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WELL-BEHAVED AND TOTALLY BOUNDED APPROXIMATE IDENTITIES FOR $C_0(X)$

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Let X denote a locally compact Hausdorff space, $C_0(X)$ the Banach algebra of continuous complex-valued functions on X which vanish at infinity. An approximate identity for $C_0(X)$ is a net $(f_\lambda)_{\lambda \in \Lambda}$ such that (1) $\|f_\lambda\| \leqq 1 \forall \lambda$; and (2) if $h \in C_0(X)$, then $\lim_\lambda \|hf_\lambda - h\| = 0$. Here the norm is the sup norm, and multiplication is the usual pointwise product.

This paper contains an analysis of approximate identities for $C_0(X)$ of two special types: totally bounded in the strict topology, and well-behaved in the sense of Taylor. In each case, existence of an approximate identity of the stated type is shown to be equivalent to paracompactness of X. A constructive, somewhat lengthy proof of the first equivalence has been given by Collins and Fontenot; here a short non-constructive proof is presented. That well-behaved implies paracompact is shown using a set-theoretic lemma of Hajnal. In the course of the argument certain spaces X which can be embedded in Stone-Čech compactifications of discrete spaces are considered. Using a result of Rosenthal on relatively disjoint families of measures, it is shown that the strict topology on $C^*(X)$ is the Mackey topology for some of these X, not all of which are paracompact. This indicates that σ-compact spaces can be pasted together in fairly complicated ways while still retaining the Mackey property.

The strict topology β on $C^*(X)$ was introduced by Buck [1]. Collins and Dorroh [3, Th. 4.2] noted that if X is paracompact, then $C_0(X)$ admits an approximate identity which is β-totally bounded. Collins and Fontenot [4] proved the converse, and gave a systematic classification of approximate identities according to their topological and set-theoretic properties. The notion of a well-behaved approximate identity (WAI) is due to Taylor [12].

An approximate identity $(f_\lambda)_{\lambda \in \Lambda}$ for $C_0(X)$ is a WAI if (3) $f_\lambda \geqq 0 \forall \lambda$; (4) $\lambda_1 < \lambda_2 \Rightarrow f_{\lambda_1} f_{\lambda_2} = f_{\lambda_1}$; and (5) if $\lambda_0 \in \Lambda$ and (λ_n) is a strictly increasing sequence in Λ, then there is a positive integer n_0 such that $f_{\lambda_0} f_{\lambda_n} = f_{\lambda_0} f_{\lambda_m}$ for $m, n \geqq n_0$. Condition (4) implies that each f_λ must have compact support. Moreover, given x, there is a λ such that $f_\lambda(x) = 1$. It is known that X paracompact $\Rightarrow C_0(X)$ has a $WAI \Rightarrow$ $(C^*(X), \beta)$ is a Mackey space.

1. β-totally bounded approximate identities for $C_0(X)$.

THEOREM 1.1. *If $C_0(X)$ contains an equicontinuous subset H such* *that, for some* $\alpha > 0$, $\sup\{|f(x)|: f \in H\} > \alpha \forall x \in X$, *then* X *is paracompact.*

Proof; If D is an equicontinuous subset of $C^*(X)$ and ε is a positive number, let $U(D, \varepsilon) = \{(x, y) \in X \times X: \sup\{|f(x) - f(y)|: f \in D\} \leqq \varepsilon\}$. Then $\{U(D, \varepsilon): D$ equicontinuous, $\varepsilon > 0\}$ is a base for a uniformity which is compatible with the topology of X. Fix $x_0 \in X$, and choose $f_0 \in H$ with $|f_0(x_0)| > \alpha$. Then, since $\{y \in X: |f_0(y)| \geqq \alpha/2\}$ is compact, so is $U(H, \alpha/2)[x_0] = \{x \in X : |f(x) - f(x_0)| \leqq \alpha/2 \forall f \in H\}$. Thus X is uniformly locally compact, hence paracompact [9, p. 215].

COROLLARY 1.2. *If* $C_0(X)$ *has a* β*-totally bounded approximate identity, then* X *is paracompact.*

Proof. A subset of $C^*(X)$ is β-totally bounded if and only if it is uniformly bounded and equicontinuous [2, Lemma 3.1]. Thus the conditions of 1.1 are satisfied.

2. Well-behaved approximate identities for $C_0(X)$.

In this section we prove the following result.

THEOREM 2.1. *If* $C_0(X)$ *has a well-behaved approximate identity, then* X *is paracompact.*

The proof relies on a sequence of lemmas. A space X is *zero-dimensional* if the topology has a base of clopen sets. A map ψ: $Y \rightarrow X$ is *perfect* if it is a continuous closed surjection such that the inverse image of each point of X is compact.

LEMMA 2.2. *If* Y *is a zero-dimensional locally compact Hausdorff space, and* $C_0(Y)$ *has a WAI* $(f_\lambda)_{\lambda \in \Lambda}$, *then there is a corresponding family* $(K_\lambda)_{\lambda \in \Lambda}$ *of compact-open subsets of* Y *such that* (1′) $\bigcup_{\lambda \in \Lambda} K_\lambda = Y$; (2′) $\lambda_1 < \lambda_2 \rightarrow K_{\lambda_1} \subset K_{\lambda_2}$; *and* (3′) *if* $\lambda_0 \in \Lambda$ *and* (λ_n) *is a strictly increasing sequence in* Λ, *then there is a positive integer* n_0 *such that* $K_{\lambda_0} \cap K_{\lambda_m} = K_{\lambda_0} \cap K_{\lambda_n}$ *for* $m, n \geqq n_0$.

Proof. For each λ, $A_\lambda = \{x \in X: f_\lambda(x) = 1\}$ is a compact subset of the open set $B_\lambda = \{x \in X: f_\lambda(x) > 1/2\}$. Choose a compact-open set K_λ with $A_\lambda \subset K_\lambda \subset B_\lambda$. Then (1′) holds, and, since $B_{\lambda_1} \subset A_{\lambda_2}$ for $\lambda_1 < \lambda_2$, so does (2′). If n_0 satisfies (5), it is not difficult to verify that $n_0 + 1$ will satisfy (3′).

LEMMA 2.3. *If* X *and* Y *are locally compact Hausdorff spaces,*

$C_0(X)$ has a WAI, and there is a perfect map of Y onto X, then $C_0(Y)$ has a WAI.

Proof. The inverse of a compact set under a perfect map is compact. Thus if $(f_\lambda)_{\lambda\in\Lambda}$ is a *WAI* for $C_0(X)$, and $\phi: Y \to X$ is perfect, it can be shown that $(f_\lambda \circ \phi)_{\lambda\in\Lambda}$ is a *WAI* for $C_0(Y)$.

In order to motivate the final (and central) lemma, we begin the

Proof of Theorem 2.1. Suppose $C_0(X)$ has a *WAI* $(f_\lambda)_{\lambda\in\Lambda}$. Let D denote the underlying set of X, endowed with the discrete topology. Then the identity map $i: D \to X$ has a unique continuous extension $\psi: \beta D \to \beta X$. Let $Y = \psi^{-1}(X)$, and let $\phi = \psi | Y$. Then we have: (1) Y is locally compact Hausdorff, since Y is open in βD; (2) $D \subset Y \subset \beta D$; thus Y is extremally disconnected [8, 6M], and therefore zero-dimensional; and (3) ϕ is a perfect map of Y onto X, since ψ is perfect and ϕ is its restriction to a complete inverse image. From 2.2 and 2.3 we obtain a family $(K_\lambda)_{\lambda\in\Lambda}$ of compact-open subsets of Y satisfying (1′), (2′) and (3′) of 2.2. For each λ, let $H_\lambda = K_\lambda \cap D$; then $\mathrm{cl}_{\beta D} H_\lambda = K_\lambda$. Let $\mathscr{H} = (H_\lambda)_{\lambda\in\Lambda}$. Then $\mathscr{H}$ is a well-behaved cover of D in the sense of the following definition.

DEFINITION 2.4. Let D be a set, A a directed set, and $\mathscr{U} = (U_\alpha)_{\alpha\in A}$ a family of subsets of D. Then $\mathscr{U}$ is a *well-behaved cover* of D if (1″) $\bigcup_{\alpha\in A} U_\alpha = D$; (2″) $\alpha_1 < \alpha_2 \Rightarrow U_{\alpha_1} \subset U_{\alpha_2}$; and (3″) if $\alpha_0 \in A$ and (α_n) is a strictly increasing sequence in A, then there is a positive integer n_0 such that $U_{\alpha_0} \cap U_{\alpha_m} = U_{\alpha_0} \cap U_{\alpha_n}$ for $m, n \geqq n_0$.

There is a simple way of producing well-behaved covers of a set D. Indeed let $(V_\beta)_{\beta\in B}$ be any decomposition of D into pairwise disjoint nonempty subsets. Let A be the collection of all finite subsets of B, directed by inclusion. For each $\alpha = (\beta_1, \cdots, \beta_n) \in A$, define $U_\alpha = \bigcup_{i=1}^n V_{\beta_i}$. Then (1″), (2″) and (3″) are easily seen to hold for $\mathscr{U} = (U_\alpha)_{\alpha\in A}$. Let us call a well-behaved cover produced in this special way a *decomposable cover* of D.

DEFINITION 2.5. Two covers $\mathscr{U}$ and $\mathscr{W}$ of a set D are equivalent if (1) given $U \in \mathscr{U}$, $\exists W \in \mathscr{W}$ such that $U \subset W$; and (2) given $W \in \mathscr{W}$, $\exists U \in \mathscr{U}$ such that $W \subset U$.

The motivation for these two definitions is as follows. Suppose we can show that our well-behaved cover $\mathscr{H}$ of D is equivalent to some decomposable cover $\mathscr{U}$ arising from a decomposition $(V_\beta)_{\beta\in B}$ of D. Then $Y = \bigcup_{\lambda\in\Lambda} K_\lambda = \bigcup_{\lambda\in\Lambda} \mathrm{cl}_{\beta D} H_\lambda = \bigcup_{\alpha\in A} \mathrm{cl}_{\beta D} U_\alpha = \bigcup_{\beta\in B} \mathrm{cl}_{\beta D} V_\beta$ (the third equality will hold because $\mathscr{H}$ and $\mathscr{U}$ are equivalent, the last because each U_α is a finite union of sets V_β). But the sets

$(\text{cl}_{\beta D} V_\beta)_{\beta \in B}$ are pairwise disjoint compact-open subsets of βD, and this implies that Y is paracompact. Since $\phi: Y \to X$ is perfect, X will then be paracompact also [7, p. 165].

Thus the proof of 2.1 reduces to a purely set-theoretic question: given a well-behaved cover $\mathscr{H} = (H_\lambda)_{\lambda \in \Lambda}$ of a set D, is there a decomposable cover $\mathscr{U} = (U_\alpha)_{\alpha \in A}$ of D which is equivalent to $\mathscr{H}$? Professor András Hajnal has kindly furnished the author with a proof that this is indeed the case. The author expresses his deep appreciation to Professor Hajnal for his permission to record the argument in the following lemma, which may be of independent interest.

LEMMA 2.6 (Hajnal). *A well-behaved cover of a set D is always equivalent to some decomposable cover of D.*

Proof. Let $\mathscr{U}$ be a family of nonempty subsets of a set S which covers S. We shall say that $\mathscr{U}$ is a *good cover* of S if there is a function f which assigns to each finite collection $\{A_1, \cdots, A_n\}$ of distinct members of $\mathscr{U}$ a member $f(A_1, \cdots, A_n)$ of $\mathscr{U}$ in such a way that (a) $\bigcup_{i=1}^n A_i \subset f(A_1, \cdots, A_n)$; (b) $f(A_1, \cdots A_{n-1}) \subset f(A_1, \cdots A_{n-1}, A_n)$; and (c) if $B \in \mathscr{U}$ and $\mathscr{U}' \subset \mathscr{U}$, there is a finite subcollection $\{A_1, \cdots, A_n\}$ of $\mathscr{U}'$ such that $B \cap (\bigcup\{W: W \in \mathscr{U}'\}) \subset f(A_1, \cdots, A_n)$. In this case f is said to be a *good function* for $\mathscr{U}$.

Claim 1. *A well-behaved cover $\mathscr{H} = (H_\lambda)_{\lambda \in \Lambda}$ of a set D is a good cover of D.* Define a function g from the collection of finite subsets of Λ to Λ so that for any $\{\lambda_1, \cdots, \lambda_n\} \subset \Lambda$, $\lambda_i < g(\lambda_1, \cdots \lambda_n) \forall i$ and $g(\lambda_1, \cdots, \lambda_{n-1}) < g(\lambda_1, \cdots, \lambda_n)$. This is easily done by induction on n, the number of elements in the finite subset. We would like to define $f(H_{\lambda_1}, \cdots, H_{\lambda_n})$ to be $H_{g(\lambda_1, \ldots, \lambda_n)}$, but there is a difficulty in that $H_\lambda = H_\mu$ for $\lambda \neq \mu$ might occur, leading to an ambiguity in the definition. Proceed as follows: well-order Λ as $(\lambda(\alpha))_{\alpha < \alpha_0}$ (this well-ordering of course has nothing to do with the partial order which Λ already possesses). If $P_1, \cdots, P_n$ are distinct members of $\mathscr{H}$, choose, for each i, the least α_i such that $H_{\lambda(\alpha_i)} = P_i$. Then define $f(P_1, \cdots, P_n)$ to be H_μ where $\mu = g(\lambda(\alpha_1), \cdots, \lambda(\alpha_n))$. It follows easily that (a) and (b) hold. Suppose (c) fails for some $P_0 \in \mathscr{H}$ and $\mathscr{H}' \subset \mathscr{H}$. By induction we can find a sequence (P_n) in $\mathscr{H}'$ such that $P_0 \cap P_n \not\subset f(P_1, \cdots, P_{n-1}) \forall n$ (P_1 is an arbitrary member of $\mathscr{H}'$). Let $P_n = H_{\lambda(\alpha_n)}$ as above. Then property (3″) of a well-behaved cover is violated for the indices $\lambda(\alpha_0)$ and $g(\lambda(\alpha_1)) < g(\lambda(\alpha_1), \lambda(\alpha_2)) < \cdots < g(\lambda(\alpha_1), \cdots, \lambda(\alpha_n)) < \cdots$, a contradiction. Thus (c) holds, and so f is a good function for $\mathscr{H}$.

Claim 2. *If $\mathscr{U}$ is a good cover of D, T is a non-empty subset of D, and $\mathscr{U}_T = \{A \cap T: A \in \mathscr{U},\ A \cap T \neq \emptyset\}$, then $\mathscr{U}_T$ is a good cover of T.* Indeed $\mathscr{U}$ can be well-ordered in some way as $(A_\alpha)_{\alpha<\alpha_0}$. If $B \in \mathscr{U}_T$, let $\alpha(B)$ be the least α such that $A_{\alpha(B)} \cap T = B$. If $B_1, \cdots, B_n \in \mathscr{U}_T$, define $h(B_1, \cdots, B_n) = f(A_{\alpha(B_1)}, \cdots, A_{\alpha(B_n)}) \cap T$. It can be verified that h is a good function for $\mathscr{U}_T$ if f is a good function for $\mathscr{U}$.

Claim 3. *A good cover of a set is equivalent to a decomposable cover of that set.* We induct on the cardinality of a good cover $\mathscr{U}$. If card $U \leqq \aleph_0$, the claim is easily established. Now suppose the result holds for good covers $\mathscr{U}$ of arbitrary sets, where card $\mathscr{U} < \kappa$ and $\kappa > \aleph_0$. Let card $\mathscr{U} = \kappa$, where $\mathscr{U}$ is a good cover of a set D, and let α_κ be the least ordinal of cardinal κ. Let f be a fixed good function for $\mathscr{U}$.

If $\mathscr{C} \subset \mathscr{U}$, we shall say that $\mathscr{C}$ is *closed* if $A_1, \cdots, A_n \in \mathscr{C} \Rightarrow f(A_1, \cdots, A_n) \in \mathscr{C}$. We construct a transfinite sequence $(\mathscr{U}_\alpha)_{\alpha<\alpha_k}$ of closed subfamilies of $\mathscr{U}$ such that (1) $\alpha < \beta \Rightarrow \mathscr{U}_\alpha \subset \mathscr{U}_\beta$; (2) $\mathscr{U}_\alpha = \bigcup_{\beta<\alpha} \mathscr{U}_\beta$ for limit ordinals α; (3) $\bigcup_{\alpha<\alpha_\kappa} \mathscr{U}_\alpha = \mathscr{U}$; and (4) card $\mathscr{U}_\alpha \leqq$ card $\alpha + \aleph_0 \forall \alpha$.

Now $\mathscr{U}$ can be indexed as (A_α) where α runs over the set of *nonlimit* ordinals less than α_κ. If $\mathscr{F} \subset \mathscr{U}$, there is a smallest closed subfamily $\mathscr{C}(\mathscr{F})$ of $\mathscr{U}$ which contains $\mathscr{F}$, and it is not difficult to show that card $\mathscr{C}(\mathscr{F}) \leqq \aleph_0 +$ card $\mathscr{F}$. Let $\mathscr{U}_1 = \mathscr{C}(\{A_1\})$. Suppose $\mathscr{U}_\alpha$ has been chosen for all $\alpha < \alpha_0$ so that $A_\alpha \in \mathscr{U}_\alpha$ for nonlimit ordinals α and (1), (2), (4) hold for $\alpha < \alpha_0$. If α_0 is a limit ordinal, let $\mathscr{U}_{\alpha_0} = \bigcup_{\alpha<\alpha_0} \mathscr{U}_\alpha$. If $\alpha_0 = \alpha_1 + 1$, let $\mathscr{U}_{\alpha_1} = \mathscr{C}(\mathscr{U}_{\alpha_0} \cup \{A_{\alpha_0}\})$. In this way the desired transfinite sequence is obtained.

Now let $S_\alpha = \bigcup \{U: U \in \mathscr{U}_\alpha\}$, $Z_\alpha = S_{\alpha+1} \backslash S_\alpha$ for $\alpha < \alpha_\kappa$ (let $\mathscr{U}_0 = \emptyset$). Note that no member of $\mathscr{U}_\alpha$ meets Z_α. Let $\mathscr{W}_\alpha = \{B \cap Z_\alpha: B \in \mathscr{U}_{\alpha+1} \backslash \mathscr{U}_\alpha\}$. As in Claim 2, one can show that $\mathscr{W}_\alpha$ is a good cover of Z_α for each nonempty Z_α. Since card $\mathscr{W}_\alpha \leqq$ card $\mathscr{U}_{\alpha+1} < \kappa$, we have by induction that $\mathscr{W}_\alpha$ is equivalent to a decomposable cover $\mathscr{V}_\alpha$ of Z_α. Since D is the disjoint union of the sets Z_α, the family $\mathscr{V}$ of finite unions of all members of the collections $\mathscr{V}_\alpha$ is a decomposable cover of D.

Finally we show that $\mathscr{U}$ and $\mathscr{V}$ are equivalent. If $V \in \mathscr{V}$, then $V = \bigcup_{i=1}^n V_{\alpha_i}$ where $V_{\alpha_i} \in \mathscr{V}_{\alpha_i}$. Then each $V_{\alpha_i} \subset U_{\alpha_i}$ for suitable $U_{\alpha_i} \in \mathscr{U}_{\alpha_i+1} \backslash \mathscr{U}_{\alpha_i}$, and $V \subset f(U_{\alpha_1}, \cdots, U_{\alpha_n}) \in \mathscr{U}$. Conversely, we show by induction that if $W \in \mathscr{U}$, and α is the least ordinal such that $W \in \mathscr{U}_\alpha$, then $W \subset V$ for some $V \in \mathscr{V}$. For $\alpha = 1$ this is clear, since $\mathscr{U}_1$ and $\mathscr{V}_1$ are equivalent covers of Z_1. Suppose the result holds for all $\alpha < \alpha_0$. If α_0 is a limit ordinal, then $\mathscr{U}_{\alpha_0} = \bigcup_{\alpha<\alpha_0} \mathscr{U}_\alpha$ and the result holds. Suppose $\alpha_0 = \alpha_1 + 1$. Applying property (c) of a good cover to $\mathscr{U}$, with $B = W$, $\mathscr{U}' = \mathscr{U}_{\alpha_1}$, there exist $A_1, \cdots, A_n \in \mathscr{U}_{\alpha_1}$ such that $W \cap \bigcup\{V: V \in \mathscr{U}_{\alpha_1}\} = W \cap S_{\alpha_1} \subset f(A_1, \cdots, A_n)$. Since $\mathscr{U}_{\alpha_1}$ is a closed family, $f(A_1, \cdots, A_n) \in \mathscr{U}_{\alpha_1}$. Thus, by induction, $W \cap S_{\alpha_1} \subset V_1$ for some $V_1 \in \mathscr{V}$. Since $W \in \mathscr{U}_{\alpha_1+1} \backslash \mathscr{U}_{\alpha_1}$, we have $W \cap (S_{\alpha_1+1} \backslash S_{\alpha_1}) = W \cap Z_{\alpha_1} \in \mathscr{W}_{\alpha_1}$. Since $\mathscr{W}_{\alpha_1}$ and $\mathscr{V}_{\alpha_1}$ are equivalent covers of Z_{α_1}, there is a member V_2 of $\mathscr{V}$ with $W \cap (S_{\alpha_1+1} \backslash S_{\alpha_1}) \subset V_2$. Then $U \subset V_1 \cup V_2$, which is in $\mathscr{V}$ because a decomposable cover is closed under finite unions. This completes the proof.

3. An application to the Mackey problem for the strict topology. A well-known result of Conway [6] states that if X is paracompact locally compact, then $(C^*(X), \beta)$ is a Mackey space. Considerable effort has been expended in attempting to find a larger class of spaces for which this is true. The condition that X be measure-compact is sufficient [11], but no example of a nonparacompact measure-compact locally compact space is known. An isolated example of a locally compact non-paracompact space with the Mackey property is presented in [13], under the assumption of the continuum hypothesis. Theorem 2.1 shows that the concept of a well-behaved approximate identity does not enlarge the class of paracompact spaces. However, the proof of 2.1 does suggest consideration of spaces X such that $X \subset \beta D$, where D is discrete. Some of these possess the Mackey property without being paracompact as we now show.

The following lemma is probably well-known; we include a proof, for completeness. If β_0 is an ordinal, and $(\alpha_\beta)_{\beta<\beta_0}$ is a set of ordinals such that $\beta_1 < \beta_2 < \beta_0 \Rightarrow \alpha_{\beta_1} < \alpha_{\beta_2}$, we shall refer to $(\alpha_\beta)_{\beta<\beta_0}$ as an increasing transfinite sequence with order type β_0.

LEMMA 3.1. *Let α_0 be a limit ordinal, and let $B = \{\beta_0$: there is an increasing transfinite sequence $(\alpha_\beta)_{\beta<\beta_0}$ of ordinals with $\sup_{\beta<\beta_0} \alpha_\beta = \alpha_0\}$. Then B has a smallest member β', and β' is the smallest ordinal whose cardinal is card β'.*

Proof. B is a nonempty set of ordinals, since $\alpha_0 \in B$, and therefore has a smallest member β'. Let $(\alpha_\beta)_{\beta<\beta'}$ be a fixed increasing transfinite sequence with $\sup_{\beta<\beta'} \alpha_\beta = \alpha_0$. Let β'' be the initial ordinal of cardinal card β'. Let $\phi: \{\beta: \beta < \beta''\} \to \{\alpha_\beta\}_{\beta<\beta'}$ be a $1-1$ correspondence (not assumed to preserve order). For each $\beta < \beta''$, let $\lambda_\beta = \sup\{\phi(\gamma): \gamma \leqq \beta\}$. Then $(\lambda_\beta)_{\beta<\beta''}$ is a non-decreasing transfinite sequence of ordinals satisfying $\lambda_\beta < \alpha_0 \forall \beta$ and $\sup_{\beta<\beta''} \lambda_\beta = \alpha_0$. We can construct from $(\lambda_\beta)_{\beta<\beta''}$ a strictly increasing transfinite sequence whose supremum is α_0. The order type of this sequence cannot exceed β'' and has cardinal card β', hence must be β''.

THEOREM 3.2. *Let D be a discrete space of cardinal γ, represented as $\{\alpha: \alpha < \alpha_0\}$ where α_0 is the least ordinal of cardinal γ. Suppose X is an open subset of βD such that* (1) *if $x \in X$, $\exists \alpha < \alpha_0$ such that $x \in \mathrm{cl}_{\beta D}\{\beta: \beta \leqq \alpha\}$; and* (2) *if $\alpha < \alpha_0$, then $X \cap \mathrm{cl}_{\beta D}\{\beta: \beta \leqq \alpha\}$ is paracompact. Then $(C^*(X), \beta)$ is a Mackey space.*

Proof. If γ is the supremum of an increasing sequence of smaller cardinals γ_n, let α_n be the least ordinal of cardinal γ_n. Then, using (1), $X = \bigcup_{n=1}^{\infty}(X \cap \mathrm{cl}_{\beta D}\{\alpha: \alpha \leqq \alpha_n\})$ is the union of an increasing sequence of open and closed paracompact subspaces. It follows that X is paracompact, so Conway's theorem applies.

Now assume that γ is not the supremum of any sequence of smaller cardinals. For each $\alpha < \alpha_0$, let $D_\alpha = \{\beta: \beta \leqq \alpha\}$ and $U_\alpha = X \cap \mathrm{cl}_{\beta D} D_\alpha$. Then the collection $(U_\alpha)_{\alpha < \alpha_0}$ is an increasing cover of X by open and closed paracompact subspaces. Let $M(X)$ denote the space of bounded regular Borel measures on X (the dual space of $(C^*(X), \beta)$). If $\mu \in M(X)$, the support of μ is contained in the union of countably many U_α; hence spt $\mu \subset U_\beta$ for some $\beta < \alpha_0$.

Let H be a weak*-compact (hence uniformly bounded) subset of $M(X)$. If $\exists \alpha < \alpha_0$ such that spt $\mu \subset U_\alpha \forall \mu \in H$, then (2) implies that H is uniformly tight. If this fails apply 3.1. Let $(\alpha_\beta)_{\beta < \beta'}$ be a fixed increasing transfinite sequence of smallest order type with $\alpha_\beta < \alpha_0 \forall \beta$ and $\sup \alpha_\beta = \alpha_0$. Choose δ_1' with $\alpha_1 < \delta_1' < \alpha_0$. Then choose $\mu_1 \in H$ and δ_1 such that $\delta_1' < \delta_1 < \alpha_0$ and $|\mu_1|(U_{\delta_1} \backslash U_{\delta_1'}) > 0$. Suppose β_0 is an ordinal less than β', and (μ_β), (δ_β), and (δ_β') have been chosen for all $\beta < \beta_0$. Then $\sup_{\beta < \beta_0} \delta_\beta < \alpha_0$. Choose δ_{β_0}' with $\sup\{\alpha_{\beta_0}, \sup_{\beta < \beta_0} \delta_\beta\} < \delta_{\beta_0}' < \alpha_0$. Then choose $\mu_{\beta_0} \in H$ and δ_{β_0} such that $\delta_{\beta_0}' < \delta_{\beta_0} < \alpha_0$ and $|\mu_{\beta_0}|(U_{\delta_{\beta_0}} \backslash U_{\delta_{\beta_0}'}) > 0$. By transfinite induction we obtain $(\delta_\beta)_{\beta < \beta'}$ with $\sup_{\beta < \beta'} \delta_\beta = \alpha_0$. For each $\beta < \beta'$, use regularity of μ_β and the fact that X is zero-dimensional to obtain a compact-open subset K_β of $U_{\delta_\beta} \backslash U_{\delta_\beta'}$ with $|\mu_\beta|(K_\beta) > 0$. Let $B_n = \{\beta < \beta': |\mu_\beta|(K_\beta) > 1/n\}$. Some B_{n_0} is cofinal in $\{\beta: \beta < \beta'\}$, for if $\sup B_n = \beta_n < \beta' \forall n$, then $\alpha_0 = \sup \delta_{\beta_n}$, contradicting our assumption that $\gamma = \mathrm{card}\, \alpha_0$ is not the supremum of a sequence of smaller cardinals. Then B_{n_0} must have order type β' because of the minimal property of β'.

Thus there exist a uniformly bounded family $(\mu_\beta)_{\beta \in B_{n_0}}$, each member of which can be regarded as a regular Borel measure on βD, and family $(K_\beta)_{\beta \in B_{n_0}}$ of disjoint clopen subsets of βD such that $|\mu_\beta|(K_\beta) > 1/n_0 \forall \beta$. Since βD is Stonian, we can apply Lemma 1.1(a) of [10] to deduce the existence of a subset C of B_{n_0} with $\mathrm{card}\, C = \mathrm{card}\, B_{n_0} = \mathrm{card}\, \beta'$ such that $|\mu_\beta|(\bigcup\{K_\gamma: \gamma \in C, \gamma \neq \beta\})^- < 1/2n_0 \forall \beta \in C$. Note that *$C$ must be cofinal in $\{\beta: \beta < \beta'\}$*: If $\sup C = \lambda < \beta'$, then λ is an ordinal with cardinal card β', a contradiction. Consequently, $\sup_{\beta \in C} \delta_\beta = \alpha_0$.

We now show that the net $(\mu_\beta)_{\beta\in C}$, directed by the well-ordering of C, has no weak*-cluster point in $M(X)$. Let $\mu_0 \in M(X)$, and find $\alpha < \alpha_0$ such that spt $\mu_0 \subset U_\alpha$.

Let β_0 be a fixed member of C such that $\delta'_{\beta_0} > \alpha$. For each $\beta \in C$ such that $\beta \geqq \beta_0$, let $H_\beta = K_\beta \cap D$. Then $H_\beta \subset \{\gamma \in D: \delta'_\beta < \gamma \leqq \delta_\beta\}$, and $K_\beta = \mathrm{cl}_{\beta D} H_\beta$ is the Stone-Čech compactification of H_β. Since $|\mu_\beta|(K_\beta) > 1/n_0$, there is a function $h_\beta \in C^*(K_\beta)$ with $||h_\beta|| \leqq 1$ and $\left|\int_{K_\beta} h_\beta d\mu_\beta\right| > 1/n_0$. Define $f_0: D \to R$ by $f_0(x) = h_\beta(x)$ if $x \in H_\beta$ for $\beta \in C$, $\beta \geqq \beta_0$, and $f_0(x) = 0$ otherwise. Now extend f_0 continuously to βD, and let f be the restriction to X. Note that $f|U_\alpha \equiv 0$, $||f|| \leqq 1$, and $f|K_\beta = h_\beta$ for all $\beta \in C$ such that $\beta \geqq \beta_0$.

We certainly have $\mu_0(f) = 0$; however, $|\mu_\beta(f)| > 1/2n_0$ for each $\beta \in C$ such that $\beta \geqq \beta_0$. To see this, fix such a β, and write D as the disjoint union of the sets H_β, $\bigcup\{H_\gamma: \gamma \in C, \gamma \neq \beta\}$, and F (what is left). Then X is the disjoint union of the sets K_β, $\mathrm{cl}_X(\bigcup\{K_\gamma: \gamma \in C, \gamma \neq \beta\})$, and $\mathrm{cl}_X F$. The integrals of f with respect to μ_β over these three sets are, in absolute value, greater than $1/n_0$, less than $1/2n_0$, and 0. The conclusion follows. Hence $(\mu_\beta)_{\beta\in C}$ has no weak*-cluster point, and we have contradicted weak*-compactness of H. This completes the proof.

For the special case where card $D = \aleph_1$, some of the technical difficulties in this argument can be avoided, and the result can be stated in modified form.

COROLLARY 3.3. *Let D be a discrete space of cardinal $\aleph_1$, and let X be an open subset of βD such that* (1) *if $x \in X$, there is a countable subset F of D such that $x \in \mathrm{cl}_{\beta D} F$; and* (2) *if F is a countable subset of D, then $X \cap \mathrm{cl}_{\beta D} F$ is σ-compact. Then $(C^*(X), \beta)$ is a Mackey space.*

EXAMPLE 3.4. Let D be any uncountable discrete space such that card D is not the supremum of a sequence of smaller cardinals. Let $D = \{\alpha: \alpha < \alpha_0\}$ as in 3.2, and let $X = \bigcup_{\alpha<\alpha_0} \mathrm{cl}_{\beta D}\{\beta: \beta \leqq \alpha\}$. Then X is extremally disconnected locally compact and sham-compact (every σ-compact subset is relatively compact), hence countably compact and pseudocompact. But X is not compact, since $\{\mathrm{cl}_{\beta D}\{\beta: \beta \leqq \alpha\}\}$ is an open cover with no finite subcover; thus X cannot be paracompact. However, according to 3.2, $(C^*(X), \beta)$ is Mackey.

REMARK 3.5. $(C^*(X), \beta)$ is said to be a *strong Mackey space* if the following is true: whenever H is a subset of $M(X)$ such that every sequence in H has a weak*-cluster point in $M(X)$, then H is uniformly tight. Conway's proof shows that if X is paracompact, then the strong Mackey property holds; the same is true for the space considered in [13]. However this cannot be true for any of the spaces X described in 3.4. Indeed if H consists of all point masses corresponding to points of X, then H is weak*-countably compact in $M(X)$, but not uniformly tight.

EXAMPLE 3.6. It is easy to show that if $(C^*(X), \beta)$ is Mackey and T is a closed subspace of X, then $(C^*(T), \beta)$ is Mackey. Assume the continuum hypothesis, and let p be a P-point of $\beta N \backslash N$. As the author pointed out in [14], $(C^*(\beta N \backslash \{p\}), \beta)$ is *not* a Mackey space. However, if X is the space of 3.4, with card $D = \aleph_1$, then, by a result of Comfort and Negrepontis [5], $\beta N \backslash N \backslash \{p\}$ is homeomorphic to the closed subspace $X \backslash D$ of X. Thus $(C^*(\beta N \backslash N \backslash \{p\}), \beta)$ is a Mackey space. This gives some indication of the apparent subtlety of the Mackey problem for the strict topology.

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