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AVERAGING STRONGLY SUBADDITIVE SET FUNCTIONS IN UNIMODULAR AMENABLE GROUPS II

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This paper continues work initiated in Part I. The central question is one of characterizing a net $\{A_\alpha\}$ of Borel sets in the group G which averages a so-called regular set function S on G in the sense that $\lambda(A_\alpha)^{-1}S(A_\alpha)$ has a limit (depending only on S), where λ is Haar measure. In Part I a sufficient condition for $\{A_\alpha\}$ to always average was derived; here we show that a "natural" relaxation of this condition is no longer sufficient for all regular S, at the same time essentially characterizing those S which may still be averaged. Moreover, the role of Følner summing sequences is considered in this context. Finally, properties of regular set functions are derived which may be of independent interest.

1. Definitions, notation, and some generalities. The notation and terminology is the same as **[3]**. We also will have frequent cause to refer to the results and techniques of **[3]**, and consequently there is a strong dependence. Throughout the topological group G is assumed locally compact and noncompact and equipped with a fixed right invariant Haar measure λ, $\mathcal{K} = \mathcal{K}(G)$ is the set of all precompact Borel subsets of G, $\mathcal{K}_+ = \{K \in \mathcal{K}\colon \lambda(K) > 0\}$, $\mathcal{K}_0 = \{K \in \mathcal{K}\colon K \text{ open}\}$, and for $A, K \subseteq G$ we define $[A]_K = \{g \in G\colon Kg \subseteq A\}$. Finally as in **[3]**, $\lambda(E)$ denotes *inner* measure in case E is not measurable, e.g. some $[A]_K$ or KA.

DEFINITION 1.1. A net $\{A_\alpha\}$ of sets in $\mathcal{K}_+$ is said to be:

1. Admissible, $\{A_\alpha\} \in \mathcal{A}$, iff $\lim_\alpha \lambda(\bar{A}_\alpha)^{-1}\lambda([A_\alpha]_K) = 1$ for all $K \in \mathcal{K}_0$,
2. Full, $\{A_\alpha\} \in \mathcal{F}$, iff $\lim_\alpha \lambda(A_\alpha)^{-1}\lambda([A_\alpha]_K) = 1$ for all $K \in \mathcal{K}_0$,
3. Stable, $\{A_\alpha\} \in \mathcal{S}$, iff $\lim \lambda(KA_\alpha)^{-1}\lambda(A_\alpha) = 1$ for all $K \in \mathcal{K}_0$, $K \neq \emptyset$,
4. Translative, $\{A_\alpha\} \in \mathcal{T}$, iff $\lambda(A_\alpha)^{-1}\lambda(kA_\alpha \Delta A_\alpha) \to 0$ uniformly for $k \in K$, for all $K \in \mathcal{K}_0$.

Comment. $\mathcal{K}_0$ in the above definition may be equivalently replaced by any subfamily of $\mathcal{K}$ which is cofinal under inclusion.

The following result gives some basic properties of these nets, some already proved in **[3]** and the others straightforward and left to the reader:

PROPOSITION 1.2. (i) *$\mathcal{A} \neq \emptyset$ iff $\mathcal{F} \neq \emptyset$ iff $\mathcal{S} \neq \emptyset$ iff $\mathcal{T} \neq \emptyset$ iff G is unimodular and amenable.*

(ii) *If G is discrete, $\mathcal{A} = \mathcal{F} = \mathcal{S} = \mathcal{T}$, and otherwise $\mathcal{A} \subset \mathcal{F}$ and $\mathcal{S} \subset \mathcal{T}$ properly and no other inclusions hold.*

(iii) *$\{A_\alpha\} \in \mathcal{S}(\mathcal{T})$ iff $\{A_\alpha^{-1}\}$ is strong* (weak) *summing* (in the sense of [2]) in unimodular G.

(iv) *$\{A_\alpha\} \in \mathcal{A}$ and $\{B_\alpha\}$ from $\mathcal{K}$ such that $\lambda(\bar{A}_\alpha)^{-1}\lambda(\bar{B}_\alpha) \to 0$ implies $\{C_\alpha\} \in \mathcal{A}$ where $C_\alpha \doteqdot A_\alpha \cup B_\alpha$.*

(v) *$\{A_\alpha\} \in \mathcal{F}$ and $\{B_\alpha\}$ from $\mathcal{K}$ such that $\lambda(A_\alpha)^{-1}\lambda(B_\alpha) \to 0$ implies $\{C_\alpha\} \in \mathcal{F}$ where $C_\alpha \doteqdot A_\alpha \cup B_\alpha$.*

The following definition from [3] is needed:

DEFINITION 1.3. $\mathcal{K}_c \doteqdot \{K \in \mathcal{K}: \lambda(K) = \lambda(\bar{K})\}$ is called the class of weak continuity sets in G.

NOTE. Every compact set is a weak continuity set.

We conclude this section with the definition of the set functions S which we shall consider as well as the basic "rearrangement" inequality for such S:

DEFINITION 1.4. (a) A set function $S: \mathcal{K} \to R$ is said to be regular iff

(i) $S \leqq 0$, $S(\emptyset) = 0$,

(ii) $S(A \cup B) + S(A \cap B) \leqq S(A) + S(B)$ for all $A, B \in \mathcal{K}$,

(iii) $S(Ag) = S(A)$ for all $A \in \mathcal{K}$, $g \in G$.

(b) A set function $S: \mathcal{K} \to R$ is said to be upper continuous at $K_0 \in \mathcal{K}$ iff $\{K_n\} \subseteq \mathcal{K}$ and $\lambda(K_0 \Delta K_n) \to 0$ implies

$$\underline{\lim}\, S(K_n) \leqq S(K_0).$$

(c) A set function $S: \mathcal{K} \to R$ is said to be upper continuous on $\mathcal{K}$ iff S is upper continuous at each $K_0 \in \mathcal{K}$.

(d) $S: \mathcal{K} \to R$ is said to be continuous at $\emptyset$ iff $\{K_n\} \subset \mathcal{K}$ such that $K_n \supseteq K_{n+1}$ and $\lambda(K_n) \to 0$ implies $S(K_n) \to S(\emptyset)$.

Equivalent formulations for regular S are contained in:

PROPOSITION 1.5. (i) *The regular set function S is upper continuous at K_0 iff for every $\epsilon > 0$ there corresponds a $\delta = \delta(K_0, \epsilon) > 0$ such that $K \in \mathcal{K}$, $K \subseteq K_0$ and $\lambda(K_0 - K) < \delta$ implies $S(K) < S(K_0) + \epsilon$.*

(ii) *The regular set function S is continuous at $\emptyset$ iff given $K \in \mathcal{K}$ and $\epsilon > 0$ there corresponds a $\delta = \delta(K, \epsilon) > 0$ such that $E \subseteq K$, $E \in \mathcal{K}$, and $\lambda(E) < \delta$ implies $S(E) > -\epsilon$.*

Proof. If (i) is violated at K_0 for $\epsilon = \epsilon_0 > 0$ then for $\delta = 1/n$ we may choose $K = K_n \subseteq K_0$ with $\lambda(K_0 - K_n) < 1/n$ and $S(K_n) \geqq S(K_0) + \epsilon_0$. But clearly $\lambda(K_0 \Delta K_n) \to 0$ and $\underline{\lim}\, S(K_n) \geqq S(K_0) + \epsilon_0$ which violates (b). Conversely, if $\lambda(K_0 \Delta K_n) < \delta$ then $K = K_n \cap K_0 \subseteq K_0$ and also $\lambda(K_0 - K) \leqq \lambda(K_0 \Delta K_n) < \delta$ implying $S(K) < S(K_0) + \epsilon$ by (i). But by the monotonicity of S, $S(K_n) \leqq S(K)$ implying $\underline{\lim}\, S(K_n) < S(K_0) + \epsilon$ (since $\lambda(K_0 \Delta K_n) < \delta$ for n sufficiently large, where $\delta = \delta(K, \epsilon)$) valid for all $\epsilon > 0$ and thus $\underline{\lim}\, S(K_n) \leqq S(K_0)$ and (b) is verified.

The proof that condition (ii) implies (d) is similar ($K \doteq K_1$, $E = K_n$ for n sufficiently large). Conversely if (ii) is violated for $K \in \mathcal{K}$ and $\epsilon = \epsilon_0 > 0$, for each integer n there corresponds an $E = E_n \subseteq K$, $E_n \in \mathcal{K}$, and satisfying $\lambda(E_n) < 1/n^2$ whereas $S(E_n) \leqq -\epsilon_0$. Set $K_n = \cup\{E_j : j \geqq n\}$, implying $S(K_n) \leqq S(E_n) \leqq -\epsilon_0$, $K_n \supseteq K_{n+1}$, and $\lambda(K_n) \to 0$, violating (d) since $S(\emptyset) = 0$.

PROPOSITION 1.6. *Let $S: \mathcal{K} \to R$ satisfy $S(K \cup T) + S(K \cap T) \leqq S(K) + S(T)$ for all $K, T \in \mathcal{K}$. Then for any $K_j \in \mathcal{K}$, $1 \leqq j \leqq n$,*

$$\sum_{j=1}^{n} S(I_j) \leqq \sum_{j=1}^{n} S(K_j),$$

where

$$I_j = \left\{ g \in G : \sum_{i \leqq n} \chi_{K_i}(g) \geqq j \right\}.$$

Proof. Well known, readily shown by induction on n the given condition being the case $n = 2$.

2. Full nets and regular set functions S. In this section we investigate necessary and sufficient conditions on S to insure that it is "successfully" averaged on full nets. First a notationally convenient definition and a basic result from **[3]**:

DEFINITION 2.1. For any function $S: \mathcal{K}_+ \to R$ let $M_S(K) \doteq \lambda(K)^{-1} S(K)$, the average of S on K, for $K \in \mathcal{K}_+$.

THEOREM 2.2. *If S is any regular set function on $\mathcal{K}$ and $\{A_\alpha\} \in \mathcal{A}$,*

$$\lim_\alpha M_S(A_\alpha) = \inf\{M_S(K) : K \in \mathcal{K}_c \cap \mathcal{K}_+\}$$
$$= \inf\{M_S(K) : K \in \mathcal{K}_c \cap \mathcal{K}_0\}.$$

Now since $\mathcal{A} \subsetneqq \mathcal{F}$ (for $\mathcal{A} \neq \emptyset$ and G nondiscrete) the question remains as to whether Theorem 2.2 (or some modification) remains true for $\{A_\alpha\} \in \mathcal{F}$ without further restriction on S. The following example due to Banach **[1]**, after simple reworking, shows this is not the case:

EXAMPLE 2.3. There exists a finitely additive translation invariance measure m_0 on the Borel sets of the circle — realized as $[0, 1)$ — and a Borel subset N of Lebesgue measure zero, $\lambda(N) = 0$, for which $m_0(N) = m_0([0, 1)) = 1$.

COROLLARY 2.4. *There exists a finitely additive translation invariant measure* m *on* $\mathcal{K} = \mathcal{K}(R)$ *such that* $m(N) = m([0, 1)) = 1$ *for a Lebesgue null Borel set* $N \subseteq [0, 1)$.

Proof. For $A \in \mathcal{K}$, define $m(A) \doteqdot \Sigma\{m_0(\{(A + n) \cap [0, 1)\})$: $n \in Z\}$, m_0 as in Example 2.3. That m has the desired properties is readily verified, where N is the set referred to in Example 2.3.

PROPOSITION 2.5. *There is a regular set function* S *on* $\mathcal{K} = \mathcal{K}(R)$ *such that given any* $l \leqq -1$ *there exists a sequence* $\{C_n\} = \{C_n(l)\}$ *in* $\mathcal{F}$ *satisfying* $\lim_n M_S(C_n) = l$. *Moreover, it follows that we may also take* $l = -\infty$ *above, and in general* $M_S(C_n)$ *need not tend to a limit at all for appropriate* $\{C_n\} \in \mathcal{F}$.

Proof. Of course $A_n = [0, n]$ is a full sequence in R. Consequently by Proposition 1.2 (v) if $B_n = B_n(l) = \cup\{i + N\colon\ i = 1, 2, \cdots, [-ln]\}$, where $[x]$ is the greatest integer $\leqq x$ and N is from Corollary 2.4, $C_n = C_n(l) = A_n \cup B_n$ is also a full sequence since B_n is a (Haar) null set. Now if $S \doteqdot -m$ where m is from Corollary 2.4 it is immediate that $S(C_n) = -[-ln]$, $\lambda(C_n) = n$, and consequently $M_S(C_n) = -[-ln]/n \to l$ as asserted. The final sentence of Proposition 2.5 follows easily from the first.

Whether an analogue of Proposition 2.5 is valid in a general amenable, unimodular, nondiscrete G has not been investigated but should follow from known methods of creating translation invariant functionals in amenable groups. At any rate, we see that in general full nets do not "average" a general regular S and some restriction is necessary. Our basic result is the following sufficient condition, and a corollary:

THEOREM 2.6. *If* S *is regular and continuous at* $\varnothing$, *then for any* $\{A_\alpha\} \in \mathcal{F}$ *we have*

$$(1)\qquad \lim_\alpha M_S(A_\alpha) = \inf\{M_S(K)\colon K \in \mathcal{K}_+\} = \inf\{M_S(K)\colon K \in \mathcal{K}_0 \cap \mathcal{K}_c\}.$$

COROLLARY 2.7. *If S is regular and* $\inf\{M_S(K)\colon K \in \mathcal{K}_+\}$ *is finite, then equation* (1) *above is valid for any* $\{A_\alpha\} \in \mathcal{F}$.

The proof of Theorem 2.6 proceeds by a series of lemmas:

LEMMA 2.8. *Let X be a set, Σ a field of subsets of X, and μ a finitely additive nonnegative measure on Σ normalized by $\mu(X) = 1$. Then for any $T_i \in \Sigma$, $1 \leqq i \leqq k$,*

(i) *there exists a sequence* $\{e_n\} \subseteq X$ (not necessarily distinct) *such that for* $1 \leqq i \leqq k$,

$$\mu_N(T_i) \doteqdot N^{-1}\Sigma\{\chi_{T_i}(e_n)\colon n \leqq N\} \to \mu(T_i) \quad \textit{as} \quad N \to +\infty.$$

(ii) *Moreover if μ gives all points measure* 0, *the terms e_n in* (i) *may be chosen to all be distinct, and in fact we may find a countably infinite family of such sequences $\{e_n^{(j)}\}$ satisfying* (i) *with $e_n^{(j)} = e_m^{(r)}$ iff $n = m$ and $j = r$ and such that $\mu_N(T_i)$ is the same when computed on all sequences $\{e_n^{(j)}\}$.*

Comment. The proof is in fact constructive, informally speaking.

Proof. Partition $\cup\{T_i\colon 1 \leqq i \leqq k\}$ into $2^k - 1$ disjoint subsets $\cap\{C_i\colon 1 \leqq i \leqq k\}$ where $C_i \in \{T_i, T_i'\}$ and not all $C_i = T_i'$. Ignoring those subsets which have measure 0, list the remaining in some order: $E_1, \cdots, E_r$ ($r \leqq 2^k - 1$). Therefore all E_j are pairwise disjoint, of positive μ measure, and any T_i is the union of an appropriate subfamily of the E_j and a set of μ measure 0. Now to the $\{E_j\}$ correspond disjoint intervals I_j of $[0, 1)$ of the same (Lebesgue) measure, e.g. $E_1 \leftrightarrow [0, \mu(E_1))$ and

$$E_j \leftrightarrow [\mu(\cup\{E_i\colon i < j\}), \mu(\cup\{E_i\colon i \leqq j\})) \quad \text{for} \quad 1 < j \leqq r.$$

Note that $\mu(\cup\{E_i\colon 1 \leqq i \leqq r\}) = \mu(\cup\{T_i\colon 1 \leqq i \leqq k\})$, and consequently if the I_j do not cover $[0, 1)$ we must have $X - \cup\{T_i\colon 1 \leqq i \leqq k\}$ of positive μ measure (and therefore $\neq \emptyset$). Now let $\{x_n\}$ be a sequence in $[0, 1)$ uniformly distributed in the classical sense, e.g. the fractional part of $n\alpha$ for any irrational α. The sequence $\{e_n\}$ is chosen as follows: if $x_n \in I_j$ choose e_n to be any point in E_j, and if x_n is not in any I_j (implying the I_j do not cover $[0, 1)$) choose e_n to be any point in $X - \cup\{T_i\colon 1 \leqq i \leqq k\} \neq \emptyset$. It is immediate that any sequence $\{e_n\}$ chosen in this fashion satisfies (i). Part (ii) follows since any set of positive measure must now contain an infinity of points and consequently may be written as the disjoint union of a countable infinity of infinite sets. Now it is not difficult to see that if the selection of the $\{e_n\}$ as above is done recursively we need not repeat any term, and moreover, by the preceding sentence, in a countable infinity of ways with no term used twice in the entire array. Finally, that $\mu_N(T_i)$ is independent of the sequence is trivial from the construction since it only depends on the distribution of $\{x_1, \cdots, x_N\}$ in $[0, 1)$.

LEMMA 2.9. *Using the notation of Lemma* 2.8, *let* Σ' *be a countable subfamily of* Σ. *Then*

(i) *there exists a sequence* $\{e_n\} \subseteq X$ (not necessarily distinct) *such that* $\mu_N(T) \to \mu(T)$ *as* $N \to +\infty$ *for all* $T \in \Sigma'$;

(ii) *moreover if* μ *gives all points measure* 0, *the terms* e_n *in* (i) *may be chosen to all be distinct.*

Comment. The full analog of Lemma 2.8 (ii) is in fact seen to be valid but is of no use here.

Proof. List the sets in Σ' sequentially as $\{T_i\}$, and by Lemma 2.8 for each $k = 1, 2, \cdots$. Choose a sequence $\{e_{n,k}\}$ such that $\mu_N(T_i) \to \mu(T_i)$ for $1 \leqq i \leqq k$ (computed with respect to $\{e_{n,k}\}$). Therefore there is an index N_k such that for $1 \leqq i \leqq k$,

$$\left| \mu(T_i) - N_k^{-1} \sum \{\chi_{T_i}(e_{n,k}) \colon n \leqq N_k\} \right| < \frac{1}{k} . \tag{2}$$

Next let $\{w_k\}$ be any sequence tending to $+\infty$, and let $\{M_k\}$ be any sequence of positive integers chosen such that $\Sigma_{k<n} M_k N_k \geqq w_n N_n$ $(n = 2, 3, \cdots)$ with N_k as in (2). This may be arranged by letting M_k tend to $+\infty$ sufficiently rapidly, e.g. if $M_{n-1} \geqq w_n N_n / N_{n-1}$ $(n = 2, 3, \cdots)$. Then the desired sequence $\{e_n\}$ is obtained by successively running through the first N_1 terms of $\{e_{n,1}\}$ M_1 times, the first N_2 terms of $\{e_{n,2}\}$ M_2 times, etc. The verification of (i) is now a simple exercise in Cesaro means whereas (ii) follows from (ii) of Lemma 2.8 upon modifying the construction above appropriately.

LEMMA 2.10. *Let* $K \in \mathcal{K}_0$ *and* $B \in \mathcal{K}$. *Then there is a countable subfamily* $\mathcal{K}' \subseteq \mathcal{K}_0$ *such that given any* $b \in B$, *and* $\epsilon > 0$ *there corresponds a* $T = T(b, \epsilon) \in \mathcal{K}'$ *such that* $T \subseteq Kb$ *and* $\lambda(Kb - T) < \epsilon$.

Proof. Fix a positive integer n, and by [3, 1.7 (i)] choose $U = U_n \in \mathcal{K}_0 \cap \mathcal{K}_c$ such that $\bar{U} \subseteq K$ and $\lambda(K - U) < 1/n$. As in the proof of [3, 1.5], choose $O = O_n$ an open symmetric neighborhood of the identity e such that $UO^2 \subseteq K$. Since $\bar{B} \subseteq OB$ is compact there is a finite cover $\cup\{Ob_i \colon 1 \leqq i \leqq k\}$ of B, $b_i \in B$ for $1 \leqq i \leqq k$. Consider the sets $UOb_1, \cdots, UOb_k$: for $b \in B$, $b_i \in Ob$ iff $b \in Ob_i$ and consequently $Ub \subseteq UOb_i \subseteq UO^2 b \subseteq Kb$ implying $\lambda(Kb - UOb_i) \leqq \lambda(Kb - Ub) = \lambda(K - U) < 1/n$, and thus the finite family UOb_i, $1 \leqq i \leqq k$, inner-approximates all Kb, $b \in B$, to within $1/n$ in measure. The family obtained by taking all the UOb_i for $n = 1, 2, \cdots$ is the required countable family $\mathcal{K}'$.

LEMMA 2.11. *Let $\{f_n\}$ be a uniformly bounded sequence of μ measurable functions on a finite measure space (X, μ) such that*:

$$\text{(i)} \quad \int f_n \, d\mu = 0, \text{ all } n, \quad \text{(ii)} \ \underline{\lim}_n f_n \geqq 0 \ \mu - \text{a.e.};$$

then

$$\text{(1)} \quad \int |f_n| \, d\mu \to 0, \quad \text{(2)} \ |f_n| \to 0 \text{ in measure on } X.$$

Proof. Fix $\epsilon > 0$ and let $E_n = E_n(\epsilon)$ be defined by $E_n = \{x \in X : f_k(x) > -\epsilon \text{ for all } k \geqq n\}$. Then since $\underline{\lim}_n f_n(x) \geqq 0$ μ – a.e. we must have $\mu(E_n) \uparrow \mu(X)$ (or $\mu(E'_n) \downarrow 0$) by Fatou. Also if $X_n^- \doteqdot \{x \in X : f_n(x) < 0\}$ we obtain

$$0 \geqq \int_{X_n^-} f_n \, d\mu \geqq -\epsilon \mu(E_n) - M \mu(E'_n)$$

where M is a uniform bound for all $|f_n|$. Consequently $\underline{\lim}_n \int_{X_n^-} f_n \, d\mu \geqq -\epsilon \mu(X)$ is valid for all $\epsilon > 0$, and therefore $\underline{\lim}_n \int_{X_n^-} f_n \, d\mu \geqq 0$. But this implies $\int_{X_n^-} f_n \, d\mu \to 0$ and also $\int_{X_n^+} f_n \, d\mu = -\int_{X_n^-} f_n \, d\mu \to 0$ and thus $\int_X |f_n| \, d\mu \to 0$. Of course (2) is an immediate consequence of (1).

LEMMA 2.12. *Let $K \in \mathcal{K}_0$ and $A \in \mathcal{K}_+$. Then there is a sequence $\{e_n\} \subseteq A$ such that if for $T \subseteq A$ we set*

$$\mu_N(T) \doteqdot N^{-1} \sum \{\chi_T(e_n) : n \leqq N\} \quad \text{and} \quad g_N(b) = \mu_N(K^{-1}b \cap A)$$

then

$$g_N(b) \to \lambda(A)^{-1} \lambda(K^{-1}b \cap A) \text{ in measure on } KA.$$

Proof. Upon applying Lemma 2.10 with K replaced by K^{-1} and B by KA we obtain a countable $\mathcal{K}' \subseteq \mathcal{K}_0$ such that given any $b \in KA$ and $\epsilon > 0$ there is a set $T \in \mathcal{K}'$ such that $T \subseteq K^{-1}b$ and $\lambda(K^{-1}b - T) < \epsilon$. Now set $\Sigma' = \{T \cap A : T \in \mathcal{K}'\}$. Then since $\lambda((K^{-1}b \cap A) - T \cap A) = \lambda((K^{-1}b - T) \cap A) \leqq \lambda(K^{-1}b - T) < \epsilon$, we see that the countable family Σ' approximates $\{K^{-1}b \cap A : b \in KA\}$ from within arbitrarily closely in measure. Now apply Lemma 2.9 (with $\mu = \lambda(A)^{-1}\lambda$, $X = A$, $\Sigma = \{K \cap A : K \in \mathcal{K}\}$) and let $\{e_n\} \subseteq A$ be a sequence such that $\mu_N(T) \to \mu(T) =$

$\lambda(A)^{-1}\lambda(T)$ for all $T \in \Sigma'$. Since $g_N(b) = \mu_N(K^{-1}b \cap A) \geqq \mu_N(T)$ for $T \subseteq K^{-1}b \cap A$, the inner approximating property of Σ' implies $\underline{\lim}_N g_N(b) \geqq \lambda(A)^{-1}\lambda(K^{-1}b \cap A)$ for all b in KA. Now set $f_N(b) \doteqdot g_N(b) - \lambda(A)^{-1}\lambda(K^{-1}b \cap A)$, and apply Lemma 2.11 (with $X = KA$, $\mu = \lambda \mid KA$): first trivially $|f_N| \leqq 1$, and hypothesis (ii) has just been verified above. It remains to verify (i):

$$\int_{KA} f_N(b)\,d\lambda(b) = \int_{KA} g_N(b)\,d\lambda(b) - \int_{KA} \lambda(A)^{-1}\lambda(K^{-1}b \cap A)\,d\lambda(b)$$

$$= N^{-1}\sum_{n \leqq N}\int_{KA} \chi_{K^{-1}b\cap A}(e_n)\,d\lambda(b) - \lambda(A)^{-1}\int_{KA}\left\{\int_A \chi_{K^{-1}b}(a)\,d\lambda(a)\right\} d\lambda(b)$$

$$= N^{-1}\sum_{n \leqq N}\int_{KA} \chi_{Ke_n}(b)\,d\lambda(b) - \lambda(A)^{-1}\int_{A}\left\{\int_{KA} \chi_{Ka}(b)\,d\lambda(b)\right\} d\lambda(a)$$

(since $e_n \in K^{-1}b \cap A$ iff $b \in Ke_n$ and $a \in K^{-1}b$ iff $b \in Ka$)

$$= N^{-1}\sum_{n \leqq N}\lambda(Ke_n \cap KA) - \lambda(A)^{-1}\int_A \lambda(Ka \cap KA)\,d\lambda(a)$$

$$= N^{-1}\sum_{n \leqq N}\lambda(Ke_n) - \lambda(A)^{-1}\int_A \lambda(Ka)\,d\lambda(a) = \lambda(K) - \lambda(K) = 0.$$

We now conclude that $|f_N| \to 0$ in measure on KA, which is the assertion of Lemma 2.12.

We are now in a position to prove the following strengthened form of [**3**, 2.3].

PROPOSITION 2.13 (The Fundamental Inequality, Strong Form): *If $A \in \mathcal{K}_+$ and $K \in \mathcal{K}_0$ and S is continuous at $\varnothing$, then*

$$\lambda(A)^{-1}S(KA) \leqq \lambda(K^{-1})^{-1}S(K).$$

Proof. Fix $K \in \mathcal{K}_0$ and $A \in \mathcal{K}_+$ and let $\{e_n\} \subseteq A$ be the sequence described in Lemma 2.12. We consider

$$N^{-1}\sum_{j \leqq N} S(Ke_j) = S(K);$$

by Proposition 1.6, since $\cup\{Ke_j : j \leqq N\} \subseteq KA$,

$$N^{-1}\sum_{j \leqq N} S(I_j) \leqq S(K)$$

where

$$I_j = I_j(N) = \left\{b \in KA: \sum_{n \leq N} \chi_{Ke_n}(b) \geqq j\right\}$$

$$= \left\{b \in KA: \sum_{n \leq N} \chi_{K^{-1}b \cap A}(e_n) \geqq j\right\}$$

$$= \{b \in KA: g_N(b) \geqq j/N\},$$

using the notation of Lemma 2.12. But by the same Lemma, $g_N(b) \to \lambda(A)^{-1}\lambda(K^{-1}b \cap A) \leqq \lambda(A)^{-1}\lambda(K^{-1})$ in measure on KA, and consequently for any $\epsilon > 0$ we must have

$$(3) \quad \lambda(\{b \in KA: g_N(b) \geqq \lambda(A)^{-1}\lambda(K^{-1}) + \epsilon\}) \to 0 \quad \text{as} \quad N \to +\infty,$$

i.e. for $j/N \geqq \lambda(A)^{-1}\lambda(K^{-1}) + \epsilon$, we have $\lambda(I_j) \to 0$ as $N \to +\infty$ (uniformly for $j \geqq N(\lambda(A)^{-1}\lambda(K^{-1}) + \epsilon)$. Now write

$$N^{-1} \sum_{j \leq N} S(I_j) = N^{-1} \sum \{S(I_j): j/N < \lambda(A)^{-1}\lambda(K^{-1}) + \epsilon, j \leqq N\}$$

$$+ N^{-1} \sum \{S(I_j): j/N \geqq \lambda(A)^{-1}\lambda(K^{-1}) + \epsilon, j \leqq N\}.$$

Since $I_j \subseteq KA$ always and consequently $S(KA) \leqq S(I_j)$ always, the first term on the right is bounded below by $N^{-1}[N(\lambda(A)^{-1}\lambda(K^{-1}) + \epsilon)]S(KA)$ where $[X]$ is the greatest integer less than X. On the other hand, since the I_j are nested the second sum is trivially bounded below by $S(I_j)$ where $j = [N(\lambda(A)^{-1}\lambda(K^{-1}) + \epsilon)]$. Therefore

$$(4) \quad N^{-1}[N(\lambda(A)^{-1}\lambda(K^{-1}) + \epsilon]S(KA) + S(I_{[N(\lambda(A)^{-1}\lambda(K^{-1}) + \epsilon)]}) \leqq S(K)$$

for all positive integers N and $\epsilon > 0$. But for fixed $\epsilon > 0$ upon letting $N \to +\infty$ in (4) we obtain

$$(\lambda(A)^{-1}\lambda(K^{-1}) + \epsilon)S(KA) \leqq S(K)$$

since S is continuous at $\varnothing$ and $\lambda(I_{[N(\lambda(A)^{-1}\lambda(K^{-1}) + \epsilon)]}) \to 0$ by (3). The proposition follows upon letting $\epsilon \downarrow 0$.

Theorem 2.6 follows immediately from Proposition 2.13 since an analogue of [**3**, 2.4] may now be proved with $\mathcal{K}_0$ instead of $\mathcal{K}_c \cap \mathcal{K}_0$ in the last inequality. Corollary 2.7 is immediate since $\inf\{M_S(K): K \in \mathcal{K}_+\}$ finite trivially implies S must be continuous at $\varnothing$. Note in particular that the Corollary shows that if $\inf\{M_S(K): K \in \mathcal{K}_+\}$ is finite then so is $\inf\{M_S(K): K \in \mathcal{K}_0 \cap \mathcal{K}_c\}$ (this much is trivial) *and* they are equal in case $\mathcal{F} \neq \varnothing$, i.e. if G is unimodular and amenable.

Moreover, we have the following partial converse to Theorem 2.6:

THEOREM 2.14. *If S is regular and for all $\{A_\alpha\} \in \mathcal{F} \neq \emptyset$ $\lim_\alpha M_S(A_\alpha)$ exists* (possibly depending on $\{A_\alpha\}$), *then the limit is in fact independent of $\{A_\alpha\} \in \mathcal{F}$ and equals* $\inf\{M_S(K): K \in \mathcal{K}_+\} = \inf\{M_S(K): K \in \mathcal{K}_0 \cap \mathcal{K}_c\}$. *Moreover* (assuming $\mathcal{F} \neq \emptyset$) *this is the case iff*

(i) $\inf\{M_S(K): K \in \mathcal{K}_0 \cap \mathcal{K}_c\} = -\infty$ *or*

(ii) *S is continuous at $\emptyset$.*

Proof. From the comment immediately preceding the statement of Theorem 2.14, we see that

$$\inf\{M_S(K): K \in \mathcal{K}_+\} < \inf\{M_S(K): K \in \mathcal{K}_0 \cap \mathcal{K}_c\}$$

iff the former is $-\infty$ and the latter is finite. If this is the case let $\{K_n\} \subseteq \mathcal{K}_+$ be chosen such that $M_S(K_n) = \lambda(K_n)^{-1}S(K_n) \to -\infty$. Next, fix any $\{A_\alpha\} \in \mathcal{A} \subseteq \mathcal{F}$, implying $\lim_\alpha M_S(A_\alpha) = \inf\{M_S(K): K \in \mathcal{K}_0 \cap \mathcal{K}_c\}$ $(> -\infty)$ by Theorem 2.2. Now since $[A_\alpha]_K \neq \emptyset$ implies $\lambda(A_\alpha) \geqq \lambda(K)$ it follows that $\lim_\alpha \lambda(A_\alpha) = +\infty$. We now wish to expand $\{A_\alpha\}$ à-la Proposition 1.2 (v) by $\{B_\alpha\}$ where each B_α is an appropriately chosen disjoint union of right translates of a set K_n for "large" n while $\lambda(B_\alpha)$ is of smaller order than $\lambda(A_\alpha)$ whereas $S(B_\alpha)$ is of larger order than $S(A_\alpha)$. The technical details follow, where for simplicity and without loss of generality we assume $M_S(K_n) < -n$ and $\lambda(K_n) \uparrow +\infty$ (if Kg_i, $1 \leqq i \leqq r$, are disjoint

$$M_S(\cup\{Kg_i: 1 \leqq i \leqq r\}) = (r\lambda(K))^{-1}S(\cup\{Kg_i: 1 \leqq i \leqq r\})$$
$$\leqq (r\lambda(K))^{-1}\left(\sum\{S(Kg_i): 1 \leqq i \leqq r\}\right) = \lambda(K)^{-1}S(K) = M_S(K)).$$

Now consider the real sequence $\{2n^{3/2}\lambda(K_n)\}$ which tends monotonically to $+\infty$, and choose α_0 such that $\alpha > \alpha_0$ implies $\lambda(A_\alpha) \geqq 2\lambda(K_1)$. Then for each $\alpha > \alpha_0$ define the positive integer $n = n(\alpha)$ by

$$2n^{3/2}\lambda(K_n) \leqq \lambda(A_\alpha) < 2(n+1)^{3/2}\lambda(K_{n+1})$$

and note in particular that $\lim_\alpha n(\alpha) = +\infty$ since $\lim_\alpha \lambda(A_\alpha) = +\infty$. For $\alpha > \alpha_0$ then choose B_α to consist of N_α disjoint right translates of K_n where $N_\alpha = [\lambda(A_\alpha)/n^{1/2}\lambda(K_n)] \geqq 2n \geqq 2$. For completeness take $B_\alpha = \emptyset$ for all other α. Consequently, for $\alpha > \alpha_0$,

$$\lambda(B_\alpha) = N_\alpha\lambda(K_n) = [\lambda(A_\alpha)/n^{1/2}\lambda(K_n)]\lambda(K_n)$$
$$\leqq (\lambda(A_\alpha)/n^{1/2}\lambda(K_n))\lambda(K_n) = \lambda(A_\alpha)/n^{1/2},$$

and therefore $\lim_\alpha \lambda(A_\alpha)^{-1}\lambda(B_\alpha)=0$. Thus by Proposition 1.2 (v) we have $\{C_\alpha\}\in\mathcal{F}$ where $C_\alpha = A_\alpha \cup B_\alpha$ for all α. Also by the subadditivity of S, for $\alpha > \alpha_0$

$$S(B_\alpha)\leqq N_\alpha S(K_n)\leqq \frac{1}{2}(\lambda(A_\alpha)/n^{1/2})(\lambda(K_n)^{-1}S(K_n)),$$

since $[x]\geqq \frac{1}{2}x$ for $[x]\geqq 2$, and consequently

$$\begin{aligned} M_S(C_\alpha) &= \lambda(C_\alpha)^{-1}S(C_\alpha)\leqq \lambda(C_\alpha)^{-1}S(B_\alpha)\\ &= (\lambda(C_\alpha)^{-1}\lambda(A_\alpha))(\lambda(A_\alpha)^{-1}S(B_\alpha))\\ &\leqq (\lambda(C_\alpha)^{-1}\lambda(A_\alpha))(M_S(K_n)/2n^{1/2}) < \lambda(C_\alpha)^{-1}\lambda(A_\alpha)(-n^{1/2}/2), \end{aligned}$$

since $M_s(K_n) < -n$. Therefore $\lim M_s(C_\alpha) = -\infty$ since $\lambda(C_\alpha)^{-1}\lambda(A_\alpha) \to 1$ and $n = n(\alpha)\to +\infty$ as α "gets large". It is now easy to combine $\{A_\alpha\}$ and $\{C_\alpha\}$ into a single net in $\mathcal{F}$ on which M_S does not converge by stipulating that A_α (or C_α) is "further out" than $C_{\alpha'}$ (and $A_{\alpha'}$) iff $\alpha > \alpha'$.

We have shown that if $\inf M_S$ is different on $\mathcal{K}_+$ and $\mathcal{K}_0 \cap \mathcal{K}_c$ then $\lim_\alpha M_S(A_\alpha)$ does not necessarily exist for $\{A_\alpha\}\in\mathcal{F}$. But conversely, by [**3**, 2.4], if they are equal then $\lim_\alpha M_S(A_\alpha)$ always equals this common value for $\{A_\alpha\}\in\mathcal{F}$. Thus $\lim_\alpha M_S(A_\alpha)$ exists for all $\{A_\alpha\}\in\mathcal{F}$ iff $\inf M_S$ is equal on $\mathcal{K}_+$ and $\mathcal{K}_0\cap\mathcal{K}_c$, and in this case $\lim_\alpha M_S(A_\alpha)$ always equals the common value.

The last assertion is now also clear since (i) or (ii) imply $\inf M_S$ is the same on $\mathcal{K}_+$ and $\mathcal{K}_0\cap\mathcal{K}_c$ (see Theorem 2.6 for (ii)). Conversely, if the common value is $-\infty$ then (i) is true whereas if they are both finite and equal (ii) is trivially true.

We conclude this section with the following:

PROPOSITION 2.15.

1. (strong converse to Proposition 2.13): *If* $\lambda(A_0)^{-1}S(KA_0)\leqq \lambda(K^{-1})^{-1}S(K)$ *for all* $K\in\mathcal{K}_0$ *and one* $A_0\in\mathcal{K}_+$ *then S is continuous at* $\varnothing$.

2. *S is continuous at $\varnothing$ iff* $\{K_n\}\subseteq\mathcal{K}_+$ *and* $K_n\supseteq K_{n+1}$ *always implies* $\inf M_S(K_n) > -\infty$.

Comment. The second assertion essentially shows that S is continuous at $\varnothing$ iff a "Lipschitz condition" holds at $\varnothing$.

Proof. 1. Choose $\{K_n\}\subset\mathcal{K}$ such that $K_n\supseteq K_{n+1}$ and $\lambda(K_n)\to 0$. By the regularity of λ we may find $\{U_n\}\subset\mathcal{K}_0$ such that $K_n\subseteq U_n$ and $\lambda(U_n - K_n)\to 0$. We may assume $U_n\supseteq U_{n+1}$ also by considering $U_n^* = \cap\{U_k\colon k\leqq n\}$ if necessary. Clearly $\lambda(U_n)\to 0$ and $S(U_n)\leqq S(K_n)$, so if we initially assume $\{K_n\}\subset\mathcal{K}_0$ and show $S(K_n)\to S(\varnothing)=0$ then the general case follows. Upon taking $K = K_n$ in the inequality,

$$\lambda(A_0)^{-1}S(K_1A_0) \leqq \lambda(A_0)^{-1}S(K_nA_0) \leqq \lambda(K_n^{-1})^{-1}S(K_n),$$

since $K_n \subseteq K_1$ and S is monotonic. Since $K_n \subseteq K_1 \subseteq \bar{K}_1$ (compact), the modular function Δ is bounded away from 0 and $+\infty$ on K_n uniformly in n and consequently

$$\lambda(K_n^{-1})^{-1} \geqq C\lambda(K_n)^{-1}$$

for an appropriate $C > 0$ independent of n. Consequently (since $S(K_n) \leqq 0$)

$$\lambda(A_0)^{-1}S(K_1A_0) \leqq C\lambda(K_n)^{-1}S(K_n),$$

or

$$\lambda(A_0)^{-1}S(K_1A_0)\lambda(K_n)/C \leqq S(K_n) \leqq 0.$$

But this clearly implies $S(K_n) \rightarrow 0$ since $\lambda(K_n) \rightarrow 0$ with n (and the other terms on the left are independent of n) and we are done.

2. Assume S is continuous at $\varnothing$ and $\{K_n\} \subseteq \mathcal{K}_+$ with $K_n \supseteq K_{n+1}$. As in the proof of 1 take $\{U_n\} \subseteq \mathcal{K}_0$ with $U_n \supseteq K_n$ and $\lambda(U_n) < 2\lambda(K_n)$ and without loss of generality assume $U_n \supseteq U_{n+1}$. Then

$$M_s(U_n) = \lambda(U_n)^{-1}S(U_n) \leqq \lambda(U_n)^{-1}S(K_n) < \tfrac{1}{2}\,\lambda(K_n)^{-1}S(K_n) = \tfrac{1}{2}M_s(K_n).$$

Consequently $M_S(U_n)$ bounded below implies $M_S(K_n)$ bounded below. Therefore we need only verify (2) for $\{K_n\} \subset \mathcal{K}_0$. The fundamental inequality implies, upon fixing any $A = A_0 \in \mathcal{K}_+$,

$$\begin{aligned}\lambda(A_0)^{-1}S(K_1A_0) \leqq \lambda(A_0)^{-1}S(K_nA_0) &\leqq \lambda(K_n^{-1})^{-1}S(K_n)\\ &\leqq C\lambda(K_n)^{-1}S(K_n) = CM_S(K_n)\end{aligned}$$

for some $C > 0$ independent of n by the same argument as in the proof of (1). Thus $\lambda(A_0)^{-1}S(K_1A_0)/C \leqq \inf M_S(K_n)$, and we are done as the converse implication is trivial.

3. Regular set functions S and their relation to summing sequences. The question arises as to how useful the Følner sequences (or nets) are with respect to averaging regular set functions. One might initially suspect that they would be rather successful but the following simple example shows otherwise:

EXAMPLE 3.1. It is readily verified that $S(E) = -\lambda(E^0)$ is regular where E^0 is the interior of E. Moreover, it is easy to see that $\inf\{M_S(K): K \in \mathcal{K}_+\} = \inf\{M_S(K): K \in \mathcal{K}_0 \cap \mathcal{K}_c\} = -1$ and S is continuous at $\varnothing$. However, if we take $G = R$ and consider the strong summing sequence $\{A_n\}$ where

$$A_n = \begin{cases} [0, n] \quad , & n \text{ even} \\ [0, n] \cap I, & n \text{ odd, where } I \text{ denotes all irrational numbers.} \end{cases}$$

Then $M_S(A_n)$ alternates between 0 and 1 and, of course, has no limit as $n \to +\infty$.

Consequently, even if S is continuous at $\varnothing$, strong Følner summing sequences need not average S. This appears to be since S may be rather "discontinuous" with respect to λ whereas the Følner condition is not quite so sensitive. Nevertheless, in Theorem 3.3 we shall state conditions on S which are sufficient for the utilization of Følner-like sequences in the averaging of S. For technical simplicity we assume G is σ-compact and consequently consider sequences rather than nets [**2**, Theorem 4 and Proposition 1]. Before proceeding to the theorem we need the following:

LEMMA 3.2. (i) *If K and A are in $\mathcal{K}_+$, $\delta > 0$, and*

$$\lambda(KA \,\Delta\, A) < \alpha\lambda(A),$$

there exists $A^* = A(\delta, \alpha) \in \mathcal{K}$, $A^* \subseteq A$ such that $\lambda(Ka - A) < \delta$, for all $a \in A^*$ and $\lambda(A^*) \geqq \lambda(A)(1 - (\alpha/\delta)\lambda(K^{-1}))$.

(ii) *Also, if $K \in \mathcal{K}_+$ and $\delta > 0$ are fixed and $\{A_n^{-1}\}$ is a* (right) *strong Følner summing sequence in the unimodular group G, there exists $\{A_n^*\} \subseteq \mathcal{K}$, $A_n^* \subseteq A_n$ such that $\lambda(A_n)^{-1}\lambda(A_n^*) \to 1$ and*

$$\lambda(Ka - A_n) < \delta \quad \textit{for all} \quad a \in A_n^*.$$

Proof. Let $E = \{a \in A : \lambda(Ka - A) \geqq \delta\}$. E is clearly measurable and moreover

$$\int_E \lambda(Ka - A)\, d\lambda(a) = \int_E \left\{\int_{KA-A} \chi_{Ka}(b)\, d\lambda(b)\right\} d\lambda(a)$$

$$= \int_{KA-A} \left\{\int_E \chi_{Ka}(b)\, d\lambda(a)\right\} d\lambda(b)$$

$$= \int_{KA-A} \left\{\int_E \chi_{K^{-1}b}(a)\, d\lambda(a)\right\} d\lambda(b)$$

$$\leqq \int_{KA-A} \lambda(K^{-1}b)\, d\lambda(b) = \lambda(KA - A)\lambda(K^{-1}) \leqq \lambda(KA \,\Delta\, A)\lambda(K^{-1})$$

$$< \alpha\lambda(A)\lambda(K^{-1}).$$

But clearly

$$\int_E \lambda(Ka - A)\,d\lambda(a) \geqq \int_E \delta\, d\lambda(a) = \delta\lambda(E),$$

and consequently

$$\lambda(E) \leqq (\alpha/\delta)\lambda(K^{-1})\lambda(A),$$

implying if $A^* \doteqdot A - E$, $\lambda(Ka - A) < \delta$ for $a \in A^*$ and

$$\lambda(A^*) = \lambda(A) - \lambda(E) \geqq (1 - (\alpha/\delta)\lambda(K^{-1}))\lambda(A),$$

as needed. Statement (ii) follows upon taking $A = A_n$ and choosing $A^* = A_n^*$ as indicated in (i) since for any K in $\mathcal{K}_+$

$$\lambda(KA_n \Delta A_n) < \alpha_n \lambda(A_n)$$

where $\alpha_n \to 0$ by inversion invariance and the defining property of $\{A_n^{-1}\}$ and consequently $\lambda(A_n)^{-1}\lambda(A_n^*) \geqq (\alpha_n/\delta)\lambda(K^{-1})) \to 1$.

We are now ready to prove:

THEOREM 3.3. *Let G be unimodular and amenable, S be a regular set function which is continuous at $\varnothing$ and upper continuous on $\mathcal{K}$, and $\{A_n\} \subseteq \mathcal{K}_+$ satisfy $\lambda(A_n)^{-1}\lambda(gA_n \Delta A_n) \to 0$ for all $g \in G$* (i.e. $\{A_n^{-1}\}$ *is a* (right) weak Følner summing sequence in G). *Then*

$$\lim M_S(A_n) = \inf\{M_S(K): K \in \mathcal{K}_+\} = \inf\{M_S(K): K \in \mathcal{K}_0 \cap \mathcal{K}_c\}.$$

Proof. The second equality follows from continuity at $\varnothing$. We first prove the result for a (right) strong Følner summing sequence $\{A_n^{-1}\}$ in G, i.e.

$$\lambda(A_n)^{-1}\lambda(KA_n \Delta A_n) = \lambda(A_n^{-1})^{-1}\lambda(A_n^{-1}K^{-1} \Delta A_n^{-1}) \to 0$$

for all $K \in \mathcal{K}$, $K \neq \varnothing$, where unimodularity has been used to obtain inversion invariance of λ. Fix $K \in \mathcal{K}_+$ and $\epsilon > 0$. Since S is upper continuous at K by Proposition 1.5 (i) there is a $\delta = \delta(\epsilon, K) > 0$ such that $K^* \in \mathcal{K}$, $K^* \subseteq K$, and $\lambda(K - K^*) < \delta$ implies $S(K^*) < S(K) + \epsilon$. Now for this K and $\delta > 0$, apply Lemma 3.2 (ii) to obtain the sequence $\{A_n^*\}$. First, since

$$\lambda(Ka - A_n) = \lambda(Ka - (Ka \cap A_n)) = \lambda(K - (Ka \cap A_n)a^{-1}) < \delta$$

for all $a \in A_n^*$ (setting $K^* = (Ka \cap A_n)a^{-1}$) we have

$$S(Ka \cap A_n) = S((Ka \cap A_n)a^{-1}) < S(K) + \epsilon.$$

Next let $\{e_n\}$ be the sequence in Lemma 2.12 corresponding to K and $A = A_n$. Moreover, in light of Lemma 2.9 and the proof of Lemma 2.12, the $\{e_n\}$ also may be chosen so that $\mu_n(A_n^*) \to \mu(A_n) = \lambda(A_n^*)^{-1}\lambda(A_n^*)$ (in the notation of Lemma 2.8). Following the proof of Proposition 2.13 (and using similar notation),

$$\begin{aligned} N^{-1}\sum_{j\leqq N} S(I_j) &\leqq N^{-1}\sum_{j\leqq N} S(Ke_j \cap A_n) \\ &\leqq N^{-1}\sum_{j\leqq N,\, e_j\in A_n^*} S(Ke_j\cap A_n) \leqq N^{-1}\sum_{j\leqq N,\, e_j\in A_n^*} (S(K)+\epsilon) \\ &= \mu_N(A_n^*)(S(K)+\epsilon), \end{aligned}$$

where

$$\begin{aligned} I_j &= \left\{g\in G\colon \sum_{i\leqq N} \chi_{Ke_i\cap A_n}(g) \geqq j\right\} \\ &= \left\{a \in A_n\colon \sum_{i\leqq N} \chi_{Ke_i}(a) \geqq j\right\} \\ &= \{a\in A_n\colon g_N(a)\geqq j/N\} \subseteq A_n. \end{aligned}$$

Consequently for any positive integer j_0,

$$(j_0/N)S(A_n) + S(I_{j0}) \leqq \mu_n(A_n^*)(S(K)+\epsilon).$$

Now fix $\rho > 0$ and choose $j_0 = [N(\lambda(A_n)^{-1}\lambda(K)+\rho)]$. Upon letting $N\to+\infty$ and reasoning as at the end of the proof of Proposition 2.13 we obtain

$$(\lambda(A_n)^{-1}\lambda(K)+\rho)S(A_n) \leqq \lambda(A_n)^{-1}\lambda(A_n^*)(S(K)+\epsilon).$$

Next letting $\rho\downarrow 0$ first and then $n\to+\infty$ we finally obtain (since $\lambda(A_n)^{-1}\lambda(A_n^*)\to 1$)

$$\left(\overline{\lim_n}\ \lambda(A_n)^{-1}S(A_n)\right)\lambda(K) < S(K)+\epsilon,$$

and since the left side does not depend on ϵ,

$$\overline{\lim_n}\ M_S(A_n) \leqq \lambda(K)^{-1}S(K) = M_S(K) \quad \text{for all} \quad K\in\mathcal{K}_+,$$

and consequently,

$$\lim_n M_S(A_n) = \inf\{M_S(K)\colon K\in\mathcal{K}_+\} = \inf\{M_S(K)\colon K\in\mathcal{K}_0\cap\mathcal{K}_c\}$$

and the theorem is proved if $\{A_n^{-1}\}$ is strong Følner. However, if $\{A_n^{-1}\}$ is weak Følner by [**2**, Theorem 15] there exist $A_n^* \subseteq A_n$ such that

$$\lambda(A_n)^{-1}\lambda(A_n^*) = \lambda(A_n^{-1})^{-1}\lambda(A_n^{*-1}) \to 1$$

and $\{A_n^{*-1}\}$ is strong Følner. Now $S(A_n) \leqq S(A_n^*)$ implies

$$\begin{aligned} M_S(A_n) &= \lambda(A_n)^{-1}S(A_n) \leqq \lambda(A_n)^{-1}S(A_n^*) \\ &= (\lambda(A_n^*)^{-1}S(A_n^*))(\lambda(A_n)^{-1}\lambda(A_n^*)) \\ &= M_S(A_n^*)(\lambda(A_n)^{-1}\lambda(A_n^*)), \end{aligned}$$

and consequently,

$$\overline{\lim}\, M_S(A_n) \leqq \lim M_S(A_n^*)$$

which equals $\inf\{M_S(K): K \in \mathcal{K}_+\}$ as we have already shown and the theorem is proved in general.

Comment. Lemma 3.2 and Theorem 3.3 were proved assuming implicitly that KA is measurable. Otherwise, as by our convention, λ denotes *inner* measure and straightforward modifications of the proofs cover this contingency.

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