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SUBORDINATING FACTOR SEQUENCES AND CONVEX FUNCTIONS OF SEVERAL VARIABLES

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In this paper we consider univalent holomorphic maps of E^n, the unit disk in C^n. We generalize Wilf's subordinating factor sequences to functions on E^n and use this characterization to obtain a covering theorem and bounds for convex mappings in C^n.

1. Introduction. Let K^n denote the class of functions F which are holomorphic and univalent in $E^n = \{z = (z_1, \cdots, z_n): \mathrm{Max}_{1 \leq i \leq n} |z_i| < 1\}$, maps E^n onto a convex region in C^n, and satisfy $F(0) = 0$ and the Jacobian J of the mapping F is nonsingular. Let G and H be holomorphic in E^n. If $G(E^n) \subset H(E^n)$, then G is subordinate to $H(G \prec H)$. If $F = (F_1, \cdots, F_n) \in K^n$ then each F_i has an expansion of the form

$$F_i(Z) = \sum_{k=1}^{\infty} \sum_{\nu_1 + \cdots + \nu_n = k} a_{\nu_1 \cdots \nu_n}(i) z_1^{\nu_1} \cdots z_n^{\nu_n} .$$

In this paper we characterize the sequences $\{c_{\nu_1 \cdots \nu_n}(i)\}$ $(i = 1, \cdots, n)$ such that the mapping

$$H = (H_1, \cdots, H_n)$$

where

$$H_i(Z) = \sum_{k=1}^{\infty} \sum_{\nu_1 + \cdots + \nu_n = k} c_{\nu_1 \cdots \nu_n}(i) a_{\nu_1 \cdots \nu_n}(i) z^{\nu_1} \cdots z^{\nu_n}$$

is subordinate to F, for all $F \in K^n$. Then we obtain a covering theorem and bounds for convex mappings.

For $n = 1$, the class K^1 is the classical family of univalent functions $F(z) = \sum_{k=1}^{\infty} a_k z^k$ which maps the unit disk onto a convex domain. Wilf [4] has characterized the sequences $\{c_k\}$ (subordinating factor sequences) such that $h(z) = \sum c_k a_k z^k$ is subordinate to $f(z) = \sum_{k=1}^{\infty} a_k z^k$ whenever $f \in K^1$. For $n > 1$, Suffridge [3] has given the following characterization of the class K^n.

THEOREM A. *Suppose $F: E^n \to C^n$ is holomorphic, $F(0) = 0$, and that J is nonsingular for all $Z \in E^n$. Then F is a univalent map of E^n onto a convex domain if and only if there exists univalent mappings $f_j \in k^1 (1 \leq j \leq n)$ such that $F(Z) = T(f_1(z_1), \cdots, f_n(z_n))$ where T is a nonsingular linear transformation.*

From Theorem A we see that if $F=(F_1,\cdots,F_n)\in K^n$ then

$$F_i(z_1,\cdots,z_n)=\sum_{k=1}^{\infty}(a_{i1}^k z_1^k+\cdots+a_{in}^k z_n^k)\ .$$

Thus we could represent $F\in K^n$ by the column vector

$$F(Z)=\sum_{K=1}^{\infty}A_k Z^k$$

where

$$A_k=\begin{bmatrix}a_{i1}^k & \cdots & a_{in}^k\\ \vdots & & \\ a_{n1}^k & & a_{nn}^k\end{bmatrix}\qquad Z^k=\begin{bmatrix}z_1^k\\ \vdots\\ z_n^k\end{bmatrix}.$$

2. Subordinating factor sequences. An infinite sequence $\{C_k\}$ of $n\times n$ matrices of complex numbers will be called a subordinating factor sequence if for each $F(Z)=\sum A_k Z^k\in K^n$ we have $\sum C_k \circledast A_k Z^k \prec F(Z)$, where $C_k\circledast A_k$ is the Hadamard product. If $C=(c_{ij})$ and $A=(a_{ij})$ then $C\circledast A=(c_{ij}a_{ij})$. Let $\mathscr{F}^n$ denote the collection of subordinating factor sequences.

THEOREM 1. *If $\{C_k\}\in\mathscr{F}^n$, then for each k the rows of $C_k=(c_{ij}^k)$ are identical, that is, for each k $(k=1,2,\cdots)$ and each j $(j=1,\cdots,n)$ we have $c_{1j}^k=c_{2j}^k=\cdots=c_{nj}^k$.*

Proof. Let $\{C_k\}\in\mathscr{F}^n$. First consider $k=1$. Pick $\zeta=(\zeta_1,\cdots,\zeta_n)\in E^n$ where $\zeta_i\neq 0$ and if $c_{jj}^1\neq 0$ then $\zeta_j=1/2e^{-i\alpha}$ with $\alpha=\arg c_{jj}^1$ if $c_{jj}^1=0$ then $\zeta_j=0$. Let $\delta=(c_{ji}^1-c_{ii}^1)\zeta_i$. If $\delta=0$, then $c_{ji}^1=c_{ii}^1$. If $\delta\neq 0$, let $M=1/\delta$. Then define the mapping $F=(F_1,\cdots,F_n)$ where $F_i(Z)=Mz_i$, $F_j(Z)=Mz_i+z_j$, and $F_k(Z)=z_k$ when neither $k\neq i$ or $k\neq j$. The mapping F is a convex univalent map by Theorem A. Thus since $\{C_k\}\in\mathscr{F}^n$ the mapping $H=(H_1,\cdots H_n)$, where $H_i(Z)=Mc_{ii}^1 z_i$, $H_j(Z)=Mc_{ji}^1 z_i+c_{jj}^1 z_j$ and $H_k(Z)=c_{kk}^1 z_K$ for $k\neq i$ or $k\neq j$, is subordinate to F. In particular, there is a $Z\in E^n$ such that $H(\zeta)=F(Z)$, which says

$$Mz_i=Mc_{ii}^1\zeta_i$$

and

$$Mz_i+z_j=Mc_{ji}^1\zeta_i+c_{jj}^1\zeta_j\ .$$

Solving for z_j we obtain

$$z_j=M(c_{ji}^1-c_{ii}^1)\zeta_i+c_{jj}^1\zeta_j=1+\frac{1}{2}|c_{ij}^1|\geqq 1\ .$$

This contradicts the fact that $|Z| < 1$. Thus we have $\delta = 0$ or $c^1_{1j} = c^1_{2j} = \cdots = c^1_{nj}$ for $j = 1, \cdots, n$.

For $k > 1$ we define the mapping $F = (F_1, \cdots, F_n)$ where

$$F_i(Z) = Mz_i + \frac{Mz_j^k}{k^2},\ F_j(Z) = Mz_i + \frac{Mz_i^k}{k^2} + z_j, \quad \text{and} \quad F_k(Z) = z_k$$

for neither $k \neq i$ or $k \neq j$. Then the proof that $c^k_{1j} = c^k_{2j} = \cdots = c^k_{nj}$ is similar to the proof for $k = 1$.

From Theorem 1 we have that if $\{C_k\} \in \mathscr{F}^n$, then for each k the rows of C_k are indentical. For the $n \times n$ matrices C_k we will use the notation

$$C_k = \begin{bmatrix} c_1^k & \cdots & c_n^k \\ \vdots & & \\ c_1^k & \cdots & c_n^k \end{bmatrix} = (c_1^k, \cdots, c_n^k) .$$

Using Theorem 1 we are now able to characterize class $\mathscr{F}^n$.

THEOREM 2. *The following are equivalent*:

(i) $\{C_K\} \in \mathscr{F}^n$ *where* $C_K = (c_1^k, \cdots, c_n^k)$.

(ii) *For each* $j = 1, \cdots, n$ *we have*

$$\operatorname{Re}\left\{1 + 2\sum_{k=1}^{\infty} c_j^k z_j^k\right\} > 0 \quad \textit{for} \quad |z_j| < 1 .$$

(iii) *For each* $j = 1, \cdots, n$ *there is a nondecreasing function* Ψ_j *on* $[0, 2\pi]$ *such that*

$$c_j^k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\theta} d\Psi_j(\theta) \quad \textit{and} \quad c_j^0 = 1 .$$

Proof. The Herglotz's integral representation for positive harmonic functions proves that (ii) and (iii) are equivalent. Let $\{C_k\} \in \mathscr{F}^n$, where $C_k = (c_1^k, \cdots, c_n^k)$. Let $f_i(z_i) = z_i/(1 - z_i)$. Then by Theorem A the mapping F is in K^n. We may write

$$F(Z) = \sum_{k=1}^{\infty} A_k Z^k$$

where $A_k = (a_{ij}^k)$ and $a_{ji}^k = 0$ if $i \neq j$ and $a_{ii}^k = 1$ then the mapping

$$H(Z) = \sum_{k=1}^{\infty} C_k \odot A_k Z^k$$

is subordinate to F. The mapping H has components $H_i(Z) = \sum_{k=1}^{\infty} c_i^k z_i^k$. Since $H \prec F$ we have that $H_i(F_i) \subset f_i(E_i)$ or $\operatorname{Re}\{H_i(E_i)\} \geqq -1/2$ where $E_i = \{z_i : |z_i| < 1\}$. Thus $\operatorname{Re}\{\sum_{k=1}^{\infty} c_i^k z_i^k\} > -1/2$ for $i =$

$1, \cdots, n$, Now suppose (iii) holds. Let $F \in K^n$. Then by Theorem A there exists a nonsingular matrix T and functions $f_1, \cdots, f_n \in K^1$, where $f_i(z_i) = \sum_{k=1}^{\infty} a_k(i) z_i^k$, such that

$$F(Z) = T\begin{bmatrix} f_1(z_1) \\ \vdots \\ f_n(z_n) \end{bmatrix}$$

where F is a column vector. Then

$$H(Z) = \sum C_k \odot A_k z^k = T\begin{bmatrix} \sum_{k=1}^{\infty} c_1^k a_k(1) z_1^k \\ \vdots \\ \sum_{k=1}^{\infty} c_n^k a_k(n) z_n^k \end{bmatrix}$$

$$= T\begin{bmatrix} \sum_{k=1}^{\infty} \frac{1}{2\pi}\int_0^{2\pi} e^{ik\phi} d\Psi_1(\phi) a_k(1) z_1^k \\ \vdots \\ \sum_{k=1}^{\infty} \frac{1}{2\pi}\int_0^{2\pi} e^{ik\phi} d\psi_n(\phi) a_k(n) z_n^k \end{bmatrix}$$

$$= T\begin{bmatrix} \frac{1}{2\pi}\int_0^{2\pi} \sum_{k=1}^{\infty} a_k(1) r_1^k e^{ij(\theta_1+\phi)} d\Psi_1(\phi) \\ \vdots \\ \frac{1}{2}\int_0^{2\pi} \sum_{k=1}^{\infty} a_k(n) r_n^k e^{ik(\theta_n+\phi)} d\Psi_n(\phi) \end{bmatrix}$$

$$= T\begin{bmatrix} \frac{1}{2\pi}\int_0^{2\pi} f_1(r_1 e^{i(\theta_1+\psi)}) d\Psi_n(\phi) \\ \vdots \\ \frac{1}{2\pi}\int_0^{2\pi} f_n(r_n e^{i(\theta_n+\phi)}) d\Psi_n(\phi) \end{bmatrix}$$

where $z_j = r_j e^{i\theta_j}$. Since each integral in the left hand side is the centroid of a nonnegative mass distribution of total mass one on a convex curve, the value of each integral must lie inside its convex curve. Further since T is a nonsingular linear transformation $H(Z)$ lies inside the image of the polydisk of radius $(r_1, \cdots, r_n)$. (A polydisk or radius $(r_1, \cdots, r_n)$ is the set $\{(z_1, \cdots, z_n): |z_i| \leqq r_i$ for $i = 1, \cdots, n\}$.) Thus $H \prec F$.

3. Convex mappings in C^n.

We now apply Theorem 2 to obtain some results for mapping in K^n.

COROLLARY 1. *For $n > 1$ let $G \in K^n$, where $G(Z) = \sum B_k Z^k$.*

Then the mapping

$$G_F^*(Z) = \sum B_k \odot A_k Z^k \,,$$

where $F(Z) = \sum A_k Z^k \in K^n$, *is not subordinate to* F *for all* $F \in K^n$.

Proof. If $G_F^* \prec F$ for all $F \in K^n$, then the sequence $\{B_k\}$ belongs to $\mathscr{F}^n$. This says that the rows of each B_k are indentical by Theorem 1. Hence the Jacobian of G will be identically zero. Thus G_F^* is not subordinate to F for all $F \in K^n$.

Let $T = (t_{ij})$ be a $n \times n$ nonsingular matrix. Let K be the functions $f \in K^1$ where $f'(0) = 1$. Let KT denote the subclass of K^n which is defined by $F \in KT$ if and only if there exist functions $f_i \in K(i = 1, 2, \cdots, n)$ such that

$$F(Z) = T\begin{pmatrix} f_1(z_1) \\ \vdots \\ f_n(z_n) \end{pmatrix}$$

where F is represented as a column vector.

COROLLARY 2. *The image of* E^n *under a mapping* $F \in KT$ *contains the polydisk* $|w| < 1/2(\sum_{j=1}^n |t_{ij}|, \cdots, \sum_{j=1}^n |t_{nj}|)$. *The radius is sharp.*

Proof. Since the sequence $\{C_k\}$ where $C_1 = (1/2, 1/2, \cdots, 1/2)$ and $C_k = (0, \cdots, 0)$ for $k \geqq 2$, belongs to $\mathscr{F}^n$, we see that the image of E^n under a mapping $F \in KT$ contains $|W| < 1/2(\sum_{j=1}^n |t_{1j}|, \cdots, \sum_{j=1}^n |t_{nk}|)$. The sharpness follows by using the function

$$F(Z) = T\begin{bmatrix} \dfrac{z_1}{1 - z_1} \\ \vdots \\ \dfrac{z_n}{1 - z_n} \end{bmatrix}.$$

Ruscheweyh and Sheil-Small [2] have proven Pólya and Schoenberg's [1] conjecture that if $f(z) = \sum_{k=1}^\infty a_k z^k$ and $g(z) = \sum b_k z^k$ are elements of K^1 then so is the function $h(z) = \sum a_k b_k z^k$. In general for K^n this is not true as shown by the example $F(Z) = \begin{pmatrix} z_1 - z_2 \\ z_1 + z_2 \end{pmatrix} = G(Z)$. However, we do have the following Pólya and Schoenberg tpye of theorem.

THEOREM 3. *Let* $T_1 = (p_{ij})$ *and* $T_2 = (q_{ij})$ *be* $n \times n$ *nonsingular matrices such that* $T = T_1 \odot T_2 = (p_{ij} q_{ij})$ *is nonsingular. If* $F(Z) =$

$\sum_{k=1}^{\infty} A_k Z^k \in KT_1$ and $G(Z) = \sum_{k=1}^{\infty} B_k Z^k \in KT_2$, then $H(Z) = \sum_{k=1}^{\infty} A_k \odot B_k Z^k$ belongs to KT.

Proof. Let $F \in KT_1$ and $G \in KT_2$. Then there exists functions $f_i, g_i \in K(i = 1, \cdots, n)$ such that

$$F(Z) = T_1 \begin{bmatrix} f_1(z_1) \\ \vdots \\ f_n(z_n) \end{bmatrix}$$

and

$$G(Z) = T_2 \begin{bmatrix} g_1(z_1) \\ \vdots \\ g_n(z)_n \end{bmatrix}$$

The mapping $H(Z) = \sum_{k=1}^{\infty} A_k \odot B_k z^k$ may be written as

$$H(Z) = T \begin{pmatrix} z_1 + \sum_{k=1}^{\infty} a_k(1) b_k(1) z_1^k \\ \vdots \\ z_n + \sum_{k=2}^{\infty} a_k(n) b_k(n) z_k^n \end{pmatrix}.$$

Thus $H \in KT$ since $z_i + \sum a_k(i) b_k(i) z_i^k$ belongs to K for each i [2].

4. **Bounds on Mapping in K_n.** Let $F \in K^n$. Then by Suffridge's representation of mappings in K^n (Theorem A), there exist an $n \times n$ nonsingular matrix $T = (t_{ij})$ and functions $f_i(z_i) = \sum_{k=1}^{\infty} a_k(i) z_i^k (i = 1, \cdots, n)$ in K^1 with $f_1'(0) = 1$ such that

$$F(Z) = T \begin{pmatrix} f_1(z_1) \\ \vdots \\ f_n(z_n) \end{pmatrix}.$$

Then

$$A_k = (a_{ij}) = T \begin{pmatrix} a_k(1) \\ \vdots \\ a_k(n) \end{pmatrix}$$

where $F(z) = \sum_{k=1}^{\infty} A_k Z^k$. Since

$$|a_k(i)| < 1 \quad \text{and} \quad \frac{|z_i|}{1 + |z_i|} < |f_i(z_i)| < \frac{|z_i|}{1 - |z_i|},$$

we have the following theorem.

THEOREM 4. *Let* $F(z) = \sum_{k=1}^{\infty} A_k Z^k$ *belongs to* K^n. *Let* T *be an* $n \times n$ *nonsingular matrix and let* $f_1, \cdots, f_n \in K^1$ *such that*

$$F(Z) = T\begin{pmatrix} f_1(z_1) \\ \vdots \\ f_n(z_n) \end{pmatrix}.$$

Then

$$|a_{ij}^k| < |t_{ij}|$$

for each k, i, *and* j, *where* $A_k = (a_{ij}^k)$. *Let* $F = (F_1, \cdots, F_n)$. *Then*

$$\sum_{j=1}^{n} |t_{ij}| \frac{|z_j|}{1 + |z_j|} \leqq |F_i(Z)| < \sum_{j=1}^{n} |t_{ij}| \frac{|z_j|}{1 - |z_j|} .$$

Both inequality are sharp.

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