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ON ALMOST EVERYWHERE COVERGENCE OF ABEL MEANS OF CONTRACTION SEMIGROUPS

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Let (X, Σ, μ) be a σ-finite measure space and $L_p(X, \Sigma, \mu)$, $1 \leqq p \leqq \infty$, the usual Banach spaces of complex valued functions. Let $\{T_t : t \geqq 0\}$ be a strongly continuous semigroup of contractions of $L_p(X, \Sigma, \mu)$ for some $1 \leqq p < \infty$ and set $R_\lambda f = \int_0^\infty e^{-\lambda t} T_t f dt$, $\lambda > 0$. If $\|T_t\|_\infty \leqq 1$ for all $t \geqq 0$, then $\lim_{\lambda \to \infty} \lambda R_\lambda f(x) = f(x)$ a.e. for all $f \in L_p(X, \Sigma, \mu)$.

A strongly continuous contraction semigroup on $L_p(X, \Sigma, \mu)$ satisfies the following: (i) $T_{s+t} = T_s \cdot T_t$, $s, t \geqq 0$; (ii) $\|T_t\|_p \leqq 1$, $t \geqq 0$; (iii) $\|T_t f - T_s f\|_p \to 0$ as $s \to t$ for any $f \in L_p = L_p(X, \Sigma, \mu)$. Merely as a notational convenience, we assume that $T_0 = I$. Before proceeding further it is necessary to clarify the definition of $R_\lambda f(x)$. By Theorem III.11.17 in [3], given $f \in L_p$ there exists a scalar function $g(t, x)$, measurable with respect to the usual product measure on $[0, \infty) \times X$, such that (i) for a.e. t, $g(t, \cdot) = T_t f$ and (ii) there exists a μ-null set $E(f)$, independent of λ, such that $x \notin E(f)$ implies $\int_0^\infty e^{-\lambda t} g(t, x) dt$, as a function of x, is in the equivalence class of $\int_0^\infty e^{-\lambda t} g(t, x) dt$. The scalar representation $g(t, x)$ is uniquely determined up to a set of product measure zero. Defining $R_\lambda f(x) = \int_0^\infty e^{-\lambda t} g(t, x) dt$, we see that $R_\lambda f(x)$ is in the equivalence class of $R_\lambda f = \int_0^\infty e^{-\lambda t} T_t f dt$ for all $\lambda > 0$. This justifies our definition of $R_\lambda f(x)$. Note that for $x \notin E(f)$, $R_\lambda f(x)$ is a continuous function of $\lambda > 0$.

The main result of this note (Theorem 4) extends a special case of a theorem of N. Dunford and J. T. Schwartz [2, p. 178]. If $p = 1$ in our theorem then the assumption $\|T_t\|_\infty \leqq 1$ for $t \geqq 0$ is unnecessary [5].

Preliminary results.

LEMMA 1. *Let $\{T_t : t \geqq 0\}$ be a strongly continuous semigroup of L_p contractions for some $1 \leqq p < \infty$. Set $\mathscr{M} = \{\lambda R_\lambda f : 0 < \lambda < \infty, f \in L_p\}$. Then $\mathscr{M}$ is dense in L_p and* $\lim_{\lambda \to \infty} \lambda R_\lambda f(x) = f(x)$ a.e. *for any $f \in L_p$.*

The denseness of $\mathscr{M}$ follows from the fact that $s - \lim_{\lambda \to \infty} \lambda R_\lambda f = f$ [4, p. 321], and the existence of the pointwise limit follows from the resolvent equation. The details appear in [5]. The next result is proved in [1].

LEMMA 2. *Let $\{T_t: t \geqq 0\}$ be a strongly continuous semigroup of L_p contractions for some $1 \leqq p < \infty$. Suppose that $\|T_t\|_\infty \leqq 1$ for all $t \geqq 0$. Then*

$$\lim_{\varepsilon \downarrow 0} \left(\frac{1}{\varepsilon}\right)\int_0^\varepsilon T_t f(x) dt = f(x) \text{ a.e.}$$

for every $f \in L_p$.

For a given L_p semigroup $\{T_t: t \geqq 0\}$, define $T'_t = e^{-t}T_t$. Then $\{T'_t: t \geqq 0\}$ is a semigroup; if $\{T_t: t \geqq 0\}$ is strongly continuous so is $\{T'_t: t \geqq 0\}$. We shall denote the resolvent of $\{T'_t\}$ by R'_λ. For $f \in L_p$, set $f^* = \sup_{\lambda > 0} |\lambda R'_\lambda f|$.

LEMMA 3. *Suppose $\{T_t: t \geqq 0\}$ is a strongly continuous contraction semigroup on L_p for some $1 \leqq p < \infty$. If, in addition, $\|T_t\|_\infty \leqq 1$ for all $t \geqq 0$, then $f^* < \infty$ a.e. for any $f \in L_p$.*

Proof. Fix $f \in L_p$ and choose $\{\varepsilon_n\}$ such that $\varepsilon_n \downarrow 0$. Set

$$g_n = \inf_{\varepsilon \leqq \varepsilon_n} \left[\frac{1}{\varepsilon}\int_0^\varepsilon T'_t f(x) dt\right],$$

$$h_n = \sup_{\varepsilon \leqq \varepsilon_n} \left[\frac{1}{\varepsilon}\int_0^\varepsilon T'_t f(x) dt\right],$$

$$f_\varepsilon^* = \sup_{\delta \leqq \varepsilon} \left|\frac{1}{\delta}\int_0^\delta T'_t f(x) dt\right| .$$

Let A be a measurable subset of X with $0 < \mu(A) < \infty$. Since $\{T'_t: t \geqq 0\}$ satisfies the conditions of Lemma 2, we have

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\int_0^\varepsilon T'_t f(x) dt = f(x) \text{ a.e. on } X .$$

Hence $\lim_{n\to\infty} g_n = \lim_{n\to\infty} h_n = f(x)$ a.e. By Egoroff's theorem, given $0 < \delta < \mu(A)/2$, there exists a measurable subset B of A such that $\mu(B) > \mu(A) - 2\delta$ and $\{g_n\}$, $\{h_n\}$ converge uniformly on B to $f(x)$. Therefore, for some K, $n \geqq K$ implies $|g_n - f| \leqq 1$ and $|h_n - f| \leqq 1$ for all $x \in B$. Consequently $|g_n| \leqq |f| + 1$ and $|h_n| \leqq |f| + 1$ on B for all $n \geqq K$. For given n, we have

$$g_n(x) \leqq \frac{1}{\varepsilon}\int_0^\varepsilon T'_t f(x) dt \leqq h_n(x)$$

for any $\varepsilon \leqq \varepsilon_n$. Thus for any $x \in B$ and $n \geqq K$,

$$\begin{aligned} f_\varepsilon^*(x) &\leqq |g_n(x)| + |h_n(x)| \\ &\leqq 2|f(x)| + 2 , \end{aligned}$$

provided $\varepsilon \leqq \varepsilon_n$. For some fixed $n \geqq K$, set $\delta = \varepsilon_n$.

By an integration by parts, we have

$$\lambda\int_0^\infty e^{-\lambda t}T'_t f(x)dt = \lambda^2\int_0^\infty e^{-\lambda t}t\left[\frac{1}{t}\int_0^t T'_s f(x)ds\right]dt \text{ a.e. on } X .$$

For $t \geqq \delta$ we have

$$\left|\frac{1}{t}\int_0^t T'_s f(x)ds\right| \leqq \frac{1}{\delta}\int_0^\infty |T'_s f(x)|\, ds < \infty \text{ a.e. on } X$$

since $\left\|\int_0^\infty |T'_s f(x)|\, ds\right\|_p \leqq \|f\|_p$. Hence for a.e. $x \in B$,

$$\begin{aligned}
&\left|\lambda^2\int_0^\infty e^{-\lambda t}t\left[\frac{1}{t}\int_0^t T'_s f(x)ds\right]dt\right| \\
&\leqq \lambda^2\int_0^\delta e^{-\lambda t}t[2|f(x)| + 2]dt \\
&\quad + \left(\frac{\lambda^2}{\delta}\right)\int_0^\infty e^{-\lambda t}t\left[\int_0^\infty |T'_s f(x)|\, ds\right]dt \\
&\leqq [2|f(x)| + 2]\left[\lambda^2\int_0^\infty te^{-\lambda t}dt\right] \\
&\quad + \left[\frac{1}{\delta}\int_0^\infty |T'_s f(x)|\, ds\right]\left[\lambda^2\int_0^\infty te^{-\lambda t}dt\right] \\
&\leqq \{2|f(x)| + 2\} + \left(\frac{1}{\delta}\right)\int_0^\infty |T'_s f(x)|\, ds
\end{aligned}$$

for all $\lambda > 0$. Hence $f^* < \infty$ a.e. on B. Since the set A was an arbitrary set of finite measure and B is a measurable subset of A having positive measure, we conclude that $f^* < \infty$ a.e. on X.

Main results.

THEOREM 4. *Let $\{T_t : t \geqq 0\}$ be a strongly continuous semigroup of L_p contractions for some $1 \leqq p < \infty$. Suppose that $\|T_t\|_\infty \leqq 1$ for all $t \leqq 0$. If $f \in L_p$, then*

$$\lim_{\lambda\to\infty} \lambda R_\lambda f(x) = f(x) \text{ a.e.}$$

Proof. By Lemmas 1 and 3 and Banach's convergence theorem [3, p. 332-333], $\lim \lambda R'_\lambda f(x)$ exists and is finite a.e. as $\lambda \to \infty$ through some countable set, say Q^+(= set of positive rationals). We recall that $\lambda R'_\lambda f(x)$ depends continuously on λ for x outside some null set. Since Q^+ is dense in R^+ it follows that $\lim_{\lambda\to\infty} \lambda R'_\lambda f(x)$ exists and is finite a.e. for all $f \in L_p$. Since $s - \lim_{\lambda\to\infty} \lambda R'_\lambda f = f$, we must have $\lim_{\lambda\to\infty} \lambda R'_\lambda f(x) = f(x)$ a.e. Upon noting that $\lim_{\lambda\to\infty} R_\lambda f(x) = 0$ a.e. for any $f \in L_p$, we see that

$$\lim_{\lambda\to\infty} \lambda R_\lambda f(x) = \lim_{\lambda\to\infty} (\lambda + 1)R_{\lambda+1}f(x)$$
$$= \lim_{\lambda\to\infty} \lambda R'_\lambda f(x)$$
$$= f(x) \text{ a.e.}$$

The following result which generalizes Theorem 4 follows from (4.9) in [1] and the arguments used in obtaining Theorem 4.

THEOREM 5. *Let $\{T_t: t \geqq 0\}$ be a strongly continuous semigroup of L_p contractions for some $1 \leqq p < \infty$. Suppose there exists a measurable function h on $[0, \infty) \times X$ such that*

(i) *$h > 0$ on $[0, \infty) \times X$, and*

(ii) *$f \in L_p$, $|f(x)| \leqq h(t, x)$ μ-a.e. implies*

$$|T_s f(x)| \leqq h(t + s, x) \quad \text{for all} \quad s, t \geqq 0 .$$

Then $\lim_{\lambda\to\infty} \lambda R_\lambda f(x) = f(x)$ a.e. *for* $f \in L_p$.

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