in \{x: g(x) \geqq \alpha\} \cap (X \cap U)$ so $f(z) \geqq f(w)$. But $y - x = \lambda(z - x)$ where $\lambda > 1$. Hence, $f(y - x) = \lambda f(z - x) > f(z - x)$ which implies $f(y) > f(z) \geqq f(w)$. Let $\{y_\alpha\}$ be a net in X such that $f(y_\alpha) \to f(x)$ in R. Since $\{y_\alpha\} \subseteq X$ and $f(y) \geqq f(w) > f(x)$ for each $y \in X \backslash U$, we may assume that $\{y, y_a\} \subseteq U \cap X$. Since $U \cap X$ is compact, it follows from the previous argument that $y_\alpha \to x$ in E.

As V. Klee has shown in [**1**] and [**2**], it is possible to extend the Krein–Milman theorem to certain noncompact convex sets with the aid of the notion of extreme ray. An extreme ray of a closed convex set X

is a closed half-line $\rho \subseteq X$ such that whenever $x, y \in X$ and $\lambda x + (1-\lambda)y \in \rho$ for some λ with $0 < \lambda < 1$, $x, y \in \rho$.

DEFINITION 3. A ray $\rho = \{x + \lambda z : \lambda \geqq 0, z \neq 0\}$ of a convex set X in a topological linear space E is a denting (strongly extreme) ray of X if for any nbhd U of 0, $\rho' \cap \text{cl-conv}[X' \backslash (x + \langle z \rangle + U)] = \emptyset$, where X' is any bounded convex subset of X, $\rho' = \rho \cap X'$ and $\langle z \rangle$ denotes the one-dimensional linear subspace generated by z. Denote the union of all denting rays of X by rdent X.

It is easy to show that every denting ray of a convex set X is an extreme ray of X. The following theorem and example show that extreme rays and denting rays coincide in some instances and are distinct in others.

THEOREM 3. *Let X be a locally compact closed convex subset of a locally convex space E, then every extreme ray of X is a denting ray of X.*

Proof. Let ρ be an extreme ray of X. We may assume without loss of generality that $\rho = \{\lambda x_0 : \lambda \geqq 0\}$, $x_0 \neq 0$. Let X' be a bounded convex subset of X and let f_0 be in E^* such that f_0 is positive on $K \backslash \{0\}$, where K is the union of all rays in X which emanate from 0, and $X \cap \{x : f_0(x) \leqq t\}$ is compact, for each $t \in R$. Such a functional exists by Theorem 3.2 in [**1**]. Since X' is bounded and convex, cl(X') is bounded and convex. According to a result of Klee [**1**, p. 236], cl(X') is compact which implies $\sup f_0(\text{cl}(X')) < \infty$. Then we may assume $X' \subseteq \{x : f_0(x) \leqq 1\} \cap X = X''$. Let $W = \{x : f_0(x) = 1\} \cap X$ and assume $f_0(x_0) = 1$. Then $x_0 \in \text{ext}(W)$ and W is compact, since X'' is compact. By Theorem 1, x_0 is a denting point of W. Let U be a nbhd of zero and let $g \in E^*$ and $\alpha > 0$ such that $x_0 \in \{x : g(x) < \alpha\} \cap W \subseteq (x_0 + U) \cap W$. Let $T = \{x : g(x) = \alpha\} \cap W$. Then T is compact, convex and $T \cap \langle x_0 \rangle = \emptyset$. Let $f \in E^*$ and $\beta > 0$ such that $f(\langle x_0 \rangle) < \beta < \inf f(T)$. Since $0 \in \langle x_0 \rangle$, we have $0 = f(\langle x_0 \rangle) < \beta < \inf f(T)$.

If $y \in W$ such that $f(y) < \beta$, then $f_0(y) = 1$ and $[x_0, y] \cap T = \emptyset$, since $f(x_0) < \beta$. It follows that $g(y) < \alpha$ and hence, $y \in (x_0 + U) \cap W \subseteq \langle x_0 \rangle + U$.

On the other hand, if $y \in X$ such that $f_0(y) < 1$ and $f(y) < \beta$, then there is a unique $\lambda > 0$ such that $f_0(y + \lambda x_0) = 1$. Again from Klee [**1**, p. 235] we have $y + \lambda x_0 \in X$. Hence, $y + \lambda x_0 \in W$ and $f(y + \lambda x_0) = f(y) < \beta$. By the previous argument, it follows that $y + \lambda x_0 \in x_0 + U$ and so $y \in (1-\lambda)x_0 + U \subseteq \langle x_0 \rangle + U$.

In both cases we have $y \in \{x : f(x) < \beta\} \cap X''$ implies $y \in \langle x_0 \rangle + U$. Hence, $X'' \backslash (\langle x_0 \rangle + U) \subseteq X'' \backslash \{x : f(x) < \beta\} \subseteq \{x : f(x) \geqq \beta\}$. Thus, $\text{cl-conv}[X' \backslash (\langle x_0 \rangle + U)] \subseteq \{x : f(x) \geqq \beta\}$. Now $f(\rho') < \beta$, since

$f(\langle x_0\rangle) < \beta$ and $\rho' = (X' \cap \rho)$, so $\rho' \cap \text{cl-conv}[X' \backslash (\langle x_0\rangle + U)] = \emptyset$. Therefore, ρ is a denting ray of X.

EXAMPLE 1. Let the space be ℓ_2 with the canonical basis $\{e_n\}$, and $X = \text{cl-conv}(\{e_n; n = 2, 3, \cdots\})$. Then $0 \in X$ and e_1 is in $\ell_2 \backslash X$. Let C be the cone generated by X with vertex e_1, then C is a closed convex subset of ℓ_2. Let ρ be the ray of the cone through 0. Clearly, ρ is an extreme ray of C. Let $S_{\frac{1}{2}}(0)$ be the open ball of radius 1/2 centered on 0. Clearly, $e_n \notin S_{\frac{1}{2}}(0)$ so $e_n \notin \langle e_1\rangle + S_{\frac{1}{2}}(0)$ and it follows that $e_n \in \text{cl-conv}[X \backslash (\langle e_1\rangle + S_{\frac{1}{2}}(0))]$ for $n \geqq 2$. However, $\{e_n\}$ converges weakly to 0 and $\text{cl-conv}[X \backslash (\langle e_1\rangle + S_{\frac{1}{2}}(0))]$ is weakly closed so $0 \in \text{cl-conv}[X \backslash (\langle e_1\rangle + S_{\frac{1}{2}}(0))]$. Hence ρ is not a denting ray of C.

A ray ρ in X, where $X \subseteq E$, is an exposed ray of X if there exist $f \in E^*$ and $\alpha \in R$ such that $\rho = \{x: f(x) = \alpha\} \cap X$ and $f(X \backslash \rho) > \alpha$. The next definition was given by V. Zizler in [**7**, p. 55] for subsets of a Banach space.

DEFINITION 4. Let X be a convex set in a locally convex space E and ρ a closed ray in X. Then ρ is a strongly exposed ray of X if (i) there exist $f \in E^*$ and $r \in R$ such that $f(x) = r$ for $x \in \rho$ and $f(x) > r$ for $x \in X \backslash \rho$, and (ii) $\{x_\alpha\}$ is eventually in $\rho + U$, whenever U is a nbhd of 0 and $\{x_\alpha\}$ is a bounded net in X such that $f(x_\alpha) \to r$. The set of all strongly exposed rays will be denoted by rstrexp X.

Clearly every strongly exposed ray is an exposed ray. The following proposition, theorem, and examples show the relationships among denting ray, exposed ray and strongly exposed ray.

PROPOSITION 1. *Let ρ be a strongly exposed ray of a convex set X in a locally convex space E. Then ρ is a denting ray of X.*

Proof. We may assume $\rho = \{\lambda x_0: \lambda \geqq 0\}$, $x_0 \neq 0$. Let $f \in E^*$ such that $\rho = \{x: f(x) = 0\} \cap X$ and $f(x) > 0$ for each $x \in X \backslash \rho$. Let U be a nbhd of zero and X' a bounded convex subset of X. Assume for each positive integer n there is an $x_n \in \{x: f(x) < (1/n)\} \cap X'$ such that $x_n \notin \langle x_0\rangle + U$. Clearly $\{x_n\}$ is bounded and $f(x_n) \to 0$. Hence, there exists a positive integer N such that $x_n \in \rho + U$ for $n \geqq N$. This is a contradiction; so there is a positive integer N' such that $\{x: f(x) < (1/N')\} \cap X' \subseteq (\langle x_0\rangle + U) \cap X'$. Thus, $\text{cl-conv}[X' \backslash (\langle x_0\rangle + U)] \subseteq \{x: f(x) \geqq (1/N')\}$ which implies $(\rho \cap X') \cap \text{cl-conv}[X' \backslash (\langle x_0\rangle + U)] = \emptyset$; so ρ is a denting ray of X.

THEOREM 4. *Let X be a locally compact closed convex subset of a locally convex space E, then every exposed ray of X is a strongly exposed ray of X.*

Proof. Let ρ be an exposed ray of X. We may assume that ρ emanates from the origin. Let $f \in E^*$ such that $\rho = X \cap \{x: f(x) = 0\}$ and $f(x) > 0$ for $x \in X \backslash \rho$. Let $\{x_\alpha\}$ be a bounded net in X such that $f(x_\alpha) \rightarrow 0$ in R and let U be a nbhd of 0. There exists a nbhd V of 0 such that V is closed, balanced and convex, $V \subseteq U$ and $V \cap X$ is compact. Let $\{x_\beta\}$ denote the set of all vectors in the net $\{x_\alpha\}$ which lie in $X \backslash U$. If $\{x_\beta\}$ is not a subnet of $\{x_\alpha\}$, then $\{x_\alpha\}$ is eventually in $U = 0 + U \subseteq \rho + U$ and the conclusion follows.

If $\{x_\beta\}$ is a subnet of $\{x_\alpha\}$, then it suffices to show that $\{x_\beta\}$ is eventually in $\rho + U$. By Theorem 1, 0 is a denting point of X, since 0 is an extreme point of X. Let $g \in E^*$ and $a > 0$ such that $\{x: g(x) < a\} \cap X \subseteq V \cap X$. Since $x_\beta \notin V$, then $g(x_\beta) \geqq a$, for each β. The net $\{x_\alpha\}$ is bounded, so there exists a number $b > 0$ such that $g(x_\beta) \leqq b$, for each β. Hence, $0 < a \leqq g(x_\beta) \leqq b$, for each β. If $y_\beta = [a/g(x_\beta)]x_\beta$, then $y_\beta \in \{x: g(x) = a\} \cap X$. Since $\{x: g(x) < a\} \cap X \subseteq V \cap X$ and $V \cap X$ is compact, then $\{x: g(x) = a\} \cap X$ is compact; so there is a subnet $\{y_\gamma\} \subseteq \{y_\beta\}$ and a point $y \in \{x: g(x) = a\} \cap X$ such that $y_\gamma \rightarrow y$ in E. Since $g(x_\beta)$ is bounded and $f(x_\beta) \rightarrow 0$ in R, we have $y \in \{x: f(x) = 0\} \cap X$ and thus, $y \in \rho$. Hence, $y \in \{x: g(x) = a\} \cap \rho$. It follows immediately that $\{y\} = \{x: g(x) = a\} \cap \rho$. Let $W = \{x: g(x) = a\} \cap X$ and $z \in W \backslash \{y\}$. Then $z \in W \backslash \rho$ which implies $f(z) > 0$. Thus, y is exposed by f on W. Since $f(y_\beta) \rightarrow 0 = f(y)$, by Theorem 2 we have $y_\beta \rightarrow y$ in E. Hence, there is a λ_0 such that $y_\beta \in y + (a/b)V$, for $\beta \geqq \lambda_0$. If $z_\beta = [g(x_\beta)/a]y$, then $z_\beta \in \rho$, for each β. But $y_\beta = [a/g(x_\beta)]x_\beta$, so $x_\beta \in [g(x_\beta)/a]y + [g(x_\beta)/a](a/b)V \subseteq \rho + V \subseteq \rho + U$, for all $\beta \geqq \lambda_0$. Therefore, the net $\{x_\beta\}$ is eventually in $\rho + U$ and it follows that ρ is a strongly exposed ray of X.

EXAMPLE 2. The ray ρ defined in Example 1 is exposed by $f = (0, \frac{1}{2}, \frac{1}{3}, \cdots, 1/n, \cdots)$ on C. Therefore ρ is an exposed ray of C that is not a denting ray of C so by Proposition 1 ρ is not a strongly exposed ray of C.

EXAMPLE 3. Let the space be R^3 and

$$X = \operatorname{conv}[\{(x, y, z): x^2 + y^2 \leqq 1, \ -1 \leqq y \leqq 0 \text{ and } z = 1\} \cup (1, 1, 1)].$$

Let C be the cone generated by X with vertex $(0, 0, 0)$. Then C is a closed convex subset of R^3. Let ρ be the ray of the cone through the point $(1, 0, 1)$. It is easy to see ρ is not an exposed ray of C, but ρ is a denting ray of C.

From the preceding work we can restate two of Klee's theorems ([**2**, Th. 2.3, p. 91], [**1**, Th. 3.4, p. 237]) as follows:

THEOREM 5. *Suppose X is a locally compact closed convex subset of a normed linear space, and X contains no line. Then* $\operatorname{ext} X \subseteq \operatorname{cl}(\operatorname{strexp} X)$ *and* $X = \operatorname{cl-conv}(\operatorname{strexp} X \cup \operatorname{rstrexp} X)$.

THEOREM 6. *If X is a locally compact closed convex subset of a locally convex space, and X contains no line, then* $X = \operatorname{cl-conv}(\operatorname{dent} X \cup \operatorname{rdent} U)$.

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