PACIFIC JOURNAL OF MATHEMATICS
Vol. 65, No. 1, 1976

REARRANGEMENTS OF FUNCTIONS ON THE RING OF INTEGERS OF A p-SERIES FIELD

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We show that every continuous function on the ring of integers of a p-series field has a rearrangement that has absolutely convergent Fourier series.

I. Introduction. Let p be a rational prime fixed throughout. K will denote the p-series field of formal Laurent series in one variable with finite principal part and with coefficients in $GF(p)$, Thus, an element $x \in K$ has representation as

$$x = \sum a_j \mathfrak{p}^j \qquad (a_j = 0, 1, \cdots, p-1)$$

and $a_j = 0$ for j sufficiently small. Addition and multiplication are defined by the usual formal sums and products of Laurent series.

The field K is topologized by taking as a basis the sets

$$V_{x,k} = \{\sum b_j \mathfrak{p}^j \colon b_j = a_j,\ j < k\}$$

where $x = \sum a_j \mathfrak{p}^j$. With this topology, K is locally compact, totally disconnected and nondiscrete.

The ring of integers $\mathfrak{O} = \{x \colon x = \sum_{j=0}^{\infty} a_j \mathfrak{p}^j\}$ is the unique maximal compact subring of K. Let dx denote Haar measure on K derived from the additive structure and normalized so that $\mathfrak{O}$ has measure 1.

As a locally compact abelian group, $\mathfrak{O}$ has a Pontryagin dual $\hat{\mathfrak{O}}$ that may be identified with $K/\mathfrak{O}$. We choose the representatives of the form

$$\sum_{-1}^{-\nu} r_j \mathfrak{p}^j \qquad (r_j = 0, 1, \cdots, p-1)$$

and use the lexicographic ordering to match the characters χ_t to the nonnegative integers. Of course, if χ is a continuous unitary character of K^+, then $\chi(x)$ is a pth root of unity for all $x \in K$.

If f is an integrable function on $\mathfrak{O}$, its Fourier coefficients are given by

$$\hat{f}(t) = \int_{\mathfrak{O}} f(x)\bar{\chi}_t(x)dx \qquad (t = 0, 1, \cdots)\,.$$

We define the class $A(\mathfrak{O})$ of continuous complex-valued functions on $\mathfrak{O}$ as those functions f for which the quantity

$$\sum_{t=0}^{\infty} |\hat{f}(t)|$$

is finite. Under the pointwise operations $A(\mathfrak{O})$ is an algebra; it is, in fact, a Banach algebra with the above taken as the norm of f.

Suppose that h is a homeomorphism of $\mathfrak{O}$, and that f and g are two functions on $\mathfrak{O}$ related by

$$g = f \circ h \ .$$

Then g is said to be a *rearrangement* of f. N. Lusin asked whether every continuous function on the circle group has a rearrangement that has absolutely convergent Fourier series (see [4] p. 8). This question is still open; however, see [3] for the best known result. Here we prove the following.

THEOREM. *Every continuous function f on $\mathfrak{O}$ has a rearrangement g that has absolutely convergent Fourier series.*

It should be noted that the setting of the theorem contains as a special case ($p = 2$) the classical dyadic group 2^{ω}.

II. Preliminaries. The principal ideal in $\mathfrak{O}$ generated by $\mathfrak{p}$, $\mathfrak{P}$, is the unique maximal ideal in $\mathfrak{O}$. There is a non-archimedean valuation $|\cdot|$ on K given by setting

$$|\mathfrak{p}| = p^{-1} \ .$$

$|\cdot|$ satisfies $|x + y| \leqq \operatorname{Max}\{|x|, |y|\}(x, y \in K)$, and therefore defines a metric on K. The topology induced by this metric coincides with that defined earlier.

The fractional ideals $\mathfrak{p}^{\nu}$ are given by

$$\mathfrak{P}^{\nu} = \{x: |x| \leqq p^{-\nu}\} \ .$$

Now for each ν, $\mathfrak{O}$ decomposes into p^{ν} pairwise disjoint spheres $\omega(\nu, j)$, each of measure $p^{-\nu}$,

$$\omega(\nu, j) = x_j + \mathfrak{P}^{\nu} \qquad (j = 1, 2, \cdots, p^{\nu}) \ .$$

We assume that the x_j are ordered lexicographically. Thus, consecutive blocks of length $p^{\nu-1}$ have the same coefficient of the $\mathfrak{P}^0$ term, consecutive blocks of length $p^{\nu-2}$ have the same coefficient of the $\mathfrak{P}^0$ and $\mathfrak{P}^1$ terms, etc.

Consequently, we have the containments

$$\omega(\nu + 1, j) \subset \omega(\nu, k) \ , \quad ((k - 1)p + 1 \leqq j \leqq kp) \ .$$

In our construction of a homeomorphism of $\mathfrak{O}$ it will be necessary to make repeated use of the fact that two compact, totally disconnected, metrizable, and perfect spaces are homeomorphic (see [1] p. 97).

From now on, since the prime number p will frequently occur exponentiated and subscripted, for typographical reasons we shall write $p(\nu)$ for p^ν.

III. LEMMA. *Suppose that g is a continuous complex-valued function defined on $\mathfrak{O}$. Then g is an A-function if the series whose nth term ($n = 0, 1, 2, \cdots$) is given by*

$$p(n)(p-1)\sum_{i=1}^{p(n)} \min_{b_i} \int_{\omega(n,i)} |g(x) - b_i|\,dx \tag{1}$$

is convergent. If M denotes the sum of this series, then $\|g\|_A \leqq M + \|g\|_\infty$.

Proof. Suppose that g is locally constant on $\mathfrak{O}$ and takes the value a_j on $\omega(\nu, j)$, ($j = 1, \cdots, p(\nu)$). Then

$$\hat{g}(t) = \int_{\mathfrak{O}} g\bar{\chi}_t dx = \sum_{k=1}^{p(\nu)} a_k \int_{\omega(\nu,k)} \bar{\chi}_t dx\,. \tag{2}$$

Now, if $t \geqq p(\nu)$, it follows from the orthogonality relatians (see [2] p. 613) that $\hat{g}(t) = 0$. Suppose that $0 \leqq t \leqq p(\nu)$, and therefore that χ_t is a character identified with a representative of $K/\mathfrak{O}$ of the form

$$\sum_{j=-1}^{-\nu} r_j \mathfrak{p}^j \qquad (r_j = 0, 1, \cdots, p-1)\,. \tag{3}$$

There are $p(n)(p-1)$ characters corresponding to the representatives (3) with $r_j = 0$, $j < -n-2$, $r_{-n-1} \neq 0$, $-1 < n < \nu$.

Consider the sum

$$\sum_{t=0}^{p(\nu)-1} |\hat{g}(t)|\,. \tag{4}$$

In order to estimate (4), let χ_t be a character corresponding to (3) with $r_{-\nu} = r_{-\nu+1} = \cdots r_{-n-2} = 0$, $r_{-n-1} \neq 0$, and $-1 < n$. From (2) we see that

$$\begin{aligned} p(\nu)\hat{g}(t) = &\{A_1^1 z^{1+q_1} + \cdots + A_p^1 z^{p+q_1}\} \\ &+ \cdots \\ &+ \{A_1^{p(n)} z_{p(n)}^{1+q} + \cdots + A_p^{p(n)} z_{p(n)}^{p+q}\}\,, \end{aligned} \tag{5}$$

where the A's are the sums of consecutive blocks of the a's of length $p(\nu-(n+1))$.

$$\begin{aligned} A_1^1 &= a_1 + \cdots + a_{p(\nu-(n+1))} \\ &\cdots \\ A_p^{p(n)} &= a_{p(\nu)-p(\nu-(n+1))+1} + \cdots + a_{p(\nu)}\,. \end{aligned}$$

Furthermore, $z \neq 1$ is a pth root of unity, and $q_1, \cdots, q_{p(n)}$ are positive integers which depend on χ_t.

Since the sum of p successive powers of of a pth root of unity $\neq 1$ is zero, we see that for arbitrary complex numbers $b_1, \cdots, b_{p(n)}$

$$\hat{g}(t) = \hat{g}(t) - p(\nu - (n+1))b_1(z + \cdots + z^p) - \cdots - p(\nu - (n+1))b_{p(n)}(z + \cdots + z^p) . \tag{6}$$

Combining (5) and (6) and applying the triangle inequality, we see that

$$|\hat{g}(t)| \leqq \left\{ \sum_{k=1}^{p(\nu-n)} |a_k - b_1| + \cdots + \sum_{k=p(\nu)-p(\nu-n)+1}^{p(\nu)} |a_k - b_{p(n)}| \right\} 1/p(\nu) . \tag{7}$$

However, the right hand side of (7) is just

$$\sum_{i=1}^{p(n)} \int_{\omega(n,i)} |g(x) - b_i| dx .$$

Since there are $p(n)(p-1)$ characters χ_t of the type under consideration, the lemma is proved in the case that g is locally constant.

Now assume that g is an arbitrary continuous function on $\mathfrak{D}$ which satisfies the hypothesis of the lemma. Let N be a fixed positive integer, and approximate g uniformly on $\mathfrak{D}$ by a sequence g_m of locally constant continuous functions. Now, for every choice of integer n and complex numbers $b_j (1 \leqq j \leqq p(n))$ we have

$$\sum_{j=1}^{p(n)} \int_{\omega(n,j)} |g_m(x) - b_j| dx \longrightarrow \sum_{j=1}^{p(n)} \int_{\omega(n,j)} |g(x) - b_j| dx \tag{8}$$

as $m \to \infty$. Since the left hand side of (8) bounds $|\hat{g}_m(t)|$, where χ_t is a character corresponding to (3) with $r_j = 0$, $j < -(n+1)$, $r_{-n-1} \neq 0$, it follows that for arbitrary $\varepsilon > 0$ that

$$\sum_{t=0}^{N} |\hat{g}_m(t)| < M + ||g||_\infty + \varepsilon$$

when m is sufficiently large. Furthermore, since for each t, $\hat{g}_m(t) \to \hat{g}(t)$ as $m \to \infty$, we conclude that

$$\sum_{t=0}^{N} |\hat{g}(t)| \leqq M + ||g||_\infty + \varepsilon .$$

Since N and ε are arbitrary, the lemma is proved.

IV. Proof of the theorem. Suppose without loss of generality that $||f||_\infty = 1$; we show how to construct a homeomorphism h of $\mathfrak{D}$ such that $g = f \circ h$ satisfies the hypothesis of the lemma. Thus we will have rearrangement of f whose Fourier series converges absolutely.

We shall construct h as a limit of homeomorphisms H_n

$$h = \lim_n H_n$$

where H_n is a composition of n homeomorphisms of $\mathfrak{O}$, $h_1 \circ h_2 \circ \cdots \circ h_n$. We begin by describing the construction of the h's.

For $U \subset \mathfrak{O}$, it will be convenient to use the following notation

$$O_f(U) = \sup_{x,y \in U} |f(x) - f(y)| .$$

The quantity $O_f(U)$ is referred to as the *oscillation* of f on U.

Choose a partition of $\mathfrak{O}$ consisting of mutually disjoint, nonvoid, open and closed sets $U_j (1 \leqq j \leqq p+1)$ such that the oscillation of f on the union of the $U_j (1 \leqq j \leqq p)$ is less than or equal $1/p(3)$. Thus,

$$O_f\left(\bigcup_{j=1}^{p} U_j\right) \leqq 1/p(3) .$$

Then take h_1 to be a homeomorphism of $\mathfrak{O}$ satisfying the following requirements

$$h_1(\omega(1, j)) = U_j \qquad (1 \leqq j \leqq p-1)$$
$$h_1(\omega(1, p) \backslash \omega(3, p(3)) = U_p$$
$$h_1(\omega(3, p(3)) = U_{p+1} .$$

Now suppose that $h_1, \cdots, h_{n-1}$ are homeomorphisms of $\mathfrak{O}$ that have been defined. Set $H_{n-1} = h_1 \circ h_2 \circ \cdots \circ h_{n-1}$.

We now turn to the definition of h_n. For $i = 1, \cdots, p(n-1)$ let $U_{i,j} (1 \leqq j \leqq p+1)$ denote a partition of $\omega(n-1, i)$ into open and closed sets such that the following are satisfied.

$$(9) \qquad O_{f \circ H_{n-1}}\left(\bigcup_{j=1}^{p} U_{i,j}\right) \leqq 1/p(2n+1) \quad (i = 1, 2, \cdots, p(n-1))$$

$$(10) \quad \omega(3(n-1), ip(2(n-1)+1) \subset U_{ip,p+1} \quad (i = 1, 2, \cdots, p(n-2)) .$$

Then take h_n to be a homeomorphism of $\mathfrak{O}$ satisfying the following requirements $(i = 1, 2, \cdots, p(n-1))$

$$(11) \qquad h_n(\omega(n, k)) = U_{i,j} \quad (k = (i-1)p + j, 1 \leqq j \leqq p-1)$$

$$(12) \qquad h_n(\omega(n, ip) \backslash \omega(3n, ip(2n+1))) = U_{i,p}$$

$$(13) \qquad h_n(\omega(3n, ip(2n+1)) = U_{i,p+1} .$$

Finally, we set $H_n = H_{n-1} \circ h_n$.

First, we observe that

$$(14) \qquad h_n \omega(n-1, i) = \omega(n-1, i) \quad (i = 1, 2, \cdots, p(n-1)) .$$

From (14) we see that for every neighborhood V of 0, $h_n(x)$ and $h_n^{-1}(x)$ belong to $V + x$ for n sufficiently large. From this follows the existence of the limits

$$\lim_n H_n = h , \quad \lim_n H_n^{-1} = h^{-1} .$$

Again, from (14) the continuity of h is clear. Therefore h is a well-defined homeomorphism of $\mathfrak{O}$.

Set $g = f \circ h$. The function g is then our rearrangement of f, and it remains to check that series described in the lemma is convergent.

Now, the inequalities

$$O_{f \circ H_n}(\omega(n, j)) \leqq 1/p(2n + 1) \quad (j = (i - 1)p + k, 1 \leqq k \leqq p - 1 , \; i = 1, \cdots, p(n - 1))$$

follow immediately from (9) and (11). Sucessive application of (14) therefore yields

$$O_g(\omega(n, j)) \leqq 1/p(2n + 1) \quad (j = (i - 1)p + k, 1 \leqq k \leqq p - 1 , \; i = 1, \cdots, p(n - 1)) \tag{15}$$

The inequalities

$$O_{f \circ H_n}(\omega(n, ip)\backslash\omega(3n, ip(2n + 1)) \leqq 1/p(2n + 1) \quad (i = 1, 2, \cdots, p(n - 1)) \tag{16}$$

follow from (9) and (12). Relation (10) (with $n - 1$ replaced by n) and the fact that $\omega(3n, ip(2n + 1)) \supset \omega(3(n + 1), i'p(2(n + 1) + 1))$, where $i' = ip$, imply that (16) holds with H_n replaced by H_{n+1}. This last step may be successively repeated to obtain

$$O_{f \circ H_m}(\omega(n, ip)\backslash\omega(3n, ip(2n + 1)) \leqq 1/p(2n + 1) \quad (n \leqq m) . \tag{17}$$

However, since $f \circ H_m$ tends uniformly to g we see that

$$O_g(\omega(n, ip)\backslash\omega(3n, ip(2n + 1)) \leqq 1/p(2n + 1) . \tag{18}$$

From (15) and the fact that the measure of $\omega(n, j)$ is $p(-n)$ we obtain the inequalities

$$\min_{b_j} \int_{\omega(n,j)} |g(x) - b_j|\, dx \leqq 1/\{p(2n + 1)p(n)\} \quad (j = (i - 1)p + k, 1 \leqq k \leqq p - 1, i = 1, \cdots, p(n - 1)) . \tag{19}$$

From (18) we deduce that

$$\min_{b_j} \int_{\omega(n,j)} |g(x) - b_j|\, dx \leqq 1/\{p(2n + 1)p(n)\} + 2/p(3n) \quad (j = ip, i = 1, \cdots, p(n - 1)) . \tag{20}$$

We consider now the nth term of the series described in the lemma. Combining (19) and (20) we obtain the inequality

$$p(n)(p-1)\sum_{j=1}^{p(n)} \min_{b_j} \int_{\omega(n,j)} |g(x)-b_j|\,dx \leqq p(n)(p-1)\{1/p(2n+1) + 2p(n-1)/p(3n)\} \leqq 3/p(n)\,.$$

Therefore, the series of the lemma is convergent. The proof of the theorem is now complete.

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Received October 10, 1974.

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