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ON EXTENDING REGULAR HOLOMORPHIC MAPS FROM STEIN MANIFOLDS

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In this paper the following results are proved. THEOREM. Let S, M be complex manifolds, S a Stein manifold, and $f: S \to M$ a holomorphic embedding. Let $K \subset S$ be compact, and let N_f be the normal bundle of f. We identify S with the zero section of N_f. Then in N_f, there is a neighborhood U of K, and a holomorphic embedding $F: U \to M$ such that $F|U \cap S = f$. If f above is an immersion, then there is an immersion F as above. There is also an analogous result for holomorphic maps f which are regular at some point p in S.

The idea of the proof is to construct a function ϕ on a neighborhood of $f(K) \subset M$ such that ϕ is strictly plurisubharmonic and $\phi^{-1}((-\infty, c])$ is compact for all c in R. Then a result of Forster and Ramspott is applied to get the final results. To construct ϕ, special coordinates are obtained near $f(S)$ in M.

1. Introduction. The central result of this paper is an analogue, in the category of complex manifolds and holomorphic maps, of the tubular neighborhood theorem. One result which extends the tubular neighborhood theorem to this category is due to Forster and Rampsott and goes as follows. Let S and M be Stein manifolds, $f: S \to M$ a holomorphic embedding, and identify S with the zero-section of the normal bundle of f. Then there is a neighborhood U of S in the normal bundle, and a biholomorphic map $\tilde{f}: U \to M$ such that $\tilde{f}|S = f$.

Results of the type mentioned above are sometimes needed when M is not assumed to be a Stein manifold. For instance, a result of this type is needed in order to prove the upper semi-continuity of the differential form of the Kobayashi metric as done by Royden in [4].

It will be shown that if $f: S \to M$ is a holomorphic embedding (resp., immersion) S is a Stein manifold, and $K \subset S$ is compact, then there is a neighborhood U of K, where K is considered as a subset of the normal bundle of f, and a biholomorphic map (resp., holomorphic immersion) $f: U \to M$ such that $\tilde{f}|S \cap U = f|S \cap U$. Also, with S, M, K as above and $f: S \to M$ a holomorphic map regular at a point $p_0 \in S$ (i.e. $df: T_{p_0}S \to T_{f(p_0)}M$ is injective), then there is a trivial bundle $\tilde{A}$ over S, $\dim_C \tilde{A} = \dim_C M$, a neighborhood U of K in $\tilde{A}$, and a holomorphic map $\tilde{f}: U \to M$ such that $\tilde{f}|S \cap U = f$ and $\tilde{f}$ is regular at p_0 (that is, in this case $d\tilde{f}: (T\tilde{A})_{p_0} \to (TM)_{f(p_0)}$ is an isomorphism). The key ingredient in the proofs of these results is the construction of an appropriate strictly plurisubharmonic function defined on a neighborhood of $f(K)$ in M, which reduces things to the result of Forster and Ramspott.

These results are contained in a thesis done at Stanford University, and were announced in [6]. The author would like to thank Professor Halsey Royden for his helpful advice.

REMARKS. For standard terminology and results in several complex variables we refer to the books of Gunning and Rossi [2] and Hörmander [3].

2. To construct the desired strictly plurisubharmonic function on M, we need to obtain coordinate systems which are related to each other in a special way. This section will contain the needed results. The main fact used here is that S is a Stain manifold and hence the one-dimensional cohomology of S with coefficients in a coherent analytic sheaf vanishes.

NOTE. (a) All coordinate neighborhoods mentioned in the remainder of this paper are assumed to be polydiscs with compact closure.

(b) A point on a manifold will be identified its with coordinates with respect to a given coordinate system under consideration unless some confusion arises from doing so.

Let S be a Stein manifold, $\dim S = m$, M is a complex manifold, $\dim M = n$, $m < n$, and let $f\colon S \to M$ be a holomorphic embedding. By the implicit function theorem, for any $p \in S$, there is a coordinate system $(\tilde{Q}_z, z)$ in M near $f(p)$ such that $S \cap \tilde{Q}_z = \{q \in \tilde{Q}_z \mid z_j(q) = 0,$ $m < j \leqq n\}$. In everything below, all coordinate systems in M near $f(S)$ will be assumed to be of this type.

NOTE. Throughout the rest of this section we assume we are given a Stein manifold S, $\dim_C S = m$, a complex manifold M, $\dim_C M = n > m$, and a holomorphic embedding $f\colon S \to M$. We let N_f denote the normal bundle of f and we let TS denote the bundle of holomorphic tangent vectors to S.

We now take the first step towards getting a collection of appropriately related coordinate systems.

PROPOSITION 1. *There is a collection $\mathscr{C}$ of coordinate systems $(\tilde{Q}_z, z)$ of the above type which cover $f(S)$, is as fine as we like, and satisfies the following property: if $(\tilde{Q}_z, z)$ and $(\tilde{Q}_w, w)$ are in $\mathscr{C}$,* *then the Jacobian of the change of coordinates in* $\tilde{Q}_z \cap \tilde{Q}_w$ *is of the form*

$$\left[\frac{\partial w_i}{\partial z_j}\right] = \left(\begin{array}{c|c} J_1^{zw} & 0 \\ \hline 0 & J_2^{zw} \end{array}\right) \quad \textit{on } f(S), \textit{ where } \quad J_1^{zw} = \frac{\partial(w_1, \cdots, w_m)}{\partial(z_1, \cdots, z_m)} .$$

Proof. By the way we have chosen our coordinates, it is clear that the Jacobian $[\partial w_i/\partial z_j]$, when restricted to $f(S)$, is of the form

$$\left(\begin{array}{c|c} J_1^{zw} & A^{zw} \\ \hline 0 & J_2^{zw} \end{array}\right) .$$

We can consider A^{zw} as a holomorphic section of the bundle $\operatorname{Hom}(N_f, TS)$ over $Q_z \cap Q_w$, by considering it as the matrix of a bundle map $N_p|_{Q_z \cap Q_w} \rightarrow TS|_{Q_z \cap Q_w}$ with respect to the basis $\{\partial/\partial z_{m+1}, \cdots, \partial/\partial z_n\}$ in N_f and the basis $\{\partial/\partial w_1, \cdots, \partial/\partial w_m\}$ in TS. Thus the collection $\{A^{zw}\}$ determines a 1-cocycle $\{\tilde{A}^{zw}\}$ on S, with respect to the covering $\{Q_z\}$, with coefficients in the sheaf, $\mathfrak{Hom}(N_f, TS)$, of germs of holomorphic sections of $\operatorname{Hom}(N_f, TS)$ (see [2] p. 256).

We have $H^1(S, \mathfrak{Hom}(N_f, TS)) = 0$ since S is a Stein manifold. Because we have taken all the Q_z's to be polydiscs, the covering $\{Q_z\}$ is a Leray covering and hence there is a 0-cochain $\{\tilde{B}^z\}$ on S, with respect to the covering $\{Q_z\}$, whose coboundary is $\{\tilde{A}^{zw}\}$, i.e. $\tilde{A}^{zw} = \tilde{B}^w - \tilde{B}^z$. Let $B^w(z, w)$ be the matrix of $\tilde{B}^w$ with respect to z-coordinates in N_f and w-corrdinates in TS etc. We now modify our old corrdinates z, w to get new coordinates ζ in $\tilde{Q}_z$ and ω in $\tilde{Q}_w$ defined by

$$\begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_n \end{pmatrix} = \left(\begin{array}{c|c} I_m & -B^z(z, z)(z_1, \cdots, z_m) \\ \hline 0 & I_{n-m} \end{array}\right) \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

and

$$\begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix} = \left(\begin{array}{c|c} I_m & -B^w(w, w)(w_1, \cdots, w_m) \\ \hline 0 & I_{n-m} \end{array}\right) \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} .$$

Then on $f(S)$ we have

$$\begin{aligned} \left[\frac{\partial \omega_i}{\partial \zeta_j}\right] &= \left[\frac{\partial \omega_i}{\partial w_j}\right]\left[\frac{\partial w_i}{\partial z_j}\right]\left[\frac{\partial z_i}{\partial z_j}\right] \\ &= \left(\begin{array}{c|c} I_m & -B^w(w, w) \\ \hline 0 & I_{n-m} \end{array}\right)\left(\begin{array}{c|c} J_1^{zw} & B^w(z, w) - B^z(z, w) \\ \hline 0 & J_2^{zw} \end{array}\right)\left(\begin{array}{c|c} I_m & B^z(z, z) \\ \hline 0 & I_{n-m} \end{array}\right) \\ &= \left(\begin{array}{c|c} J_1^{zw} & 0 \\ \hline 0 & J_2^{zw} \end{array}\right) = \left(\begin{array}{c|c} J_1^{\zeta\omega} & 0 \\ \hline 0 & J_2^{\zeta\omega} \end{array}\right) . \end{aligned}$$

Let $\mathscr{C}$ be the collection of all coordinate systems $(\tilde{Q}_z, \zeta)$ as just obtained. Clearly $\mathscr{C}$ satisfies the properties stated in the proposition.

NOTE. From now on we work within the collection $\mathscr{C}$ of coordinate systems obtained above.

Next we want coordinate systems around $f(S)$ which are related to each other by the Jacobian of the change of coordinates on $f(S)$, up to terms "third order" and higher in the variables $z_{m+1}, \cdots, z_n$. This is done inductively, so we start with an easy special case, and eliminate the details in the general case.

NOTATION. Let N_f be the normal bundle of f and let TS be the holomorphic tangent bundle of S; let $\mathscr{N}$ and τS be the correponding sheaves of holomorphic sections, and let $\mathscr{N}^*$ be the dual of $\mathscr{N}$.

COCYCLE LEMMA S. *Let $\mathscr{N}$, $\mathscr{N}^*$, τS be as above. Associate to each ordered intersection $Q_u \cap Q_w$ the section of $\mathscr{N} \otimes \mathscr{N}^* \otimes \mathscr{N}^*$* (resp. $\tau S \otimes \mathscr{N}^* \otimes \mathscr{N}^*$) *given by*

$$\sum_{r,k,l=m+1}^{n} \frac{\partial^2 u_r}{\partial w_k \partial w_l} \frac{\partial}{\partial u_r} \otimes dw_k \otimes dw_l = A^{wu}$$

$$\left(\text{resp. } \sum_{r=1}^{m} \sum_{k,l=m+1}^{n} \frac{\partial^2 u_r}{\partial w_k \partial w_l} \frac{\partial}{\partial u_r} \otimes dw_k \otimes dw_l = B^{wu}\right).$$

Then the collection $\{A^{wu}\}$ (resp. $\{B^{wu}\}$) *determines a 1-cocylce on S with respect to the covering $\{Q_z\}$, with coefficients in $\mathscr{N} \otimes \mathscr{N}^* \otimes \mathscr{N}^*$* (resp. $\tau S \otimes \mathscr{N}^* \otimes \mathscr{N}^*$).

Proof. Step a. Suppose $m < k,\ l \leq n$ and consider coordinate systems $(\tilde{Q}_u, u)$, $(\tilde{Q}_w, w)$, and $(\tilde{Q}_z, z)$. It is clear that on $\tilde{Q}_u \cap \tilde{Q}_w \cap \tilde{Q}_z \cap f(S)$ we have

$$\frac{\partial^2 w_j}{\partial w_k \partial w_l} = 0\ , \quad \text{so} \quad \sum_{j=1}^{n} \frac{\partial^2 w_j}{\partial w_k \partial w_l} \frac{\partial u_r}{\partial w_j} = 0\ ,$$

and by calculating using the chain rule we get

$$\text{(S.a)} \qquad 0 = \frac{\partial^2 u_r}{\partial w_k \partial w_l} + \sum_{i,h,a=1}^{n} \frac{\partial^2 z_a}{\partial u_h \partial u_i} \frac{\partial w_h}{\partial w_k} \frac{\partial u_i}{\partial w_l} \frac{\partial u_r}{\partial z_a} + \sum_{a,b,j=1}^{n} \frac{\partial^2 w_j}{\partial z_a \partial z_b} \frac{\partial z_a}{\partial w_k} \frac{\partial z_b}{\partial w_l} \frac{\partial u_r}{\partial w_j}\ .$$

Step b. Notice now that on $f(S)$ we have $\partial w_a / \partial z_b = 0$ for

$a \leqq m < b$ or $b \leqq m < a$. Using this with (S.a) we now get that

$$\begin{aligned}
0 &= \sum_{l,r,k=m+1}^{n} \frac{\partial^2 w_j}{\partial w_k \partial w_l} \frac{\partial u_r}{\partial w_j} \frac{\partial}{\partial u_r} \otimes dw_k \otimes dw_l \\
&= \sum_{l,k,r=m+1}^{n} \frac{\partial^2 u_r}{\partial w_k \partial w_l} \frac{\partial}{\partial u_r} \otimes dw_k \otimes dw_l \\
&\quad + \sum_{j,a,b=m+1}^{n} \frac{\partial^2 w_j}{\partial z_a \partial z_b} \frac{\partial}{\partial w_j} \otimes dz_a \otimes dz_b \qquad \text{(S.b)} \\
&\quad + \sum_{a,h,i=m+1}^{n} \frac{\partial^2 z_a}{\partial u_h \partial u_i} \frac{\partial}{\partial z_a} \otimes du_h \otimes du_j \\
&= A^{wu} + A^{zw} + A^{uz} .
\end{aligned}$$

It is now evident that the collection $\{A^{wu}\}$ determines a 1-cocycle on S with respect to the covering $\{Q_z\}$, with coefficients in $\mathscr{N} \otimes \mathscr{N}^* \otimes \mathscr{N}^*$.

The proof for the case $\tau S \otimes \mathscr{N}^* \otimes \mathscr{N}^*$ is exactly the same, except that the r index varies between 1 and m, instead of between $m + 1$ and n as above, and A^{wu} is replaced by B^{wu}.

PROPOSITION 2S. *There is a subcollection $\mathscr{D}_2$ of $\mathscr{C}$ such that the coordinate polydiscs of $\mathscr{D}_2$ cover $f(S)$, the covering is as fine as we like, and for $(\tilde{Q}_z, z)$ and $(\tilde{Q}_w, w)$ in $\mathscr{D}_2$ we have $\partial^2 w_r/\partial z_k \partial z_l = 0$ on $f(S)$ for $k, l > m$.*

Proof. Step A. Let $\{A^{zw}\}$ and $\{B^{zw}\}$ be the cocycles in the lemma above. Since S is Stein, we have that

$$H^1(S, \mathscr{N} \otimes \mathscr{N}^* \otimes \mathscr{N}^*) = H^1(\tau S \otimes \mathscr{N}^* \otimes \mathscr{N}^*) = 0 .$$

Each member of the covering $\{Q_z\}$, which consists of all coordinate polydiscs in $\mathscr{C}$, is a Stein manifold, hence there is a 0-cochain $\{\Gamma(z)\}$ with respect to the covering and with coefficients in $(\tau S \oplus \mathscr{N}) \otimes \mathscr{N}^* \otimes \mathscr{N}^*$, whose coboundary is $\{B^{zw}\} \oplus \{A^{zw}\}$, i.e. $B^{zw} \oplus A^{zw} = \Gamma(w) - \Gamma(z)$. Let $\{\gamma^r_{kl}(\genfrac{}{}{0pt}{}{u}{w})(z)\}$ denote the components of $\Gamma(z)$ with respect to the basis $\{\partial/\partial u_r \otimes dw_k \otimes dw_l\}$, that is

$$\Gamma(z) = \sum_{r=1}^{n} \sum_{k,l=m+1}^{n} \gamma^r_{kl}(\genfrac{}{}{0pt}{}{u}{w})(z) \frac{\partial}{\partial u_r} \otimes dw_k \otimes dw_l$$

in $Q_z \cap Q_u \cap Q_w$. In this notation we of course allow $u = w = z$ etc.

Step B. In each coordinate polydisc $\tilde{Q}_z$ we define the functions $\zeta_r = z_r - \sum_{k,l=m+1}^{n} \gamma^r_{kl}(\genfrac{}{}{0pt}{}{z}{z})(z) z_k z_l$. Since $z_k = 0$ on $f(S)$ for $k \geqq m + 1$, it is clear that $\partial\zeta_r/\partial z_s = \delta^r_s$ on $f(S)$, so$[\partial\zeta_r/\partial z_s] = I_n$, the identity matrix. It is now clear that if the functions ζ_r are restricted to a sufficiently small polydisc $\tilde{Q}_\zeta \subset \tilde{Q}_z$ (whose center is the same as that of $\tilde{Q}_z$), they give a set of coordinates there. Let $\mathscr{D}_2$ be the collection of all coordinate systems $\{(\tilde{Q}_\zeta, \zeta)\}$, where the $(\tilde{Q}_\zeta, \zeta)$ are obtained from the $(\tilde{Q}_z, z)$ in $\mathscr{C}$ as described above. If $(\tilde{Q}_\zeta, \zeta)$ and $(\tilde{Q}_\omega, \omega)$ are in $\mathscr{D}_2$, then we have on $f(S)$

$$\left[\frac{\partial \omega_i}{\partial \zeta_j}\right] = \left[\frac{\partial \omega_i}{\partial \zeta_j}\right]\left[\frac{\partial w_j}{\partial z_i}\right]\left[\frac{\partial z_i}{\partial \zeta_j}\right] = I_n\left[\frac{\partial w_i}{\partial z_j}\right]I_n = \left[\frac{\partial w_i}{\partial z_j}\right]$$

and hence $\mathscr{D}_2$ is a subcollection of $\mathscr{C}$. Also $\mathscr{D}_2$ is as fine as we like and clearly its coordinate polydiscs cover $f(S)$.

Step C. Now we need only show that if $(\tilde{Q}_\zeta, \zeta)$ and $(\tilde{Q}_\omega, w)$ are in $\mathscr{D}_2$, then for $k, l > m$ we have $\partial^2 w_r/\partial\zeta_k\partial\zeta_l = 0$ on $f(S)$. We consider only $r > m$, since the $r \leqq m$ case differs only by changing the range of r index below.

Using the chain rule, the fact that terms of the form $\partial w_a/\partial z_b = 0$ on $f(S)$ for $a \leqq m < b$ or $b \leqq m < a$, the definition of $\gamma^r_{ea}\binom{w}{w}(z)$, and expressing derivatives of the form $\partial^2 z_b/\partial\zeta_k\partial\zeta_l$ $(b > m)$ in terms of things like $\partial^2\zeta_c/\partial z_h\partial z_i$ using (S.a), we have on $f(S)$ that

$$\begin{aligned}\frac{\partial^2 w_r}{\partial\zeta_k\partial\zeta_l} = &\sum_{a,e=m+1}^{n} \frac{\partial w_e}{\partial\zeta_k}\frac{\partial w_a}{\partial\zeta_l}\left[-\gamma^r_{ea}\binom{w}{w}(w)\right] \\ &+ \sum_{a,b,c=m+1}^{n} \frac{\partial\omega_r}{\partial w_a}\frac{\partial z_c}{\partial\zeta_k}\frac{\partial z_b}{\partial\zeta_l}\left[\gamma^r_{cb}\binom{w}{z}(w) - \gamma^r_{cb}\binom{w}{z}(z)\right] \\ &+ \sum_{b=m+1}^{n}\frac{\partial\omega_r}{\partial z_b}\left[-\sum_{i,h,c=m+1}^{n}\frac{\partial^2\zeta_c}{\partial z_h\partial z_i}\frac{\partial z_h}{\partial\zeta_k}\frac{\partial z_i}{\partial\zeta_l}\frac{\partial z_b}{\partial\zeta_c}\right].\end{aligned} \tag{S.c}$$

It is clear that $\partial^2\omega_r/\partial\zeta_k\partial\zeta_l = 0$ on $f(S)$ for all $k, l > m$ if and only if

$$\sum_{r,k,l=m+1}^{n}\frac{\partial^2\omega_r}{\partial\zeta_k\partial\zeta_l}\frac{\partial}{\partial\omega_r}\otimes d\zeta_h\otimes d\zeta_l = 0\ .$$

By the preceding, and using the facts that $\partial\omega_r/\partial w_a = \delta^r_a$ and $\partial/\partial w_r = \partial/\partial\omega_r$ on $f(S)$, we have then

$$\begin{aligned}&\sum_{r,k,l=m+1}^{n}\frac{\partial^2\omega_r}{\partial\zeta_k\partial\zeta_l}\frac{\partial}{\partial\omega_r}\otimes d\zeta_k\otimes d\zeta_l \\ &= \sum_{a,e,k,l,r=m+1}^{n}\frac{\partial w_e}{\partial\zeta_k}\frac{\partial w_a}{\partial\zeta_l}\left[-\gamma^r_{ea}\binom{w}{w}(w)\right]\frac{\partial}{\partial\omega_r}\otimes d\zeta_k\otimes d\zeta_l \\ &\quad+\sum_{a,b,c,k,l,r=m+1}^{n}\frac{\partial\omega_r}{\partial w_a}\frac{\partial z_c}{\partial\zeta_k}\frac{\partial z_b}{\partial\zeta_l}\left[\gamma^a_{bc}\binom{w}{z}(w) - \gamma^a_{bc}\binom{w}{z}(z)\right]\frac{\partial}{\partial\omega_r}\otimes d\zeta_k\otimes d\zeta_l \\ &\quad+\sum_{b,h,i,k,l,r=m+1}^{n}\frac{\partial\omega_r}{\partial z_b}\frac{\partial z_h}{\partial\zeta_k}\frac{\partial z_i}{\partial\zeta_l}\gamma^b_{hi}\binom{z}{z}(z)\frac{\partial}{\partial\omega_r}\otimes d\zeta_k\otimes d\zeta_l = 0\ .\end{aligned}$$

This last equality is obtained after expressing in everything in terms of the basis $\{\partial/\partial z_a \otimes dz_b \otimes dz_c\}$. We have now finished the proof.

We now clear away higher order terms from the change of coordinates $w(z)$ on $\tilde{Q}_z \cap \tilde{Q}_w$. The proofs are much the same as in the lemma and Proposition 2.2S above, but the calculations become a bit longer. We will not carry out all the detail.

We have constructed the collection $\mathscr{D}_2$. We now assume that we have constructed collections $\mathscr{D}_{t-1} \subset \mathscr{D}_{t-2} \subset \cdots \subset \mathscr{D}_2$ such that if (Q_z, z) and (Q_w, w) are in $\mathscr{D}_{t-1}$, then for $2 \leqq k \leqq t-1$ we have $\partial^k w_r/\partial z_{j_1} \cdots \partial z_{j_k} = 0$ on $f(S)$ for $j_1, \cdots, j_k > m$. We now continue the induction.

COCYCLE LEMMA G. *Associate to each ordered intersection* $Q_u \cap Q_w$ *of coordinates in* $\mathscr{D}_{t-1}$, *the section of* $\mathscr{N} \otimes \mathscr{N}^* \otimes \cdots \otimes \mathscr{N}^*$ (resp. $\tau S \otimes \mathscr{N}^* \otimes \cdots \otimes \mathscr{N}^*$)—*with* t *factors of* $\mathscr{N}^*$—*given by*

$$A^{wu} = \sum_{r,k_1,\cdots,k_t=m+1}^{n} \frac{\partial^t u_r}{\partial w_{k_1} \cdots \partial w_{k_t}} \frac{\partial}{\partial u_r} \otimes dw_{k_1} \otimes \cdots \otimes dw_{k_t}$$

(resp. $B^{wu} = \{$same sum with $1 < r \leqq m$ instead$\}$). *Then the collection* $\{A^{wu}\}$ (resp. $\{B^{wu}\}$) *determines a* 1-*cocycle on* S *with respect to the covering* $\{Q_z\}$, *with coefficients in* $\mathscr{N} \otimes \mathscr{N}^* \otimes \cdots \otimes \mathscr{N}^*$ (resp. $\tau S \otimes \mathscr{N}^* \otimes \cdots \otimes \mathscr{N}^*$).

Proof. Step a. As in Step a of the previous cocycle lemma, we start with the fact that for $k_1, \cdots, k_t > m$ we have on $f(S)$

$$\text{(G.a)} \qquad \frac{\partial^t w_j}{\partial w_{k_1} \cdots \partial w_{k_t}} = 0 = \sum_{j=1}^{n} \frac{\partial^t w_j}{\partial w_{k_1} \cdots \partial w_{k_t}} \frac{\partial u_r}{\partial w_j}\,.$$

Calculating using the chain rule we get a formula analogous to (S.a), but very much messier to write down, so we don't write it here but refer to it as (G.a).

Step b. Using (G.a) and the fact that our coordinate systems are in $\mathscr{D}_{t-1}$, after calculating with the chain rule we get that

$$\begin{aligned} 0 &= \sum_{r,k_1,\cdots,k_t=m+1}^{n} \frac{\partial^t w_j}{\partial w_{k_1} \cdots \partial w_{k_t}} \frac{\partial u_r}{\partial w_j} \frac{\partial}{\partial u_r} \otimes dw_{k_1} \otimes \cdots \otimes dw_{k_t} \\ &= \sum_{r,k_1,\cdots,k_t=m+1}^{n} \frac{\partial^t u_r}{\partial w_{k_1} \cdots \partial w_{k_t}} \frac{\partial}{\partial u_r} \otimes dw_{k_1} \otimes \cdots \otimes dw_{k_t} \\ \text{(G.b)} \qquad &\quad + \sum_{r,j_1,\cdots,j_t=m+1}^{n} \frac{\partial^t w_r}{\partial z_{j_1} \cdots \partial z_{j_t}} \frac{\partial}{\partial w_r} \otimes dz_{j_1} \otimes \cdots \otimes dz_{j_t} \\ &\quad + \sum_{r,h_1,\cdots,h_t=m+1}^{n} \frac{\partial^t z_r}{\partial u_{h_1} \cdots \partial u_{h_t}} \frac{\partial}{\partial z_r} \otimes du_{h_1} \otimes \cdots \otimes du_{h_t} \\ &= A^{wu} + A^{zw} + A^{uz}\,. \end{aligned}$$

Thus, $\{A^{wu}\}$ is a cocycle. The $\tau S \otimes \mathscr{N}^* \otimes \cdots \otimes \mathscr{N}^*$ case is handled as before.

PROPOSITION 2G. *For all* $t \geqq 3$, *there is a subcollection* $\mathscr{D}_t$ *of* $\mathscr{D}_{t-1}$ *such that the coordinate polydiscs of* $\mathscr{D}_t$ *cover* $f(S)$, *the covering is as fine as we like, and for* $(\tilde{Q}_z, z)$ *and* $(\tilde{Q}_w, w)$ *in* $\mathscr{D}_t$ *we have* $\partial^t w_r/\partial z_{k_1} \cdots \partial z_{k_t} = 0$ *on* $f(S)$ *for* $k_1, \cdots, k_t > m$.

Proof. *Step* A. Starting with the cocycles $\{A^{zw}\}$ and $\{B^{zw}\}$ of the preceding lemma, we obtain the 0-cochain $\{\Gamma(z)\}$ with components $\left\{\gamma^r_{k_1\cdots k_t}\begin{pmatrix} u \\ w \end{pmatrix}(z)\right\}$ exactly as in Step A of Proposition 2S.

Step B. Exactly as in Step B of Proposition 2S we get now coordinate systems $(\tilde{Q}_\zeta, \zeta)$ in $\mathscr{C}$ from $(\tilde{Q}_z, z)$ in $\mathscr{D}_{t-1}$ by setting $\zeta_r = z_r - \sum^n_{k_1,\cdots k_t=m+1} \gamma^r_{k_1\cdots k_t}\begin{pmatrix} z \\ z \end{pmatrix}(z) z_{k_1} \cdots z_{k_t}$.

Step C. (i) Next we must show that the coordinate systems obtained in Step B form a subcollection of $\mathscr{D}_{t-1}$. Suppose $2 \leqq q \leqq t-1$ and consider $\partial^q \omega_r/\partial\zeta_{k_1} \cdots \partial\zeta_{k_q}$ where $k_1, \cdots, k_q > m$. Using the chain rule we express $\partial^q \omega_r/\partial\zeta_{k_1} \cdots \partial\zeta_{k_q}$ in terms of

$$\frac{\partial^q \omega_r}{\partial w_{j_1} \cdots \partial w_{j_q}}, \qquad \frac{\partial^q w_a}{\partial z_{h_1} \cdots \partial z_{h_q}}, \qquad \frac{\partial^a \zeta_b}{\partial z_{l_1} \cdots \partial z_{l_q}},$$

and various lower order derivatives. From the definition of ζ_b, ω_r, and the fact that the z and w coordinates are in $\mathscr{D}_{t-1}$, all these terms (and hence $\partial^q\omega_r/\partial\zeta_{k_1} \cdots \partial\zeta_{k_q}$) will be 0. So our new coordinates are in $\mathscr{D}_{t-1}$.

(ii) It only remains now to show that for $k_1, \cdots, k_t > m$ we have $\partial^t\omega_r/\partial\zeta_{k_1} \cdots \partial\zeta_{k_t} = 0$ on $f(S)$. We use the method of paragraph (i) above to get an expression for $\partial^t\omega_r/\partial\zeta_{k_1} \cdots \partial\zeta_{k_t}$. Using the fact that the ω and ζ coordinates are in $\mathscr{D}_{t-1}$, we get an $f(S)$

$$\begin{aligned} \frac{\partial^t \omega_r}{\partial\zeta_{k_1} \cdots \partial\zeta_{k_t}} &= \sum^n_{j_1,\cdots,j_t=m+1} \frac{\partial w_{j_1}}{\partial\zeta_{k_1}} \cdots \frac{\partial w_{j_1}}{\partial\zeta_{k_t}} \frac{\partial^t \omega}{\partial w_{j_1} \cdots \partial w_{j_t}} \\ &+ \sum^n_{a,j_1\cdots,j_t=m+1} \frac{\partial\omega_r}{\partial w_a} \frac{\partial^t w_a}{\partial z_{j_1} \cdots \partial z_{j_t}} \frac{\partial z_{j_1}}{\partial\zeta_{k_1}} \cdots \frac{\partial z_{j_t}}{\partial\zeta_{k_t}} \\ &- \sum^n_{a,b,j_1,\cdots,j_t=m+1} \frac{\partial\omega_r}{\partial z_b} \frac{\partial z_b}{\partial\zeta_a} \frac{\partial z_{j_1}}{\partial\zeta_{k_1}} \cdots \frac{\partial z_{j_t}}{\partial\zeta_{k_t}} \frac{\partial^t\zeta_a}{\partial z_{j_1} \cdots \partial z_{j_t}}. \end{aligned} \tag{G.C}$$

We now proceed exactly as in (S.C) and following in Step C of Proposition 2S to conclude the proof.

3. Throughout this section, unless stated to the contrary, we assume we are given complex manifolds S, M, $m = \dim_c S < n = \dim_c M$,

S is a Stein manifold, $K \subset S$ is compact, and $f: S \to M$ is a holomorphic embedding.

Having obtained all the special collections of coordinate systems we need in the previous section, we now proceed to construct a strictly pluri-subharmonic function ϕ_{λ_0} such that $\phi_{\lambda_0}^{-1}([-\infty, a])$ is compact for all $a \in \boldsymbol{R}$ and ϕ_{λ_0} is defined on a neighborhood of $f(K)$. We will thus have $f(K)$ contained in a Stein submanifold of M. The desired strictly pluri-subharmonic function will be gotten by using a certain strictly pluri-subharmonic function on N_f, and transferring it to an open subset of M by means of the maps Δ_{zw} defined below.

NOTE. Sometimes below for clarity we will use the notation $p_z(z) = p_z(z_1, \cdots, z_n)$ to mean the point q on M such that q has coordinates $(z_1, \cdots, z_n)$ with respect to the coordinate system $(\tilde{Q}_z, z)$.

Also if we are given a point $p \in \tilde{Q}_z$ with coordinates $z(p) = (z_1(p), \cdots, z_n(p))$, then we let $\hat{z} = \hat{z}(p) = (z_1, \cdots, z_m, 0, \cdots, 0)$ denote the point q on $f(S) \cap \tilde{Q}_z$ such that $z_i(q) = z_i(p)$ for $1 \leqq i \leqq m$ and $z_i(q) = 0$ for $m < i \leqq n$.

Also below we will sometimes implicitly identify a point $s \in S$ with $f(s) \in M$, and we will often identify a point $p \in M$ with its set of coordinates $(z_1(p), \cdots, z_n(p))$ with respect to a given system $(\tilde{Q}_z, z)$.

DEFINITION. Let $(\tilde{Q}_z, z)$ and $(\tilde{Q}_w, w)$ be in $\mathscr{D}_t$ and let (Q_z, z) and (Q_w, w) be the associated coordinate systems on S. Let $(T'_z; z, \partial/\partial z)$ denote the coordinate system in $T'_z = N_f | Q_z$ determined by the coordinates $(z_1, \cdots, z_m)$ in $Q_z \subset S \subset N_f$ and the basis $\partial/\partial z_{m+1}, \cdots, \partial/\partial z_n$ in the fibers of N_f over Q_z. Let $T_z \subset T'_z$ be the coordinate polydisc given by $T_z = \{v \in T'_z | (z_1(v), \cdots, z_m(v), a_{m+1}(v), \cdots, a_n(v)) \in z(Q_z) = $ (image of $Q_z \subset M$ in $\boldsymbol{C}^n$ by the coordinate functions)$\}$, where v has coordinates $(z_1(v), \cdots, z_m(v), a_{m+1}(v), \cdots, a_n(v)$ with respect to $(T'_z; z, \partial/\partial z)$. We then define

$$\Delta_{zw}: T_z \cap T_w \longrightarrow T_z \cap T_w$$

by

$$\begin{aligned}\Delta_{zw}\Big(\sum_{j=m+1}^{n} a_j \frac{\partial}{\partial z_j}(z_1, \cdots, z_m)\Big) &= \Delta_{zw}\Big(\sum_{j=m+1}^{n} a_j \frac{\partial}{\partial z_j}(\hat{z})\Big) \\ &= \sum_{j=m+1}^{n} w_j(z_1, \cdots, z_n)\frac{\partial}{\partial w_j}(w_1(z), \cdots, w_m(z)) = \sum_{j=m+1}^{n} w_j(z)\frac{\partial}{\partial w_j}(\hat{w}(z)) .\end{aligned}$$

We now describe some properties of the maps Δ_{zw}.

PROPOSITION 3. *Let $(\tilde{Q}_z, z)$ and $(\tilde{Q}_w, w)$ be in $\mathscr{D}_{t-1}$. Then we have*

$$\Delta_{zw}\Big(\sum_{j=m+1}^{n} z_j\frac{\partial}{\partial z_j}(\hat{z})\Big) = \sum_{j=m+1}^{n}[z_j + 0(z_{m+i}^t)]\frac{\partial}{\partial z_j}(\hat{z}(\hat{w}(z))) ,$$

where $0(z_{m+i}^t)$ *means of order at least* t *in the variables* $z_{m+1}, \cdots, z_n$. *Also* $w_j(z) = w_j(\hat{z}) + 0(z_{m+i}^t)$ *for* $1 \leqq j \leqq m$.

Proof. Considering what we are trying to prove, we need to express Δ_{zw} solely in terms of the basis $\partial/\partial z_{m+1}, \cdots, \partial/\partial z_n$ for N_f over $Q_z \cap Q_w$. By the definition of Δ_{zw} we have

$$\begin{aligned}\Delta_{zw}\Big(\sum_{j=n+1}^{n} z_j\frac{\partial}{\partial z_j}(\hat{z})\Big) &= \sum_{j=m+1}^{n} w_j(z)\frac{\partial}{\partial w_j}(\hat{w}(z)) \\ &= \sum_{j,k=m+1}^{n} w_j(z)\frac{\partial z_k}{\partial w_j}(\hat{w}(z))\frac{\partial}{\partial z_k}(\hat{z}(\hat{w}(z))) .\end{aligned}$$

It is easily seen that

$$\begin{aligned}\frac{\partial}{\partial z_a}\Big[\sum_{j=m+1}^{n} w_j(z)\frac{\partial z_k}{\partial w_j}(\hat{w}(z))\Big](z) &= \sum_{j=m+1}^{n}\frac{\partial w_j}{\partial z_a}(z)\frac{\partial z_k}{\partial w_j}(\hat{w}(z)) \\ &\quad + \sum_{j=m+1}^{n} w_j(z)\frac{\partial}{\partial z_a}\Big[\frac{\partial z_k}{\partial w_j}(\hat{w}(z))\Big](z) .\end{aligned}$$

Thus on S (where we consider S as the zero section of N_f), that is when $z = \hat{z} = (z_1, \cdots, z_m, 0, \cdots, 0)$,

$$\begin{aligned}\frac{\partial}{\partial z_a}\Big[\sum_{j=m+1}^{n} w_j(z)\frac{\partial z_k}{\partial w_j}(\hat{w}(z))\Big](\hat{z}) &= \sum_{j=m+1}^{n}\frac{\partial w_j}{\partial z_a}(z)\frac{\partial z_k}{\partial w_j}(\hat{w}(\hat{z})) \\ &+ \sum_{j=m+1}^{n} w_j(\hat{z})\frac{\partial}{\partial z_a}\Big[\frac{\partial z_k}{\partial w_j}(\hat{w}(z))\Big](\hat{z}) = \delta_a^k + 0 = \delta_a^k ,\end{aligned}$$

since all the coordinate systems involved are in $\mathscr{D}_{t-1}$ and $w_j(\hat{z}) = 0$ for $m < j \leqq n$.

From this it is clear that tangential derivatives (i.e. derivatives involving only $\partial/\partial z_a$'s with $1 \leqq a \leqq m$) of $\sum_{j=m+1}^{n} w_j(z)(\partial z_k/\partial w_j)(\hat{w}(z))$ of all orders vanish on S. It is also clear that any derivative of the form $\partial^r/\partial z_{k_1} \cdots \partial z_{k_r}$ with $k_1 > m$ say and $k_j \leqq m$ for $j \neq 1$ vanishes on S.

To complete the proof we need only show that normal derivatives (i.e. derivatives involving only $\partial/\partial z_a$'s with $a > m$) of order r, with $2 \leqq r \leqq t-1$, vanish on S. Suppose $2 \leqq r \leqq t-1$ and $k_1, \cdots, k_r > m$. Using the product rule we see that

$$\frac{\partial^r}{\partial z_{k_1} \cdots \partial z_{k_r}}\Big[w_j(z)\frac{\partial z_j}{\partial w_j}(\hat{w}(z))\Big]$$

consists of terms of the form

$$\frac{\partial^s w_j}{\partial z_{j_1} \cdots \partial z_{j_s}} \cdot \frac{\partial^q}{\partial z_{l_q} \cdots \partial z_{l_1}} \left[\frac{\partial z_k}{\partial w_j}(\hat{w}(z)) \right].$$

If $q = 0$ the first factor is 0 on S since all coordinate systems are in $\mathscr{D}_{t-1}$. Since

$$\frac{\partial}{\partial z_h} = \sum_{j=m+1}^{n} \frac{\partial w_j}{\partial z_h} \frac{\partial}{\partial w_j}$$

on $S(h > m)$, for $q \geqq 1$ the second factor is

$$\frac{\partial^{q-1}}{\partial z_{l_q} \cdots \partial z_{l_2}} \left[\sum_{i=1}^{m} \frac{\partial^2 z_k}{\partial w_i \partial w_j}(\hat{w}(z)) \frac{\partial w_i}{\partial z_{l_1}}(z) \right].$$

Carrying out further differentiations we see that this factor is 0 on S since our coordinates are in $\mathscr{D}_{t-1}$. This completes the proof of the main part of the proposition.

The final statement of the proposition is clear since all coordinate systems are in $\mathscr{D}_{t-1}$.

With Proposition 3 says in essence is that if we use the coordinate system (Q_z, z) in S and the basis $\partial/\partial z_{m+1}, \cdots, \partial/\partial z_n$ in $N_f \mid Q_z \cap Q_w$, then with respect to these coordinates Δ_{zw} differs from the identity map $(z_1, \cdots, z_m; z_{m+1}, \cdots, z_n) \mapsto (z_1, \cdots, z_m; z_{m+1}, \cdots, z_n)$ only by terms which are order t and higher in the variables $z_{m+1}, \cdots, z_n$.

Construction of the function ϕ_{λ_0}.

NOTE. Throughout the rest of this section we use only coordinates in $\mathscr{D}_3$.

Since S is a Stein manifold, there is a strictly plurisubharmonic function $\tilde{\rho}$ on S such that $\tilde{\rho} > 0$ and for all $c \in \boldsymbol{R}$, $\{p \in S \mid \tilde{\rho}(p) \leqq c\}$ is compact. It is clearly sufficient to assume that the set K is of the form $K = \{p \in S \mid \tilde{\rho}(p) \leqq b\}$. Choose b_1, b_2, b_3 such that $b < b_2 < b_3 < b_1$ and set $K_i = \{p \in S \mid \tilde{\rho}(p) \leqq b_i\}$.

Consider the collection of open sets $\mathscr{A} = \{\tilde{Q}_z\}$ where $(\tilde{Q}_z, z)$ is in $\mathscr{D}_3$. For each $\tilde{Q}_z$ in $\mathscr{A}$ let $\tilde{R}_z$ be a polydisc with compact closure in $\tilde{Q}_z$ (for instance if the image, by the coordinate functions $z_1, \cdots z_n$, of $\tilde{Q}_z$ is the polydisc $\Delta_{r_1} \times \cdots \times \Delta_{r_n}$, let $\tilde{R}_z$ be the set whose image by the coordinate functions is the polydisc $\Delta_{r_1/2} \times \cdots \times \Delta_{r_n/2}$). Since K_1 is compact there is a finite subcollection $\mathscr{B} = \{\tilde{R}_{z\alpha}\}$ of $\mathscr{A}$ such that $\bigcup_\alpha \tilde{R}_{z\alpha}$ covers $f(K_1)$. By very standard theorems there are C^∞ functions η_α such that $0 \leqq \eta_\alpha \leqq 1$, $\operatorname{supp} \eta_\alpha \subset \tilde{R}_{z\alpha}$, and $\sum_\alpha \eta_\alpha = 1$ on a neighborhood T of $f(K_1)$.

NOTE. For convenience, from now on we use the notation $\tilde{Q}_\alpha$ for $\tilde{Q}_{z\alpha}$, $\tilde{R}_\alpha$ for $\tilde{R}_{z\alpha}$, and $p_\alpha(d_1, \cdots, d_n)$ for $p_{z\alpha}(d_1, \cdots, d_n)$. Also

$A \subset\subset B$ means A has compact closure in B.

Since $\tilde{R}_\alpha \subset\subset \tilde{Q}_\alpha$ we have $\tilde{R}_{\alpha_1} \cap \tilde{R}_{\alpha_2} \subset\subset \tilde{Q}_{\alpha_1} \cap \tilde{Q}_{\alpha_2}$ and thus by elementary calculus there is a constant $D > 1$ (since we are dealing with only a finite number of $\tilde{R}_\alpha$'s) such that for $p \in \bigcup_\alpha \tilde{R}_\alpha$ we have $|z^{\alpha_1}_{m+j}(p)| \leqq D \sup_{1 \leqq i \leqq n-m} |z^{\alpha_2}_{m+i}(p)|$ for all $1 < j < n - m$ if $p \in \tilde{R}_{\alpha_1} \cap \tilde{R}_{\alpha_2}$.

For each point $p \in S$ there are m global (holomorphic) sections of N^*_f whose values at p form a basis for the vector space $(N^*_f)_p$ (see [3], p. 138). Thus there are global (holomorphic) sections of N^*_f, call them $\sigma_1, \cdots, \sigma_r$, such that for all $p \in f^{-1}(T)$ some m of these sections form a basis for $(N^*_f)_p$. Now consider the function μ on N_f given by $\mu(v) = \sum_{i=1}^r \sigma_i(v)\overline{\sigma_i(v)}$. Since each function $\sigma_i(v)$ is a holomorphic function on N_f, clearly μ is plurisubharmonic on N_f and vanishes on the zero section $S \subset N_f$. Let $\pi: N_f \to S$ be the projection map of the bundle N_f over S and define $\rho = \tilde{\rho} \circ \pi$. It is now easy to see that for all $\lambda \geqq 0$ the function $\psi_\lambda = \rho + (\lambda + 1)\mu$ is strictly plurisubharmonic on N_f.

Throughout the remainder of this section we use $(\tilde{Q}_w, w)$ to denote one of the coordinate systems $(\tilde{Q}_\alpha, z^\alpha)$. Also we express the forms σ_i in local coordinates by $\sigma_i(p) = \sum_{j=m+1}^n a_j^{i\alpha}(p) dz_j^\alpha$ in $\tilde{Q}_\alpha$ and by $\sigma_i(p) = \sum_{j=m+1}^n a_j^{iw}(p) dw_j$ in $\tilde{Q}_w$.

Now in $\bigcup_\alpha \tilde{R}_\alpha$ (or in fact in $\bigcup_\alpha \tilde{Q}_\alpha$) we define the function $\phi_\lambda(\lambda \geqq 0)$ by

$$\begin{aligned} \phi_\lambda(x) = \phi_\lambda(p_w(w(x))) &= \sum_\alpha \eta_\alpha(x)\psi_\lambda\Big(\Delta_{wz\alpha}\Big(\sum_{j=m+1}^n w_j(x)\frac{\partial}{\partial w_j}(p_w(w(x)))\Big)\Big) \\ &= \sum_\alpha \eta_\alpha(x)\rho(p_\alpha(\hat{z}^\alpha(x))) \\ &\quad + (\lambda + 1)\sum_\alpha \sum_{i=1}^r \eta_\alpha(x)\Big|\sigma_i\Big(\sum_{j=m+1}^n z_j(x)\frac{\partial}{\partial z_j}(p_\alpha(\hat{z}^\alpha(x)))\Big)\Big|^2 . \end{aligned} \tag{3.1}$$

Above we have written $\rho(p_\alpha(\hat{z}^\alpha(x)))$ where, strictly speaking, we mean $\rho(f^{-1}(p_\alpha(\hat{z}^\alpha(x))))$, since ρ is a function on the zero section $S \subset N_f$. We will not be so meticulous below and will write things like $\rho(\hat{z}^\alpha(x))$ where we mean $\rho(f^{-1}(p_\alpha(x^\alpha(x))))$ or $\partial/\partial z_j(\hat{z}^\alpha(x))$ where we mean $\partial/\partial z_j(p_\alpha(\hat{z}^\alpha(x)))$.

By Proposition 3 we have

$$\rho(\hat{z}^\alpha(x)) = \rho(p_w(\hat{w}(x) + O_\alpha(w^4_{m+h}))) = \rho(\hat{w}(x)) + O_\alpha(w^4_{m+h}, \bar{w}^4_{m+h}) ,$$

where $O(w^4_{m+h}, \bar{w}^4_{m+h})$ means terms which are at least fourth order in the variables $w_{m+1}, \cdots, w_n, \bar{w}_{m+1}, \cdots, \bar{w}_n$, and

$$\begin{aligned} &\sigma_i\Big(\sum_{j=m+1}^n z_j^\alpha(x)\frac{\partial}{\partial z_j}(\hat{z}^\alpha(x))\Big) \\ &\quad = \sigma_i\Big[\Big(\sum_{j=m+1}^n (w_j(x) + O_\alpha(w^4_{m+h}))\Big)\frac{\partial}{\partial w_j}(\hat{w}(x) + O_\alpha(w^4_{m+h}))\Big] \end{aligned}$$

$$= \sum_{j=m+1}^{n} a_j^{iw}(\hat{w}(x) + O_\alpha(w_{m+h}^4))[w_j + O_\alpha(w_{m+h}^4)] .$$

Therefore

$$\phi_\lambda(x) = \phi_0(x) + \lambda \sum_{i=1}^{r} \sum_{j,k=m+1}^{n} a_j^{iw}(\hat{w}(x))\overline{a_j^{iw}(\hat{w}(x))}w_j\bar{w}_k + \lambda O(w_{m+h}^5, \bar{w}_{m+h}^5) . \tag{3.2}$$

From this formula it is easy to see that on $f(S)$ we have

$$\left[\frac{\partial^2\phi_\lambda}{\partial w_a \partial \bar{w}_b}\right] = \left[\begin{array}{c|c} \dfrac{\partial^2 \rho}{\partial w_a \partial \bar{w}_b} & 0 \\ \hline 0 & (\lambda+1)\dfrac{\partial^2 \mu}{\partial w_a \partial \bar{w}_b} \end{array}\right].$$

Thus $[\partial^2\varphi_\lambda/\partial w_a\partial\bar{w}_b]$ is positive definite on $f(S)$ and hence ϕ_λi s strictly plurisubharmonic on some neighborhood of $f(K_1)$. In particular ϕ_0 is strictly plurisubharmonic on a neighborhood E' of $f(K_1)$. Let $E \subset\subset E' \cap T$ be another neighborhood of $f(K_1)$.

We now need to make some restrictions on the size of the last $n-m$ coordinates in each $\tilde{R}_\alpha$. First we assume that $c > 0$ is small enough so that if $|z_j(x)| < c$ for all $m+1 \leqq j \leqq n$ and $p_\alpha(\hat{z}^\alpha(x)) \in f(K_1)$, then $x \in E$. Second, we choose c so small that, in addition to the above property, if $|z_j^\alpha(x)| < c$ for all $m+1 \leqq j \leqq n$ and $p_\alpha(\hat{z}^\alpha(x)) \in \tilde{R}_\alpha \cap \tilde{R}_{\alpha_1}$, then we have $p_\alpha(z^\alpha(x), \cdots, z_m^\alpha(x), a_{m+1}, \cdots, a_n) \in \tilde{Q}_\alpha \cap \tilde{Q}_{\alpha_1}$ whenever $|a_i| < c$ for $m+1 \leqq i \leqq n$. Thirdly we choose c small enough so that also if for some α_0, $p_{\alpha_0}(\hat{z}^{\alpha_0}(x))$ is outside $f(K_{b_3})$ and $|z_j^{\alpha_0}(x)| < c$ for $m+1 \leqq j \leqq n$, then for any other α we have $p_\alpha(\hat{z}^\alpha(x)) \notin f(K_{b_2})$.

Before going further we remark that the forms σ_i determine a hermitian metric in N_f over $f^{-1}(T)$ given by $\langle v, w\rangle = \sum_{i=1}^r \sigma_i(v)\overline{\sigma_i(w)}$. Let $\|.\|$ denote the norm determined by this metric. Then there is a constant $L \leqq 1$ such that for all α, $1/L \sup_{j>m} |z_j^\alpha| \leqq \|\sum_{j=m+1}^n z_j^\alpha(\partial/\partial z_j^\alpha)\| \leqq L \sup_{j>m}|z_j^\alpha|$ since $\tilde{Q}_\alpha$ is compact and there are only finitely Q_α's.

By (3.2) we have in E

$$\left[\frac{\partial^2\phi_\lambda}{\partial w_a\partial\bar{w}_b}\right] = \left[\frac{\partial^2\phi_0}{\partial w_a\partial\bar{w}_b}\right] + \lambda\left[\frac{\partial^2}{\partial w_a\partial\bar{w}_b}\sum_{j,k=m+1}^{r}\sum_{i=1}^{r}\overline{a_j^{iw}(\hat{w}(x))}a_k^{iw}(\hat{w}(x))\bar{w}_j w_k\right] + \lambda[O(w_{m+j}^3, \bar{w}_{m+j}^3)] .$$

We have already noted that the first term $[\partial^2\phi_0/\partial w_a\partial\bar{w}_b]$ is positive definite in E. The second term is clearly positive semi-definite. Let $t > 0$ be such that if $|a_{ij}| < t$, then

$$\left[\frac{\partial^2\phi_0}{\partial w_a\partial\bar{w}_b}\right] + [a_{ij}]$$

is also positive definite in E. Clearly then

$$\left[\frac{\partial^2\phi_0}{\partial w_a \partial \bar{w}_b}\right] + \lambda\left[\frac{\partial^2}{\partial w_a \partial \bar{w}_b} \sum_{i=1}^{r} \sum_{j,k=m+1}^{n} a_j^{iw} \overline{a_k^{iw}} w_j \bar{w}_k\right] + [a_{ij}]$$

is also positive definite in E. Now let N be such that the term $O(w^3_{m+j}, \bar{w}^3_{m+j})$ above satisfies $|O(w^3_{m+k}, \bar{w}^3_{m+j})| \leqq N \sup_j |w^3_{m+j}|$ on the $\tilde{R}_\alpha$'s. Let $d = b_2 - \min_K \rho$. Now choose c small enough so that in addition to the three properties mentioned above, c also satisfies $4D^4dL^2/c^2 < t/Nc^3$, that is, $c < t/4D^4dNL^2$, and choose λ_0 such that $4D^4dL/c^2 < \lambda_0 < t/Nc^3$. Then clearly $d < \lambda_0 c^2/4D^4L^2 = \lambda_0(c/2D^2L)^2$ and $\lambda_0 Nc^3 < t$. Thus we now have that ϕ_{λ_0} is strictly plurisubharmonic in $G = E \cap \{x \mid \text{for some } \alpha, |z^\alpha_{m+i}(x)| < c, \text{ for all } j\}$. Let $H = \{x \mid \text{for some } \alpha, p_\alpha(\hat{z}^\alpha(x)) \in K_{b_3}$ and $|z^\alpha_j(x)| < c/2D$ for $m+1 \leqq j \leqq n\} \subset G$. Note that $\bar{H} \subset G$ and that $\bar{H}$ is compact. It is also clear by the way our constants were chosen that $f(K) \subset E \cap \phi^{-1}_{\lambda_0}([-\infty, b_2]) \subset H$ and thus $E \cap \phi^{-1}_{\lambda_0}((-\infty, b_2]$ is compact.

THEOREM 1. *If ϕ is strictly plurisubharmonic in the open set E and $\phi^{-1}([-\infty, a])$ is a compact subset of E, then $\phi^{-1}((-\infty, a))$ is a Stein manifold.*

Proof. See [3], p. 116.

Using this theorem and the strictly plurisubharmonic function ϕ_{λ_0} constructed above, we see that we have now proved the following theorem.

THEOREM 2. *If $f: S \to M$ is a holomorphic embedding and $K \subset S$ a compact set, then there is an open Stein submanifold U of M such that $f(K) \subset M$.*

4. We are now ready to prove our final results. First we need the following lemma.

LEMMA 3. *Let X, Y be Stein manifolds and $f: X \to Y$ a holomorphic embedding. Then there is a neighborhood U of X in the normal bundle of f* (where X is identified with the zero section of the normal bundle of f), *and a holomorphic embedding $F: U \to Y$ such that $F|X = f$.*

Proof. See [1], p. 162, Hilfsatz 11.

Our main result now follows easily.

THEOREM 3. *Let $f: S \to M$ be a holomorphic embedding, where S is a Stein manifold. Let K be a compact subset of S and let N_f be the normal bundle of f. Identify S with the zero section of N_f. Then there is a neighborhood U of K in N_f and a holomorphic embedding $F: U \to M$ such that $F|U \cap S = f$.*

Proof. By Theorem 2 above there is a neighborhood X of K, in S, such that X is a Stein submanifold of S, and there is a neighborhood Y of $f(K)$ in M such that Y is a Stein submanifold of M and $f(X) \subset Y$. The conclusion of the theorem now follows directly from Lemma 3 above.

REMARK. Note that Theorem 3 implies that we can extend Proposition 3 to clear away all higher order terms at the same time.

A version of Theorem 3 above is also valid for immersions. To derive this, however, we need the following well-known fact.

LEMMA 4. *Let $f: S \to M$ be a holomorphic immersion, where $\dim_C S = m$ and $\dim_C M = n$. Then there is a complex manifold W with $\dim_C W = n$, a holomorphic embedding $h: S \to W$, and a holomorphic immersion $g: W \to M$ such that $f = g \circ h$.*

Proof. This follows easily from the tubular neighborhood theorem for differentiable manifolds, because things can be given the appropriate complex analytic structure since g is a local diffeomorphism and h is an embedding.

THEOREM 4. *Let S be a Stein manifold and let $f: S \to M$ be a holomorphic immersion. Let $K \subset S$ be compact, let N_f denote the normal bundle of f, and identify S with the zero section of N_f. Then there is a neighborhood U of K in N_f and a holomorphic immersion $F: U \to M$ such that $F|U \cap S = f$.*

Proof. By Lemma 4 above there is a complex manifold W, $\dim_C W = \dim_C M$, a holomorphic embedding $h: S \to W$, and a holomorpoic immersion $g: W \to M$ such that $f = g \circ h$. Since g is locally a diffeomorphism, it is easy to see that the normal bundle of h is the same as the normal bundle of f. The conclusion of the theorem now follows easily from Theorem 3.

Our last result involves extending a map (to an equi-dimensional map) which is only assumed to be regular at a point. In general there is no normal bundle to extend things to, so we must use some other object. It turns out that an appropriate extension can be made to a trivial bundle in this case. We first prove, however, the following intermediate lemma.

LEMMA 5. *Let E be a vector bundle over a Stein manifold S. Then for any $x_0 \in S$ there is a trivial bundle $\tilde{A}$* ($\dim_C \tilde{A} = \dim_C E$), *a neighborhood U of the zero section of $\tilde{A}$, and a holomorphic map $F: U \to E$ such that $F(x) = x$ for $x \in S$ and F is regular at x_0* (here we identify S with the zero section of $\tilde{A}$ and E).

Proof. Let $m = \dim_C S$, $n = \dim_C E$. Thus the fibers of E are of dimension $n - m$. Let $f: S \to E$ be the embedding of S as the zero section of E and let $i: E \to C^q$ be a holomorphic embedding (this is possible since vector bundles over Stein manifolds are Stein manifolds). By Theorem 8, p. 257 of [2] there is a neighborhood W of $i(E)$ in C^q and a holomorphic retraction $\rho: W \to i(E)$. Let $N_{i \cdot f}$ be the normal bundle of $i \circ f$. By Lemma 3, since S is a Stein manifold, there is a neighborhood U_1 of the zero section in $N_{i \cdot f}$ and a holomorphic embedding $\alpha: U_1 \to C^q$ of U_1 onto an open set in C^q. By Lemma 3 above, there is a neighborhood $U_2 \subset U_1$ of S in $N_{i \cdot f}$ and a holomorphic embedding $F_2: U_2 \to C^q$ such that $F_2 | S = i \circ f$. Since the tangent bundle of C^q (denoted TC^q) is trivial, so is $(i \circ f)^* TC^q = S \times C^q$ and $(i \circ f)^* TC^q = N_{i \cdot f} \oplus TS$, where TS denotes the tangent bundle of S. Let $\pi: N_{i \cdot f} \oplus TS \to N_{i \cdot f}$ be the obvious projection (that is, for $n \in N_{i \cdot f}$ and $s \in TS$, $\pi(n \oplus s) = n$). Let A be an $n - m$ dimensional subspace of $(N_{i \cdot f} \oplus TS)_{x_0} = (S \times C^q)_{x_0}$ (that is, the fiber over x_0) such that $i^{-1} \circ \rho \circ F_2 \circ \pi | S \times A$ is regular at x_0, and let $\tilde{A}$ be the trivial bundle $\tilde{A} = S \times A \subset S \times C^q = N_{i \cdot f} \oplus TS$. If we set $U = \tilde{A} \cap \pi^{-1}(U_2 \cap \alpha^{-1}(W))$ and $F = i^{-1} \circ \rho \circ F_2 \circ \pi$, then clearly $\tilde{A}$, U, and F satisfy the conclusions of the lemma, that is, $F: U \to E$ is regular at x_0.

We now proceed to the general case. The proof of the following theorem is contained, in essence, in the proof of the preceding lemma.

THEOREM 5. *Let S a Stein manifold and let $f: S \to M$ be a holomorphic map which is regular at $x_0 \in S$. Let $K \subset S$ be compact with $x_0 \in K$. Then there is a trivial bundle $\tilde{A}$ over S, $\dim_C \tilde{A} = \dim_C M$, a neighborhood U of K in $\tilde{A}$* (here S is identified with the zero section of $\tilde{A}$), *and a holomorphic map $F: U \to M$ such that $F | S \cap U = f$ and F is regular at x_0.*

Proof. Let $m = \dim_C S$, $n = \dim_C M$ and let $g: S \to S \times M$ be defined by $g(x) = (x, f(x))$. Clearly g is an embedding. By Theorem 2 above there is an open Stein submanifold $X \subset S$ with $K \subset X$ and there is an open Stein submanifold $Y \subset S \times M$ such that $f(X) \subset Y$. There is a holomorphic embedding $i: Y \to C^q$ since Y is a Stein manifold. By Theorem 8, p. 257 of [2] there is a neighborhood W of $i(Y)$ and a holomorphic retraction $\rho: W \to i(Y)$. Let $N_{i\cdot g}$ denote the normal bundle (over X) of $i \circ g$. By Lemma 3 above, since X is a Stein manifold, there is a neighborhood U_1 of the zero section of $N_{i\cdot g}$ and a holomorphic embedding $\alpha: U_1 \to C^q$ of U_1 onto an open set in C^q. By Lemma 3 above there is a neighborhood $U_2 \subset U_1$ of X in $N_{i\cdot g}$ and a holomorphic embedding $F_2: U_2 \to C^q$ such that $F_2 | X \cap U_2 = i \circ g$. Since the tangent bundle TC^q is trivial, so is $(i \circ g)^* TC^q = X \times C^q = N_i . g \oplus TX$ where$TX$is the tangent bundle of X. Let $\pi_1: N_{i\cdot g} \oplus TX$ be the projection onto $N_{i\cdot g}$ along TX. Let $\pi_M: S \times M \to M$ be the projection onto M. Now let A be an $n - m$ dimensional subspace of $(X \times C^q)_{x_0}$ (that is, the fiber over x_0) such that $\pi_M \circ i^{-1} \circ \rho \circ F_2 \circ \pi_1$ is regular at x_0 (there is such an A since all the maps involved have rank at least n) and let $\tilde{A}_1$ be the trivial bundle $\tilde{A}_1 = X \times A \subset X \times C^q = N_{i\cdot g} \oplus TX$. Set $U = \tilde{A}_1 \cap \pi^{-1}(U_2 \cap \alpha^{-1}(W))$, set $F = \pi_M \circ i^{-1} \circ \rho \circ F_2 \circ \pi_1$, and set $\tilde{A} = S \times A \supset X \times A = \tilde{A}_1$. Then clearly $\tilde{A}$, U, and F satisfy the conclusions of the theorem.

References

1. O. Forster, and K. J. Ramspott, *Analytische Modulgarben und Endromisbundel*, Invent. Math., **2** (1966), 145-170.
2. R. C. Gunning, and H. Rossi, *Analytic Functions of Several Complex Variables*, Prentice-Hall, Englewood Cliffs, N. J., 1965.
3. L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, Van Nostrand, Princeton, N. J., 1966.
4. H. L. Royden, *Remarks on the Kobayashi Metric*, Proc. Maryland Conf. on Several Complex Variables, Springer Lecture Notes in Mathematics 185.
5. ———, *The extension of regular holomorphic maps*, Proc. Amer. Math. Soc., **43** (1974), 306-310.
6. C. C. Seabury, *Some extension theorems for regular maps of Stein manifolds*, Bull. Amer. Math. Soc., **80** (1974), 1223-1224.

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