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ON FACETS WITH NON-ARBITRARY SHAPES

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It is proved that the shape of a facet of a d-polytope with $d+3$ vertices can be arbitrarily preassigned. A minimal example of a 4-polytope with 8 vertices which does not have this property is described.

1. Introduction. The shape of a facet F of a polytope P is said to be *arbitrarily preassignable* if, given any polytope F' combinatorially equivalent to F there is a polytope P' combinatorially equivalent to P such that F' is a facet of P' and F' is the image of F under the combinatorial isomorphism which maps P onto P'.

In [3] Barnette and Grünbaum proved that the shape of one n-gonal 2-face F of a 3-polytope can be any preassigned convex n-gon F'. They ask to what extent their result holds in higher dimensions. They mention that there is an 8-polytope P with 12 vertices such that the shape of one of its 7-dimensional faces can not be arbitrarily chosen, and they conjecture that a similar example can be found already in four dimensions.

In [4], such a 4-polytope with 13 vertices is described. We shall describe a smaller example of this type in the proof of our first theorem:

THEOREM. *There is a* 4-*polytope with* 8 *vertices such that the shape of one of its* 3-*faces can not be arbitrarily preassigned.*

From the results in [3] and the following lemma we know that the above theorem yields a minimal example of such a polytope.

LEMMA. *Let P be a d-polytope with $d+3$ vertices. Then the shape of any facet of P can be arbitrarily preassigned.*

Proof of the theorem. We shall prove the theorem by describing a 4-polytope P, the facets of which are given by their vertices in Table 1.

P possesses 15 facets, 14 of them being tetrahedra and one an octahedron. The vertices of the octahedron are labelled like it is described in Figure 1.

First of all, we have to show that the complex described in Table 1 is isomorphic to the boundary-complex of a convex polytope. Those 3-polytopes given in Table 1 which do not contain the vertex 1, are either an octahedron (235678) or the convex hull of the vertex

TABLE 1

235678	1248
1237	2348
1347	1568
1467	1458
4567	1268
1267	3458
3457	1234
1456	

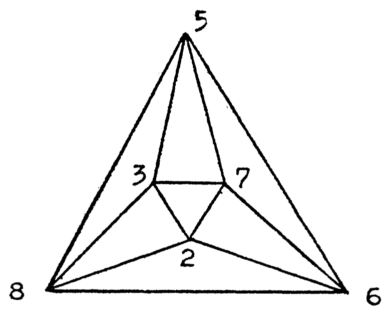

FIGURE 1

4 and a 2-face of the octahedron (4567, 3457, 2348, 3458). Consequently, the boundary-complex of a pyramid over an octahedron contains a subcomplex isomorphic to the complex formed by these five 3-polytopes. The underlying set of this complex is homeomorphic to a 3-ball.

Using the result of [2], we obtain the following special formulation of Theorem 9 in [5]: If B is a complex formed by 3-faces in the boundary-complex of a 4-pyramid, and if the underlying set of B is homeomorphic to a 3-ball, then there is a 4-polytope whose set of facets consists of a set isomorphic to B and all 3-polytopes which are the convex hull of a new vertex and the boundary cells of B.

Applying this theorem to the complex given in Table 1, we see that it is isomorphic to the boundary-complex of a 4-polytope. We now prove the following:

(1) There is no polytope P' combinatorially equivalent to P such that those vertices of P' which correspond to the vertices 2, 3, 5 and 6 lie in one plane.

We assume that there is a polytope of the type P' and regard its Schlegel-diagram $\mathscr{P}'$ with basis 1234. Easy calculations show that in $\mathscr{P}'$ the vertices 2, 3, 5 and 6 are still coplanar (we use the same symbols for vertices of P' and their images in $\mathscr{P}'$). So we restrict our attention to $\mathscr{P}'$ which we assume to lie in a 3-dimensional space.

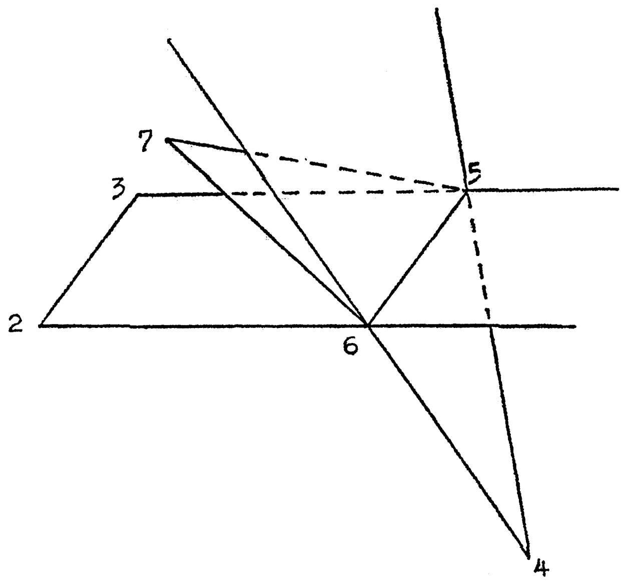

FIGURE 2

Let H_1 be the open halfspace which contains the vertex 1 and is bounded by the plane spanned by 2, 3, 5 and 6. Then the vertex 7 lies in H_1, for otherwise 1237 and 235678 would have common inner points.

Let H_2 be the open halfspace which does not contain the vertex 1 and is bounded by the plane spanned by 456. Then 7 lies in H_2, for otherwise 1456 and 4567 would have common inner points. These *arguments yield the following incidences:* 7 *lies in* $H_1 \cap H_2$, *the edge* 23 lies in H_2 and 4 does not lie in H_1.

From this we may conclude that 235678 and 4567 have common inner points (see Figure 2), a contradiction. So we have proved our assumption to be false and consequently, (1) holds. It follows immediately from (1) that an octahedron, the 3 diagonals of which meet in one point, can not be a facet of a polytope combinatorially equivalent to P. Thus the theorem is proved.

REMARKS. A polytope P is said to be *projectively unique* provided every polytope P' combinatorially equivalent to P is projectively equivalent to P. The 8-polytope with 12 vertices mentioned above is a projectively unique one and possesses a facet F which is not projectively unique. Thus in this example, the shape of F may only be arbitrarily chosen within the class of polytopes which are projectively equivalent to F.

Our example P reveals another phenomenon concerning the freedom of choice of the shape of a facet: In a polytope combinatorially equivalent to P, the intersection of the segment 36 and the triangle 257 has to be an inner point of 257. For, if the intersection were in the boundary of 257, we would have a contradiction to (1). Or, if the intersection were empty, the Schlegel-diagram of P obtained from a projection onto the facet 1234 could be subdivided in such a way that the new diagram would be isomorphic to the non-polyhedral diagram constructed in [1]. The underlying polyhedron of this diagram, however, can not be 1234 (see [1]), which contradicts our assumption.

Consequently, any metrical type of an octahedron can be preassigned to be a facet of P, only if the corresponding vertices of P and the octahedron are labelled in the right way.

It would be interesting to find other phenomena which limit the freedom of preassigning the shape of a facet.

Proof of lemma. Let P be a d-polytope with $d+3$ vertices possessing a facet F with $d+2$ vertices. Then P is a pyramid with basis F, and any polytope F' combinatorially equivalent to F can serve as a basis for a polytope combinatorially equivalent to P.

Now let P be a d-polytope with $d+3$ vertices possessing a facet F with less than $d+2$ vertices, and let F' be any polytope combinatorially equivalent to F. Then there is a projective transformation f which is permissible for F and maps F onto F'. If we extend f in a suitable way to the affine space spanned by P, we obtain a projective transformation g which is permissible for P and maps P onto a polytope P'. P' is of course combinatorially equivalent to P and possesses all the required properties. Taking k-fold pyramids over the 4-polytope described in the theorem gives $(k+4)$-polytopes with $k+8$ vertices with k-fold pyramids over an octahedron as facets whose shape can not be preassigned. Consequently, the lemma is the best possible.

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