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FIXED POINTS OF LOCALLY CONTRACTIVE AND NONEXPANSIVE SET-VALUED MAPPINGS

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Let (M, d) be a complete metric space and $S(M)$ the set of all nonempty bounded closed subsets of M. A set-valued mapping $f: M \to S(M)$ will be called (uniformly) *locally contractive* if there exist ε and λ $(\varepsilon > 0, 0 < \lambda < 1)$ such that $D(f(x), f(y)) \leqq \lambda d(x, y)$ whenever $d(x, y) < \varepsilon$ and where $D(f(x), f(y))$ is the distance between $f(x)$ and $f(y)$ in the Hausdorff metric induced by d on $S(M)$. It is shown in the first theorem that if M is "well-chained," then f has a fixed point is, that is, a point $x \in M$ such that $x \in f(x)$. This fact, in turn, yields a fixed-point theorem for locally nonexpansive set-valued mappings on a compact star-shaped subset of a Banach space. Both theorems are extensions of earlier results.

1. Locally contractive set-valued mappings. Following Assad and Kirk [1] we shall define D as follows: if $r > 0$ and $Y \in S(M)$, let

$$Z(r, Y) = \{x \in M: \operatorname{dist}(x, Y) < r\} .$$

Then for $A, B \in S(M)$ we define

$$D(A, B) = \inf\{r: A \subset Z(r, B) \text{ and } B \subset Z(r, A)\} .$$

Also noted in [1] are two lemmas:

LEMMA 1. *If $A, B \in S(M)$ and $x \in A$, then for each positive number α there exists $y \in B$ such that*

$$d(x, y) \leqq D(A, B) + \alpha .$$

LEMMA 2. *Let $\{X_n\}$ be a sequence of sets in $S(M)$, and assume that $\lim_{n\to\infty} D(X_n, X_0) = 0$ $(X_0 \in S(M))$. Then if $x_n \in X_n$ $(n = 1, 2, \cdots)$ and $\lim_{n\to\infty} x_n = x_0$, it follows that $x_0 \in X_0$.*

Finally, suppose M is well-chained in the sense that for every $\varepsilon > 0$ and $x, y \in M$ there exists an ε-chain, that is, a finite set of points

$$x = y_0, \ y_1, \cdots, \ y_n = z$$

(n may depend on both x and z) such that $d(y_i, y_{i+1}) < \varepsilon$ $(i = 0, 1, \cdots, n - 1)$.

THEOREM 1. *Suppose (M, d) is a complete well-chained metric space and $S(M)$ the set of all nonempty bounded closed subsets of M. If $f: M \to S(M)$ is locally contractive, then f has a fixed point.*

Proof. Assume that $\varepsilon < 1$ and let $x_0, y_0 \in M$ such that $d(x_0, y_0) < \varepsilon$. Then

$$D(f(x_0), f(y_0)) \leqq \lambda d(x_0, y_0) .$$

Now choose a positive number $\eta < \varepsilon - \lambda\varepsilon < 1$. Let x_1 be any element in $f(x_0)$; then there exists by Lemma 1 an element $y_1 \in f(y_0)$ such that

$$d(x_1, y_1) \leqq D(f(x_0), f(y_0)) + \eta .$$

Hence

$$d(x_1, y_1) < \lambda\varepsilon + \eta < \lambda\varepsilon + \varepsilon - \lambda\varepsilon = \varepsilon .$$

Next, let $x_2 \in f(x_1)$; then there exists $y_2 \in f(y_1)$ such that

$$\begin{aligned} d(x_2, y_2) &\leqq D(f(x_1), f(y_1)) + \eta^2 \\ &\leqq \lambda d(x_1, y_1) + \eta^2 . \end{aligned}$$

In general, for $n > 0$

$$d(x_n, y_n) \leqq D(f(x_{n-1}), f(y_{n-1})) + \eta^n ,$$

and we can show by induction that

$$d(x_n, y_n) < \lambda^n\varepsilon + \lambda^{n-1}\eta + \lambda^{n-2}\eta^2 + \cdots + \eta^n . \tag{1}$$

Indeed,

$$\begin{aligned} &\lambda^n\varepsilon + \lambda^{n-1}\eta + \lambda^{n-2}\eta^2 + \cdots + \eta^n \\ &\quad < \lambda^n\varepsilon + \lambda^{n-1}(\varepsilon - \lambda\varepsilon) + \lambda^{n-2}(\varepsilon - \lambda\varepsilon)^2 + \cdots + (\varepsilon - \lambda\varepsilon)^n \\ &\quad \leqq \lambda^n\varepsilon + \lambda^{n-1}(\varepsilon - \lambda\varepsilon) + \lambda^{n-2}(\varepsilon - \lambda\varepsilon) + \cdots + (\varepsilon - \lambda\varepsilon) \\ &\quad = \lambda^n\varepsilon + (\lambda^{n-1}\varepsilon - \lambda^n\varepsilon) + (\lambda^{n-2}\varepsilon - \lambda^{n-1}\varepsilon) + \cdots + (\varepsilon - \lambda\varepsilon) \\ &\quad = \varepsilon . \end{aligned}$$

So if (1) is valid for $n = N > 0$, let $x_{N+1} \in f(x_N)$; then there exists $y_{N+1} \in f(y_N)$ such that

$$\begin{aligned} d(x_{N+1}, y_{N+1}) &\leqq D(f(y_N), f(y_N)) + \eta^{N+1} \leqq \lambda d(x_N, y_N) + \eta^{N+1} \\ &< \lambda(\lambda^N\varepsilon + \lambda^{N-1}\eta + \lambda^{N-2}\eta^2 + \cdots + \eta^N) + \eta^{N+1} \\ &= \lambda^{N+1}\varepsilon + \lambda^N\eta + \lambda^{N-1}\eta^2 + \cdots + \lambda\eta^N + \eta^{N+1} . \end{aligned}$$

Using this information we now construct a sequence in M as follows: let $y_{0,0}$ be an arbitrary element in M and let $y_{1,0} \in f(y_{0,0})$.

Consider the ε-chain

$$y_{0,0},\ y_{0,1},\ \cdots,\ y_{0,n} = y_{1,0} \in f(y_{0,0})\ ,$$

so that $d(y_{0,i}, y_{0,i+1}) < \varepsilon$ $(i = 0, 1, \cdots, n-1)$. Since $y_{1,0} \in f(y_{0,0})$, we may choose $y_{1,1} \in f(y_{0,1})$ such that

$$(2) \qquad d(y_{1,0}, y_{1,1}) \leqq D(f(y_{0,0}), f(y_{0,1})) + \eta\ .$$

Similarly, since $y_{1,1} \in f(y_{0,1})$, choose $y_{1,2} \in f(y_{0,2})$ such that

$$d(y_{1,1}, y_{1,2}) \leqq D(f(y_{0,1}), f(y_{0,2})) + \eta\ .$$

Continuing along the ε-chain, since $y_{1,n-1} \in f(y_{0,n-1})$, there exists $y_{1,n} = y_{2,0} \in f(y_{0,n})$ (i.e., $y_{2,0} \in f(y_{1,0})$) such that

$$d(y_{1,n-1}, y_{1,n}) \leqq D(f(y_{0,n-1}), f(y_{0,n})) + \eta\ .$$

Consequently,

$$d(y_{1,0}, y_{2,0}) = d(y_{1,0}, y_{1,n}) \leqq \sum_{i=0}^{n-1} d(y_{1,i}, y_{1,i+1}) < n(\lambda\varepsilon + \eta)\ .$$

Next, referring to (2), since $y_{2,0} \in f(y_{1,0})$, there exists $y_{2,1} \in f(y_{1,1})$ for which

$$d(y_{2,0}, y_{2,1}) \leqq D(f(y_{1,0}), f(y_{1,1})) + \eta^2\ ,$$

and for $y_{2,n-1} \in f(y_{1,n-1})$, we have $y_{2,n} = y_{3,0} \in f(y_{1,n})$ (i.e., $y_{3,0} \in f(y_{2,0})$) such that

$$d(y_{2,n-1}, y_{2,n}) \leqq D(f(y_{1,n-1}), f(y_{1,n})) + \eta^2\ .$$

Proceeding in this manner, and making use of (1), we get (for $m > 0$)

$$d(y_{m,l}, y_{m,l+1}) < \lambda^m\varepsilon + \lambda^{m-1}\eta + \lambda^{m-2}\eta^2 + \cdots + \eta^m$$

$(l = 0, 1, \cdots, n-1)$. Now let $z_m = y_{m,0}$, so that $z_m \in f(z_{m-1})$, $m = 1, 2, \cdots$, and $z_{m+1} = y_{m+1,0} = y_{m,n}$. Then

$$\begin{aligned} d(z_m, z_{m+1}) &\leqq \sum_{l=0}^{n-1} d(y_{m,l}, y_{m,l+1}) \\ &< n(\lambda^m\varepsilon + \lambda^{m-1}\eta + \lambda^{m-2}\eta^2 + \cdots + \eta^m)\ . \end{aligned}$$

To show that $\{z_m\}$ is a Cauchy sequence, let $\beta = \max(\lambda, \eta)$. Then

$$d(z_m, z_{m+1}) < n(m+1)\beta^m\ ,$$

and for $0 < i < j$

$$d(z_i, z_j) \leqq \sum_{k=i}^{j-1} d(z_k, z_{k+1})$$
$$< n \sum_{k=i}^{j-1} (k+1)\beta^k$$
$$\leqq n \sum_{k=i}^{\infty} (k+1)\beta^k .$$

It is easily checked that $d(z_i, z_j) \to 0$ as $i \to \infty$, implying that $\{z_m\}$ is a Cauchy sequence, which converges to some $z \in M$ by the completeness of M.

Finally, since $z_m \in f(z_{m-1})$ and $z_m \to z$, $f(z_{m-1}) \to f(z)$ and, by Lemma 2, $z \in f(z)$.

REMARK 1. Nadler [4] proved a similar theorem by a different method under the additional assumption that each $f(x)$ is compact.

2. Locally nonexpansive set-valued mappings. Let X be a Banach space and C a subset of X. A mapping $T: C \to S(C)$ will be called *locally nonexpansive* if there exists $\varepsilon > 0$ such that

$$D(Tx, Ty) \leqq \|x - y\| ,$$

whenever $\|x - y\| < \varepsilon$ and where D is again the distance in the Hausdorff metric induced by d on $S(M)$ (as usual, $d(x, y) = \|x - y\|$ for all $x, y \in X$).

THEOREM 2. *Let X be a Banach space and C a compact star-shaped subset of X. If $T: C \to S(C)$ is locally nonexpansive, then there exists a point $x \in C$ such that $x \in Tx$.*

Proof. Let c be the star-center of C and let $\{k_n\}$ be an increasing sequence of real numbers converging to 1. Define $U_n: C \to S(C)$ by

$$U_n x = (1 - k_n)c + k_n Tx ,$$

where $k_n Tx = \{k_n y: y \in Tx\}$. Let $z, y \in C$ such that $\|z - y\| < \varepsilon$. Then $D(Tz, Ty) \leqq \|z - y\|$. Now for any two elements $z' \in Tz$ and $y' \in Ty$

$$\|(1 - k_n)c + k_n z' - (1 - k_n)c - k_n y'\| = k_n \|z' - y'\| .$$

Hence

$$D(U_n z, U_n y) \leqq k_n \|z - y\| .$$

Consequently, U_n has a fixed point $x_n \in C$ by Theorem 1. Since C is compact, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging to some $x \in C$, and because T is continuous,

$$Tx_{n_i} \longrightarrow Tx .$$

Now

$$\begin{aligned} \operatorname{dist}(x_{n_i}, Tx_{n_i}) &\leqq D(U_{n_i}x_{n_i}, Tx_{n_i}) \\ &= D((1 - k_{n_i})c + k_{n_i}Tx_{n_i}, Tx_{n_i}) \longrightarrow D(Tx, Tx) \text{ as } i \longrightarrow \infty . \end{aligned}$$

Thus

$$\operatorname{dist}(x, Tx) = 0 ,$$

which implies that $x \in Tx$, Tx being closed.

Theorem 2 and its point-to-point analogue generalize an earlier theorem due to Dotson [2]:

COROLLARY. *A nonexpansive self-mapping of a compact star-shaped subset of a Banach space has a fixed point.*

REMARK 2. Edelstein [3] has shown that a locally contractive (nonexpansive) point-to-point mapping need not be globally contractive (nonexpansive). On convex sets, however, a locally nonexpansive mapping is nonexpansive.

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