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A CLASS OF T_1-COMPACTIFICATIONS

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In this work the correspondence between T_2-compactifications, proximity relations, and families of maximal round filters is extended to the case of T_1-spaces. The major results spell out bijections between a special class of T_1-compactifications, certain proximity relations on the original space, and certain filterfamilies. Perhaps the most interesting result is the identification of a class of compactifications between T_1 and T_2-compactifications. This class consists of the principal weakly regular minimal compactifications and includes the Wallman compactification, and also the one-point compactification of a locally compact space. Moreover, the Wallman compactification is the largest weakly regular minimal compactification of a T_1-space. This improves the known result that the Wallman compactification is T_1, and is larger than any T_2-compactification.

1. Extension structures and T_1-compactifications. This section develops the notion of a compactification structure, which is a family of filters satisfying conditions strong enough to guarantee that the induced extension is a compactification. These induced compactifications constitute a class lying between T_1 and T_2-compactifications. In the language developed in this section, it is the class of principal weakly regular minimal compactifications. The main result is the $1-1$ correspondence between these compactifications and the set of compactification structures on a given T_1-space. The Wallman compactification is characterized as the largest weakly regular minimal compactification of a T_1-space. Much of the notation and terminology in this section is taken from Thron [4], Chapter 17.

DEFINITION 1.1. An *extension structure* on a topological space $(X, \mathscr{T})$ is a family Φ of open filters on X which includes all neighborhood filters. We will call the extension structure T_1 if no filter in Φ contains any other filter in Φ.

The inverse images of the neighborhood filters of an entension constitute an extension structure. We will call this the *trace system* of the extension. Conversely, each extension structure is the trace system of some extension.

This correspondence between extensions and extension structures is not 1-1, since inequivalent extensions can have the same trace system. However, for each extension structure Φ there is a preferred extension $\pi(\Phi)$, which we will call the *principal extension* associated with Φ.

This extension is obtained as follows (in the case of a T_0-space). For $A \subset X$ we define $A^\wedge = \{\mathscr{F} \in \Phi: A \in \mathscr{F}\}$. Then $\{G^\wedge: G \in \mathscr{T}\}$ is a base for a topology $\mathscr{T}^\wedge$ on Φ. We define $j(x) = \mathscr{N}_x$, the neighborhood filter at x, for $x \in X$. When $(X, \mathscr{T})$ is T_0 then $(j, (\Phi, \mathscr{T}^\wedge))$ is a T_0-extension of $(X, \mathscr{T})$ with trace system Φ. We will denote this extension by $\pi(\Phi)$.

By a *principal extension* we mean any extension equivalent to one of the $\pi(\Phi)$'s. Since equivalent extensions have the same trace system, this is the same as saying that a principal extension is equivalent to the image under π of its trace system. A principal extension is the continuous image of *every* extension with the same trace system. Thus the principal extensions are minimal in some sense. The Wallman, one-point, and T_2-compactifications are all principal extensions.

In what follows we will describe a more useful characterization of principal extensions.

DEFINITION 1.2. Let $K = (e, (Y, \mathscr{T}'))$ be an extension of $(X, \mathscr{T})$. For any subset A of Y we define

$$A^+ = \{y \in Y: A \in ee^{-1}(\mathscr{N}_y)\} .$$

REMARK 1.3. Notice that if G is open then $G \subseteqq G^+ \subseteqq G^-$ and G^+ is open. In fact, G^+ is the union of all open sets in Y which have the same restriction to X as G does.

PROPOSITION 1.4. *A T_0-extension Y of a space X is principal iff $\{G^+: G$ is open in $Y\}$ is a base for the topology on Y.*

Proof. Note first that this condition is preserved under equivalence of extensions. To see this, check that whenever $(e_i, (Y_i, \mathscr{T}_i))$ are extensions of $(X, \mathscr{T})$ equivalent under $h: Y_1 \to Y_2$ then for $G \in \mathscr{T}_1$, we have $h(G)^+ = h(G^+)$.

Now if Φ is an extension structure on X then in $\pi(\Phi)$ we have $G^{\wedge +} = G^\wedge$. From this and the preceding remark it follows that every principal extension satisfies the required condition.

Suppose now that K satisfies the given condition. Let Φ be its trace system. We wish to show K and $\pi(\Phi)$ are equivalent. Let $h: Y \to \Phi$ be defined by $y \to e^{-1}(\mathscr{N}_y)$. Then h is continuous, with $he = j$. We wish to establish that h is a homeomorphism.

To prove that h is an open map, note that if U and V are open subsets of Y with $U^+ \subseteqq V$ then $e^{-1}(U)^\wedge \subseteqq h(V)$. Now using the fact that the U'^+s form a base for $\mathscr{T}$ we can easily see that h maps open sets to open sets.

Now suppose $y \neq z$. Since Y is T_0 we can assume $\mathcal{N}_y \nsubseteq \mathcal{N}_z$. Using the condition, let U be an open set such that $y \in U^+$ and $z \notin U^+$. Then clearly $e^{-1}(U) \notin e^{-1}(\mathcal{N}_z)$ and so $h(y) \neq h(z)$.

The following definitions lead up to conditions to be imposed on an extension structure which will guarantee that the induced extension is compact. These conditions identify a class of T_1-compactifications which include the Wallman compactification and the 1-point compactification of a locally compact space.

DEFINITION 1.5. An extension structure Φ on a T_1-space $(X, \mathcal{T})$ is *totally bounded* iff every ultrafilter on X contains a member of Φ.

REMARK 1.6. It is easy to check that in the definition of totalboundedness we can replace ultrafilters by ultra-closed filters. (These are closed filters not contained properly in any other closed filter.)

Note also that this at least is a necessary condition in that the trace system of a compactification is always totally bounded.

DEFINITION 1.7. An extension structure Φ is *covered* iff every filter in Φ is contained in some ultra-closed filter.

REMARK 1.8. Requiring an extension structure to be covered is close to requiring it to be minimal in the sense that no "extra" points are being adjoined to the space. The idea is that every point adjoined is used to make an ultra-closed filter converge.

The last condition to be imposed on an extension structure is a kind of regularity condition. This will be defined in terms of an induced proximity relation.

DEFINITION 1.9. Given Φ an extension structure on X, we define Φ-containment as follows.

$A <_\Phi B$ iff for every $\mathcal{F}$ in Φ, if A is in some ultraclosed filter containing $\mathcal{F}$, then $B \in \mathcal{F}$.

This defines a map ρ from extension structures on X to relation on $\mathcal{P}(X)$. We will sometimes denote $<_\Phi$ by $\rho(\Phi)$.

REMARK 1.10. Let $\langle \mathcal{F} \rangle$ denote the intersection of all ultralosed filters containing $\mathcal{F}$. (If there are none, $\langle \mathcal{F} \rangle = \mathcal{P}(X)$.) Then for a closed set A, $A <_\Phi B$ iff for every $\mathcal{F}$ in Φ, either $B \in \mathcal{F}$ or $X \backslash A \in \langle \mathcal{F} \rangle$.

The relation $<_\Phi$ is a strong containment relation. On a T_1-space, Φ-containment implies containment. If Φ is the set of maximal round filters from an EF-proximity $\subset$, then $A \subset B$ iff $A^- <_\Phi B$. If Φ is the trace system of the Wallman compactification then $A <_\Phi B$ iff $A \subseteqq B^i$.

DEFINITION 1.11. The Φ-hull of a filter $\mathscr{F}$ on X is the filter

$$\Phi(\mathscr{F}) = \{A: F <_\Phi A \quad \text{for some} \quad F \in \mathscr{F}\} .$$

REMARK 1.12. Note that if $\mathscr{U}$ is an ultra-closed filter containing a filter $\mathscr{F}$ in Φ then $\Phi(\mathscr{U}^i) \subseteqq \Phi(\mathscr{U}) \subseteqq \mathscr{F}$. When equality holds we say that Φ is regular. This is made precise in the next definition.

DEFINITION 1.13. An extension structure Φ is *regular* iff for every $\mathscr{F}$ in Φ and every ultra-closed filter $\mathscr{U}$, if $\mathscr{F} \subseteqq \mathscr{U}$ then $\mathscr{F} \subseteqq \Phi(\mathscr{U}^i)$.

REMARK 1.14. Note that if Φ is covered and totally-bounded then Φ is regular iff it consists of the Φ-hulls of open hulls of ultra-closed filters.

Note also that a covered regular extension structure is T_1. In fact, if Φ is covered and regular then no two of its filters are contained in the same ultra-closed filter. Thus Φ determines a partition of the ultra-closed filters. Without regularity we have this separation property for any two filters in Φ as long as one of them converges. This can be established using the following lemma.

LEMMA 1.15. *Let Φ be a T_1-extension structure on X. If $\mathscr{F} \in \Phi$ and $\mathscr{U}$ is ultra-closed and $\mathscr{F} \subseteqq \mathscr{U} \to x$ then $\mathscr{F} = \mathscr{N}_x = \mathscr{U}^i$.*

Proof. Note that on a T_1-space if $\mathscr{U}$ is ultra-closed and $\mathscr{U} \to x$ then $\mathscr{U} = \dot{x}$, the point filter at x. Hence $\mathscr{U}^i = \mathscr{N}_x$. Since $\mathscr{F}$ is open, we have $\mathscr{F} \subseteqq \mathscr{U}^i = \mathscr{N}_x$. Since Φ is T_1, $\mathscr{F} = \mathscr{N}_x$.

EXAMPLES 1.16.

1. Let Ω denote the trace system of the Wallman compactification. Note that Ω consists of the open hulls of the ultra-closed filters. We have $A <_\Omega B$ iff $A \subseteqq B^i$, and hence each filter in Ω is its own Ω-hull. From Remark 1.14 it follows that Ω is regular.

2. Let $\mathscr{E}$ denote the set of filters maximal round with respect to an EF-proximity $\subset$. Recall $A \subset B$ iff $A^- < B$. Thus $\mathscr{E}$ consists

of the $\mathscr{E}$-hulls of filters in Ω, and this makes $\mathscr{E}$ regular.

3. Let $\mathscr{A}$ denote the trace system of the Alexandroff 1-point compactification of a locally compact T_1-space X (which is not compact). Here by locally compact we mean that every point of X has a compact neighborhood.

We claim that $\mathscr{A}$ consists of the neighborhood filters, together with the open hull of the intersection $\mathscr{F}_0$ of all the nonconvergent ultra-closed filters on X. To see this, let A be any subset of X. If A is compact then it cannot be a member of any noncovergent ultra-closed filter. If A is closed but not compact then using Zorn's lemma we can obtain a nonconvergent ultra-closed filter which contains A. Thus for any closed subset B of X we have that B is compact iff $X\backslash B$ is in $\mathscr{F}_0^i$.

To establish regularity, we will show that for each $\mathscr{F}$ in $\mathscr{A}$ we have $\mathscr{F} = \mathscr{A}(\mathscr{F})$. Note first that $\mathscr{A}$ is a T_1-extension structure. For if $x \in X$ then $\sim\{x\} \in \mathscr{F}_0^i$, and so $\mathscr{F}_0^i \nsubseteq \mathscr{N}_x$. On the other hand, since X is not compact, $\mathscr{F}_0^i$ cannot contain any compact sets. Since local compactness ensures that each $\mathscr{N}_x$ contains a compact set, we have $\mathscr{N}_x \nsubseteq \mathscr{F}_0^i$, for all x.

Now let $\mathscr{F} \in A$. Suppose $\mathscr{F} = \mathscr{F}_0^i$. Let G be open such that $G \in \mathscr{F}$. Then since every convergent ultra-closed filter is a point filter, we have $G \prec_{\mathscr{A}} G$. Hence $\mathscr{F}_0^i = \mathscr{A}(\mathscr{F}_0^i)$. Now suppose $\mathscr{F} = \mathscr{N}_x$. Let G be an open neighborhood of x, and let W be a compact neighborhood of x. From Lemma 1.15 it follows that $G \cap W \prec_{\mathscr{A}} G$. Hence $\mathscr{N}_x = \mathscr{A}(\mathscr{N}_x)$.

DEFINITION 1.17. A *compactification structure* is a covered totally-bounded regular extension structure.

THEOREM 1.18. *The principal extension obtained from a compactification structure is a T_1-compactification.*

Proof. Let Φ be a compactification structure on a T_1-space $(X, \mathscr{T})$. We need to show that $(\Phi, \mathscr{T}^\wedge)$ is compact. Let $\{G_\alpha : \alpha \in I\}$ be a family of open sets in X such that $\Phi \subseteq \bigcup_\alpha \hat{G_\alpha}$. Let $\mathscr{S} = \{X\backslash U : U$ is open and $U \prec_\Phi G_\alpha$ for some $\alpha \in I\}$.

We claim that some finite intersection of sets in $\mathscr{S}$ is empty. If not, then there is some ultra-closed filter $\mathscr{U}$ which contains $\mathscr{S}$. Since Φ is totally-bounded, $\mathscr{U}$ contains some filter $\mathscr{F}$ in Φ. By regularity, $\mathscr{F} = \Phi(\mathscr{U}^i)$. This, together with the definition of $\mathscr{S}$ and the fact that some G_α is in $\mathscr{F}$, leads to the contradiction that for some open set U, both U and $X\backslash U$ are in $\mathscr{U}$. Thus $\mathscr{S}$ does not have the finite intersection property.

Accordingly, let $U_1, \cdots, U_n$ be open sets with complements in $\mathscr{S}$, and such that $X = \bigcup_i U_i$ and $U_i <_\Phi G_{\alpha_i}$ for $1 \leqq i \leqq n$. Using that Φ is covered it is easy to show that $\Phi \subseteqq \bigcup_i G_{\alpha_i}\hat{}$.

This establishes that $(\Phi, \mathscr{T}\hat{})$ is compact. The regularity of Φ guarantees that no filter in Φ contains any other filter in Φ; hence $(\Phi, \mathscr{T}\hat{})$ is T_1.

We turn now to the characterization of those compactifications which are obtained from compactification structures.

DEFINITIONS 1.19. A filter $\mathscr{Y}$ on an extension (e, Y) of X is *relatively ultra-closed* iff $e(X) \in \mathscr{Y}$ and $e^{-1}(\mathscr{Y})$ is ultra-closed on X. The condition $e(X) \in \mathscr{Y}$ guarantees that $e(e^{-1}(\mathscr{Y})) = \mathscr{Y}$.

Using this we can define a weak closure operator on Y as follows. For $A \subseteqq Y$,

$A^* = A \cup \{y: \exists \mathscr{Y}$ relatively ultra-closed such that $A \in \mathscr{Y} \to y\}$.

For a filter $\mathscr{D}$ on Y, $\mathscr{D}^*$ is the filter generated by $\{A^*: A \in \mathscr{D}\}$.

We say Y is a *weakly regular* extension of X iff for every relatively ultra-closed filter $\mathscr{Y}$,

$$\mathscr{Y} \longrightarrow y \Longrightarrow (\mathscr{Y}^i)^* \longrightarrow y .$$

REMARK 1.20. Note that $A^* \subseteqq A^-$. From this it follows that if $\mathscr{N}_y \subseteqq \mathscr{U}$ then $\mathscr{N}_y^- \subseteqq (\mathscr{U}^i)^*$. Hence every extension which is regular in the usual topological sense is also weakly regular. Thus in particular every T_2-compactification is weakly regular.

LEMMA 1.21. *A principal extension is weakly regular iff its trace system is regular.*

Proof. Let $K = (e, Y)$ be a principal extension of a T_1-space X with trace system Φ. The key to establishing the result is the following relation: for any two open sets U and V in Y,

$$e^{-1}(U) <_\Phi e^{-1}(V) \quad \text{iff} \quad U^* \subseteqq V^+ .$$

Using this, the result follows easily from the definitions.

DEFINITION 1.22. An extension is *covered* iff every point has a relatively ultra-closed filter which converges to it.

REMARK 1.23. Note that an extension is covered iff its trace system is covered.

The results obtained thus far are summarized in the following theorem.

THEOREM 1.24. *The compactifications obtained from compactification structures under the map π are exactly those principal T_1-compactifications which are weakly regular and covered. Given a fixed T_1-space X, the map π is a bijection from the set of compactification structures on X to the set of* (equivalence classes of) *principal weakly regular covered T_1-compactifications of X.*

Proof. Let Φ be a compactification structure on X. Then by Theorem 1.18 we have that $\pi(\Phi)$ is a T_1-compactification of X. Recall that $\pi(\Phi)$ has trace system Φ. Hence by Lemma 1.21, $\pi(\Phi)$ is weakly regular. Finally, by Remark 1.23, $\pi(\Phi)$ is covered. Hence π maps compactification structures to principal weakly regular covered T_1-compactifications.

Conversely, suppose Φ is the trace system of a principal weakly regular covered T_1-compactification. From Remarks 1.6 and 1.23 we have that Φ is totally-bounded and covered. By Lemma 1.21, Φ is regular.

To see that π is $1-1$, let τ denote the map from an extension to its trace system. Clearly $\tau\pi$ is the identity map on the set of extension structures.

To see that π is a surjection, let K be a principal weakly regular covered T_1-compactification of X. Let $\Phi = \tau(\mathrm{K})$. By Remarks 1.6 and 1.23, Φ is totally-bounded and covered. By Lemma 1.21, Φ is regular. Hence Φ is a compactification structure. We claim that $\pi(\Phi)$ is equivalent to K. This follows from the fact that they both have the same trace system.

The next series of results gives an equivalent characterization of the compactifications induced by compactification structures. In the characterization "covered" can be replaced by "minimal" although the two concepts taken separately do not appear to be equivalent. The main result is that a weakly regular principal compactification is minimal iff it is covered and T_1.

DEFINITION 1.25. A compactification Y of a space X is *minimal* iff the only compact set between $k(X)$ and Y is Y itself.

LEMMA 1.26. *Every weakly regular minimal compactification of a T_1-space is covered.*

Proof. Assume $\mathrm{K} = (k, Y)$ is a weakly regular minimal compactification of a T_1-space X.

Let Z be the set of all points in Y which are limits of relatively ultra-closed filters. We wish to show $Z = Y$. Since K is minimal it is sufficient to show that $k(X) \subseteqq Z$ and Z is compact.

Since X is T_1, for $x \in X$ we have that $k(\dot{x})$ is a relatively ultra-closed filter converging to $k(x)$. Hence $k(X) \subseteqq Z$.

Now suppose $\{G_\alpha : \alpha \in I\}$ is an open cover of Z. Let

$$\mathscr{S} = \{X \backslash U : U \text{ is open and } k(U)^* \subseteqq G_\alpha \text{ for some } \alpha\} .$$

We claim that $\mathscr{S}$ does not have the finite intersection property. Suppose this is false. Then let $\mathscr{U}$ be an ultra-closed filter containing $\mathscr{S}$. Since Y is compact, $k(\mathscr{U})$ has a limit, $y \in Y$. Since $k(\mathscr{U})$ is relatively ultra-closed, $y \in Z$. Thus y is in some G_α. By weak regularity, $G_\alpha \in (k(\mathscr{U})^i)^*$; thus we can choose U open in Y such that $U \in k(\mathscr{U})$ and $U^* \subseteqq G_\alpha$. This allows us to conclude that $k^{-1}(U) \in \mathscr{U}$ and $X \backslash k^{-1}(U) \in \mathscr{S} \subseteqq \mathscr{U}$, which is impossible. Therefore $\mathscr{S}$ cannot have the finite intersection property.

Accordingly, let $U_1, \cdots, U_n$ be such that $X \backslash U_i \in \mathscr{S}$ and $\bigcup_i U_i = X$. Choose α_i such that $k(U_i)^* \subseteqq G_{\alpha_i}$. We claim that $Z \subseteqq \bigcup_i G_{\alpha_i}$.

Let $z \in Z$. Let $\mathscr{V}$ be relatively ultra-closed such that $\mathscr{V} \to z$. Since $k^{-1}(\mathscr{V})$ is ultra-closed, some U_i is in $k^{-1}(\mathscr{V})$. Then $k(U_i) \in \mathscr{V}(=kk^{-1}(\mathscr{V}))$ and hence by definition $z \in k(U_i)^* \subseteqq G_{\alpha_i}$.

LEMMA 1.27. *Every minimal compactification is T_0.*

Proof. Suppose $\mathscr{N}_y = \mathscr{N}_z$ with $y \neq z$. Since X is T_0, at least one of these, say y, is not in $k(X)$. Note $Y \backslash \{y\}$ is still compact, since $\mathscr{N}_y \subseteqq \mathscr{N}_z$. This violates the minimality of K.

To help establish the connection between being minimal and being covered we introduce the notion of a *separated* extension.

DEFINITION 1.28. An extension is *separated* iff each relatively ultra-closed filter has at most one limit.

LEMMA 1.29. *Every weakly regular T_0-extension is separated.*

Proof. Let $\mathscr{V}$ be a relatively ultra-closed filter such that $\mathscr{V} \to y, z$. By weak regularity, $(\mathscr{V}^i)^* \to y, z$. From $(\mathscr{V}^i)^* \to y$ and $\mathscr{V} \to z$ we obtain $\mathscr{N}_y \subseteqq \mathscr{N}_z$. Similarly, $\mathscr{N}_z \subseteqq \mathscr{N}_y$. Since Y is T_0, we obtain $y = z$.

LEMMA 1.30. *Every covered separated extension is T_1.*

Proof. Let $\mathscr{N}_y \subseteqq \mathscr{N}_z$. Since K is covered, there is a relatively ultra-closed filter $\mathscr{U}$ which converges to z and a fortiori to y. Since K is separated, $y = z$.

LEMMA 1.31. *Every covered separated compactification is minimal.*

Proof. Let Z be compact such that $k(X) \subseteqq Z \subseteqq Y$. We wish to show $Z = Y$.

Let $y \in Y$. Since Y is covered there is a relatively ultraclosed filter $\mathscr{V}$ such that $\mathscr{V} \rightarrow y$. Since Z is compact, $\mathscr{V}$ must converge to a point $z \in Z$. Since Y is separated, $y = z$.

THEOREM 1.32. *Let* K *be a weakly regular compactification of a T_1-space. The following conditions are equivalent*:

(i) K *is minimal*

(ii) K *is covered and separated*

(iii) K *is covered and T_1*

Proof. If K is minimal then by Lemma 1.26 it is covered; and by Lemmas 1.27 and 1.29 it is separated. Hence (i) $\Rightarrow$ (ii).

If K is separated and covered then by Lemma 1.30 it is T_1. Thus (ii) $\Rightarrow$ (iii). Finally, if K is covered and T_1 then by Lemma 1.29 it is separated. By Lemma 1.31, every covered separated compactification is minimal.

COROLLARY 1.33. *The compactifications obtained from compactification structures under π are those principal compactifications which are weakly regular and minimal.*

REMARK 1.34. The following proposition establishes that the compactifications induced by compactification structures fall between T_1 and T_2 compactifications.

PROPOSITION 1.35. (1) *Every T_2-compactification is weakly regular, minimal, and principal.*

(2) *Every weakly regular minimal compactification is T_1.*

Proof. Let Y be a T_2-compactification. Then Y is regular. Since for any $A \subseteqq Y$ we have $A^+ \subseteqq A^-$ and $A^* \subseteqq A^-$ we can conclude Y is principal and weakly regular. (See Definitions 1.2 and 1.19.) Since compact subsets of a T_2-space are closed, we have that Y is minimal.

The rest of the conclusion follows from Theorem 1.32.

THEOREM 1.36. *The Wallman compactification of a T_1-space is its largest weakly regular minimal compactification.*

Proof. Let $\omega = (j, (\mathscr{W}, \mathscr{T}^{\wedge}))$ denote the Wallman compactification of a T_1-space X. We have already seen that its trace system is regular (Example 1.16). Hence by Remark 1.20, ω is weakly regular.

Note that if $\mathscr{U}$ and $\mathscr{V}$ are ultra-closed on X then $j(\mathscr{U}) \to \mathscr{V}$ iff $\mathscr{U} = \mathscr{V}$. From this it follows that ω is covered and separated. Hence by Theorem 1.32, ω is minimal.

Now let $\mathrm{K} = (k, (Y, \mathscr{T}'))$ be any weakly regular minimal compactification of X. For $\mathscr{U}$ ultra-closed we note $k(\mathscr{U})$ has a unique limit in Y, since K is compact and separated (Theorem 1.32). This defines a map $h: \mathscr{W} \to Y$. Since K is covered, h maps onto Y.

To establish that h is continuous we will use the fact that K is weakly regular. Suppose G is open in Y and $\mathscr{U}$ is ultraclosed with $k(\mathscr{U}) \to y \in G$; i.e., $\mathscr{U} \in h^{-1}(G)$. By weak regularity $(k(\mathscr{U})^i)^* \to y$. Pick U open in $k(\mathscr{U})$ so $U^* \subseteq G$. It is easy to check that $k^{-1}(U)^\wedge$ is an open neighborhood of $\mathscr{U}$ contained in $h^{-1}(G)$.

2. Proximity relations and T_1-compactifications. In this section we will study the correspondence between extension structures and a class of proximity-like relations, called extension relations. These extension relations take in many of the known proximity relations, including *LO* and *EF*-proximities. We can pass from extension relations to extension structures via a map φ. Of particular interest are the extension relations which give rise to compactification structures under φ. These are the weakly dense, balanced, focused extension relations, and will be called compactification relations. The map φ is not $1-1$ on this set, and hence gives rise to an equivalence relation on the set of compactification relations. Each equivalence class contains a largest member, called a *saturated* compactification relation. The main result states that φ is a bijection from the set of saturated compactification relations on a fixed T_1-space X to the set of compactification structures on X.

DEFINITION 2.1. An *extension relation* on a space $(X, \mathscr{T})$ is a relation $<$ on the subsets of X such that the following conditions are satisfied.

(R1) $\phi < A$ for $A \subseteq X$;
(R2) if $A < B$ then $A \subseteq B$ for $A, B \subseteq X$;
(R3) if $A' \subseteq A < B \subseteq B'$ then $A' < B'$;
(R4) if $A < B$ and $A < C$ then $A < B \cap C$;
(R5) if $A < B$ then $A < B^i$;
(R6) if $x \in G \in \mathscr{T}$ then $\{x\} < G$.

REMARK 2.2. The first three axioms are very nearly the axioms for a semi-topogenous order given by Csàszàr [1]. Only the axiom $A < X$ for $A \subseteq X$ is missing. The extension relations of interest; namely, those of the form $<_\Phi$, where Φ is an extension structure, all satisfy this axiom. So the fact it is missing does not appear significant. The axiom (R4) is a weakening of the requirement for a semi-topogenous order to be a topogenous order. If $<$ is symmetric, then the first four axioms gurantee that $<$ is a topogenous order.

The last two axioms are designed to guarantee that the topology induced in the usual way by $<$ is the original topology $\mathscr{T}$. (A set A is $<$-open iff $x \in A \Rightarrow \{x\} < A$.)

Note that a symmetric extension relation is a *LO*-proximity compatible with the topology.

EXAMPLE 2.3. Note that if Φ is a T_1-extension structure then $<_{\Phi}$ is an extension relation. In particular then $<_{\mathscr{U}}$, $<_{\mathscr{E}}$, and $<_{\Omega}$ are extension relations.

To see more clearly the correspondence between extension relations and ordinary proximities we introduce the notion of the "star" of a relation.

DEFINITION 2.4. For any relation $<$ on the subsets of a topological space X we define $A <^* B$ iff for every closed subset F of A we have $F < B$.

PROPOSITION 2.5. *If $<$ is an extension relation on a T_1-space X then $<^*$ is also an extension relation on X.*

REMARK 2.6. Note that if $\mathscr{E}$ is the set of filters maximal round with respect to an Efremovich proximity $\Subset$ then $<_{\mathscr{E}} = \Subset^*$. Note also that if $\Subset$ is a Lodato proximity then $\Subset$ and $\Subset^*$ are both extension relations.

In order to obtain an extension structure back from an extension relation we make the following definitions.

DEFINITION 2.7. If $<$ is an extension relation on X and $\mathscr{F}$ is a filter on X then

$$r_<(\mathscr{F}) = \{A \colon \exists F \in \mathscr{F} \text{ such that } F < A\}.$$

Then $\Phi_< = \{r_<(\mathscr{U}) \colon \mathscr{U} \text{ is ultra-closed}\}$.

This defines a map φ from extension relations to extension structures on X.

PROPOSITION 2.8. *If $<$ is an extension relation on a T_1-space X then $\varphi(<) = \Phi_<$ is an extension structure on X.*

Proof. This follows easily from the definitions.

Next we will develop properties which will guarantee that $\Phi_<$ is a compactification structure.

DEFINITION 2.9. An extension relation $<$ is *weakly dense* iff for any two ultra-closed filters $\mathscr{U}$ and $\mathscr{V}$ we have

$$r_<(\mathscr{U}) \subseteqq \mathscr{V} \Longrightarrow r_<(\mathscr{U}) \subseteqq r_<(\mathscr{V})\ .$$

We say $<$ is *balanced* if $r_<(\mathscr{U}) \subseteqq \mathscr{V} \Longrightarrow r_<(\mathscr{V}) \subseteqq \mathscr{U}$ for any ultra-closed filters $\mathscr{U}$ and $\mathscr{V}$. The relation $<$ is *focused* if $r_<(\mathscr{U}) = r_<(\mathscr{U}^i)$ for any ultra-closed filter $\mathscr{U}$.

Finally, $<$ is *a compactification relation* iff it is weakly dense, balanced, and focused.

LEMMA 2.10. *Let $<$ be a compactification relation on a T_1-space X.*

(1) *If $\mathscr{F} \subseteqq \mathscr{U}$, where $\mathscr{F} \in \Phi_<$ and $\mathscr{U}$ is ultra-closed, then $\mathscr{F} = r_<(\mathscr{U}) = r_<(\mathscr{U}^i)$.*

(2) $< \subseteqq \rho\varphi(<)$.

Proof. Straightforward.

THEOREM 2.11. *If $<$ is a compactification relation on a T_1-space X then $\Phi_<$ is a compactification structure.*

Proof. It is easy to check that $\Phi_<$ is a covered totally-bounded extension structure.

To see that $\Phi_<$ is regular let $\mathscr{F} \in \Phi_<$ and suppose $\mathscr{F} \subseteqq \mathscr{U}$, where $\mathscr{U}$ is ultra-closed. By the preceding lemma, $\mathscr{F} = r_<(\mathscr{U}^i)$. By the same lemma, $< \subseteqq \rho\varphi(<)$ and hence $r_<(\mathscr{U}^i) \subseteqq \Phi < (\mathscr{U}^i)$. This establishes regularity.

THEOREM 2.12. *If Φ is a compactification structure on a T_1-space X then $\rho(\Phi)$ is compactification relation.*

Proof. Let $<$ denote $\rho(\Phi)$. Note that for any filter $\mathscr{F}$, $r_<(\mathscr{F}) = \Phi(\mathscr{F})$.

1. $<$ is focused. Let $\mathscr{U}$ be ultra-closed. Since Φ is totally-bounded, we can pick $\mathscr{F} \in \Phi$ such that $\mathscr{F} \subseteqq \mathscr{U}$. By regularity, $\mathscr{F} = \Phi(\mathscr{U}^i) = \Phi(\mathscr{U})$.

2. If $\mathscr{F} \in \Phi$ and $\mathscr{F} \subseteqq \mathscr{U}$ where $\mathscr{U}$ is ultra-closed then $\mathscr{F} = r_<(\mathscr{U}) = r_<(\mathscr{U}^i)$.

3. $<$ is weakly dense and balanced. Let $\mathscr{U}$ and $\mathscr{V}$ be ultra-closed such that $r_<(\mathscr{U}) \subseteqq \mathscr{V}$. It is sufficient to show $r_<(\mathscr{U}) = r_<(\mathscr{V})$.

Choose $\mathscr{F} \in \Phi$ such that $\mathscr{F} \subseteqq \mathscr{U}$. Then $\mathscr{F} = r_<(\mathscr{U})$ and hence $\mathscr{F} \subseteqq \mathscr{V}$. This implies $\mathscr{F} = r_<(\mathscr{V})$. Thus $r_<(\mathscr{U}) = \mathscr{F} = r_<(\mathscr{V})$.

REMARK 2.13. We have now defined maps ρ and φ, which map compactification structures to compactification relations, and back. These maps turn out to be inverses, provided we restrict the domain of φ to a special class of relations, called *saturated* compactification relations.

LEMMA 2.14. *The map $\varphi\rho$ is the identity on the set of compactification structures on a given T_1-space X.*

Proof. Let $\Phi_1 = \varphi\rho(\Phi)$, where Φ is a compactification structure on X. If $\mathscr{F} \in \Phi$ then since Φ is covered there is an ultra-closed filter $\mathscr{U}$ with $\mathscr{F} \subseteqq \mathscr{U}$. Since Φ is regular, $\mathscr{F} = \Phi(\mathscr{U})$. But by definition of Φ_1 we have $\Phi(\mathscr{U}) \in \Phi_1$. Hence $\Phi \subseteqq \Phi_1$.

Now let $\mathscr{F} \in \Phi_1$. Choose $\mathscr{U}$ ultra-closed such that $\mathscr{F} = \Phi(\mathscr{U})$. Since Φ is totally-bounded there is a filter $\mathscr{D} \in \Phi$ with $\mathscr{D} \subseteqq \mathscr{U}$. By regularity, $\mathscr{D} = \Phi(\mathscr{U})$ and hence $\mathscr{F} \in \Phi$.

DEFINITION 2.15. An extension relation $<$ is *saturated* iff $A \not< B$ implies the existence of an ultra-closed filter $\mathscr{U}$ such that $A \in \mathscr{U}$ and $B \notin r_<(\mathscr{U})$.

PROPOSITION 2.16. (1) *If Φ is a compactification structure then $<_\Phi = \rho(\Phi)$ is saturated.*

(2) *If $<$ is a saturated compactification relation and $<'$ is any extension relation such that $\varphi(<') = \varphi(<)$ then $<' \subseteqq <$.*

Proof. Clearly (1) follows from definitions. Now suppose $<$ is a saturated compactification relation on a T_1-space X. Let $<'$ be an extension relation on X with $\varphi(<') = \varphi(<)$.

First we will establish that for any ultra-closed filter $\mathscr{U}$ we have $r_<(\mathscr{U}) = r_{<'}(\mathscr{U})$. Note that $r_{<'}(\mathscr{U}) \in \varphi(<)$, and so $r_{<'}(\mathscr{U}) = r_<(\mathscr{V})$ for some ultra-closed filter $\mathscr{V}$. Then $r_<(\mathscr{V}) \subseteqq \mathscr{U}$. Since $<$ is weakly dense and balanced, $r_<(\mathscr{V}) = r_<(\mathscr{U})$.

Now suppose $A \not< B$. We wish to show $A \not<' B$. Let $\mathscr{U}$ be ultra-closed such that $A \in \mathscr{U}$ and $B \notin r_<(\mathscr{U})$. Then $B \notin r_{<'}(\mathscr{U})$ and hence $A \not< B$.

LEMMA 2.17. *The map $\rho\varphi$ is the identity on the set of saturated compactification relations on a fixed T_1-space.*

Proof. Let $<$ be a saturated compactification relation and let

$\lessdot_1 = \rho\varphi(<)$. Note that $\lessdot_1$ is a saturated compactification relation, by Theorems 2.11, 2.12 and Proposition 2.16. Moreover, since $\varphi\rho$ is the identity on compactification structures, we have $\varphi(\lessdot_1) = \varphi(<)$. Thus $\lessdot_1 = <$, by the preceding proposition.

The results of this section are summarized in the next theorem.

THEOREM 2.18. *The map φ is a bijection from the set of saturated compactification relation on a T_1-space X to the set of compactification structures on X. Moreover, φ and ρ are inverses on these two sets.*

Proof. Lemmas 2.14 and 2.17.

Open questions 2.19. Gagrat and Naimpally [2] proved that a separated Lodato proximity gives rise to a T_1-compactification, using maximal bunches. In this paper it was shown that a compactification relation gives rise to a T_1-compactification via an associated family of filters. What is the relation between these two constructions? A Lodato proximity compatible with a given T_1-topology is simply a symmetric extension relation on the space. Are these two compactifications equivalent for a symmetric compactification relation?

Similarly we now have two ways of obtaining an extension relation, given a T_1-compactification $\mathrm{K} = (k, (Y, \mathscr{T}'))$ of X. Let $\Subset_{\mathrm{K}}$ be the relation induced on X from the elementary proximity $\Subset$ on Y; namely, $A \Subset_{\mathrm{K}} B$ iff $k(A)^- \cap k(X\backslash B)^- = \varnothing$. Let the Φ be trace system of K. It is easily checked that $\Subset_{\mathrm{K}}^* \subseteqq \lessdot_\Phi$. Under what conditions are these two relations equal?

We have seen that different extension relations can give rise to the same family of filters (under the map φ). This divides the set of extension relations into equivalence classes. What can be learned from studying these classes? In particular, which of these classes contain compactification relations?

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