PACIFIC JOURNAL OF MATHEMATICS
Vol. 65, No. 2, 1976

A CHARACTERIZATION OF RIEMANN ALGEBRAS

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A topological algebra $\mathscr{A}$ is called Riemann algebra if it is topologically isomorphic to the Fréchet algebra $\mathscr{O}(R)$ of all holomorphic functions on some Riemann surface R.

One obtains a characterization of Riemann algebras by a theorem of R. L. Carpenter and the Oka-Weil-Cartan theorem; we show that: a uniform Fréchet algebra $\mathscr{A}$ whose spectrum is locally compact and connected, is a Riemann algebra if and only if every closed maximal ideal is principal.

1. Let R be a Riemann surface and $\mathscr{O}(R)$ that algebra of all holomorphic functions on R. $\mathscr{O}(R)$ is a uniform Fréchet algebra, if it is endowed with the topology of uniform convergence on compact sets of R. A topological algebra $\mathscr{A}$ which is topologically isomorphic to $\mathscr{O}(R)$, is called a *Riemann algebra*.

If R is compact, one obtains the trivial Riemann algebra $\boldsymbol{C}$.

In 1953 S. Kakutani posed the problem of characterizing Riemann algebras by intrinsic properties. The most far reaching result is due to I. Richards [9], who considers algebraic properties only. More natural conditions are obtained if one considers topologically algebraic properties. Some special results are found in R. F. Arens [1], F. T. Birtel [3], I. Kra [8], R. L. Carpenter [5]; a summary, in a sense, is provided by I. Richards' paper [10]. This present paper is written in a selfcontained way.

In his paper [4] R. L. Carpenter, using earlier work of A. Gleason [7], proved the following—here cited with a slight modification.

THEOREM. *Let $\mathscr{A}$ be a uniform Fréchet algebra whose spectrum $\mathbf{s}\mathscr{A}$ is locally compact and connected and does not consist of a singleton. If every closed maximal ideal in $\mathscr{A}$ is principal, then $\mathbf{s}\mathscr{A}$ can be given the structure of a Riemann surface in such a way that $\mathscr{A}$ is topologically isomorphic to a closed subalgebra of $\mathscr{O}(\mathbf{s}\mathscr{A})$. Furthermore $\mathscr{A}$ contains local coordinates for every point in $\mathbf{s}\mathscr{A}$.*

In fact, we show that $\mathscr{A}$ is even topologically isomorphic to $\mathscr{O}(\mathbf{s}\mathscr{A})$ under the above hypotheses.

2. Let us fix some notations first.

A *Fréchet algebra* (F-algebra) is a commutative, locally convex, complete algebra over the complex field $\boldsymbol{C}$ with unit whose topology is generated by a countable number of semi-norms.

Now let $\mathscr{A}$ be a F-algebra. By $\boldsymbol{s}\mathscr{A}$ we denote the *spectrum* of $\mathscr{A}$, the set of all nontrivial continuous $\boldsymbol{C}$-algebra homomorphisms $\boldsymbol{f}: \mathscr{A} \to \boldsymbol{C}$; as usual, it is given the Gelfand topology (=weak $-$ $*$ topology).

The Gelfand representation. Let $\mathscr{C}(\boldsymbol{s}\mathscr{A})$ denote the space of all continuous functions on $\boldsymbol{s}\mathscr{A}$, endowed with the compact open topology. Then the standard Gelfand representation

$$\boldsymbol{G}: \mathscr{A} \longrightarrow \mathscr{C}(\boldsymbol{s}\mathscr{A}) , \quad a \longmapsto \hat{a}$$

given by setting $\hat{a}(\boldsymbol{f}) := \boldsymbol{f}(a)$ for $a \in \mathscr{A}$, $\boldsymbol{f} \in \boldsymbol{s}\mathscr{A}$, is a continuous $\boldsymbol{C}$-algebra-homomorphism.

Call $\mathscr{A}$ a *uniform Fréchet algebra* (uF-algebra), if the Gelfand representation $\boldsymbol{G}$ induces a topological isomorphism of $\mathscr{A}$ onto a closed subalgebra $\boldsymbol{G}(\mathscr{A}) \subset \mathscr{C}(\boldsymbol{s}\mathscr{A})$.

Let $K \subset \boldsymbol{s}\mathscr{A}$ be a compact set. For $a \in \mathscr{A}$ we denote the K-sup norm as usual.

$$||a||_K := \sup_{\boldsymbol{f} \in K} |\hat{a}(\boldsymbol{f})| .$$

By $\mathscr{A}_K$ we denote the separated completion of $\mathscr{A}$ under the seminorm $||\cdot||_K$. It is well known that a uF-algebra $\mathscr{A}$ can be represented as an inverse limit of countably many uniform Banach algebras

$$\mathscr{A} = \varprojlim \mathscr{A}_{K_n} ,$$

where an appropriate sequence $\cdots \subset K_n \subset K_{n+1} \subset \cdots$ of compact subsets exhausts $\boldsymbol{s}\mathscr{A}$ such that any compact $K \subset \boldsymbol{s}\mathscr{A}$ is contained in some K_n.

It is well known (e.g. cf. [6] p. 377f.) that every Riemann algebra $\mathscr{O}(R)$ is a uF-algebra whose Gelfand space is topologically equivalent to R. The following gives a simple *characterization of Riemann algebras.*

THEOREM. *Let $\mathscr{A} \neq \boldsymbol{C}$ be a uF-algebra such that $\boldsymbol{s}\mathscr{A}$ is locally compact and connected. Then the following statements are equivalent:*

(i) *$\mathscr{A}$ is a Riemann algebra;*
(ii) *every closed maxmal ideal in $\mathscr{A}$ is principal;*
(iii) *every closed ideal in $\mathscr{A}$ is principal.*

Proof. The implication (i) $\Rightarrow$ (iii) follows e.g. from O. Forster [6], Satz 5.2. The implication (iii) $\Rightarrow$ (ii) is trivial. New we prove (ii) $\Rightarrow$ (i)[1].

Carpenter's theorem states in particular: $\boldsymbol{s}\mathscr{A}$ can be given the structure of a Riemann surface by coordinates which are the Gelfand transforms of certain elements of $\mathscr{A}$, such that $G(\mathscr{A})$ is a closed subalgebra of $\mathscr{O}(\boldsymbol{s}\mathscr{A})$.

Now we want to prove that $G(\mathscr{A}) = \mathscr{O}(\boldsymbol{s}\mathscr{A})$. We will do this by means of the Oka-Weil-Cartan theorem (cf. [2], p. 145) which we state here, together with the necessary definitions. Recall that $\boldsymbol{s}\mathscr{A}$ is a Riemann surface, and $G(\mathscr{A})$ is an algebra of holomorphic functions on $\boldsymbol{s}\mathscr{A}$. One calls $\boldsymbol{s}\mathscr{A}$:

(a) "*$G(\mathscr{A})$-convex*" if, for every compact subset $K \subset \boldsymbol{s}\mathscr{A}$, the set

$$\hat{K} = \{f \in \boldsymbol{s}\mathscr{A} \mid |\hat{a}(f)| \leqq \|a\|_K \text{ for all } \hat{a} \in G(\mathscr{A})\}$$

is also compact;

(b) "*$G(\mathscr{A})$-separable*" if $G(\mathscr{A})$ separates points on $\boldsymbol{s}\mathscr{A}$;

(c) "*$G(\mathscr{A})$-regular*" if $G(\mathscr{A})$ contains local coordinates for every point of $\boldsymbol{s}\mathscr{A}$.

The Oka-Weil-Cartan theorem states that for any Stein manifold (and hence in particular for any noncompact Riemann surface), the conditions (a), (b), (c) imply that the algebra $G(\mathscr{A})$ is *dense* in $\mathscr{O}(\boldsymbol{s}\mathscr{A})$.

Thus we wish to show that $\boldsymbol{s}\mathscr{A}$ is $G(\mathscr{A})$-convex, $G(\mathscr{A})$-separable, and $G(\mathscr{A})$-regular.

$G(\mathscr{A})$-separable. This is trivial, since by definition the algebra $\mathscr{A}$ separates points on $\boldsymbol{s}\mathscr{A}$, and $G(\mathscr{A})$ is an isomorphic copy of $\mathscr{A}$.

$G(\mathscr{A})$-regular. The existence of local coordinates is part of Carpenter's theorem.

$G(\mathscr{A})$-convex. This follows from the fact that $\mathscr{A}$ is a *uF*-algebra and from Tychonov's theorem. Take any compact set $K \subset \boldsymbol{s}\mathscr{A}$. We need to show that its $G(\mathscr{A})$-convex hull $\hat{K}$ defined in (a) above is also compact. The restriction map $G(\mathscr{A}) \rightarrow G(\mathscr{A})_K$ has dense image and therefore, the canonic map $\boldsymbol{s}\mathscr{A}_K \rightarrow \boldsymbol{s}\mathscr{A}$ is injective;

[1] There is a different, more complicated proof of the main theorem which uses the functional calculus for *B*-algebras instead of the Oka-Weil-Cartan theorem.

it is even a homomorphic embedding, as a direct calculation with a subbasis of the weak-$*$ topology of $s\mathscr{A}$ shows.

Now it is easily seen that $\hat{K}$ coincides with $s\mathscr{A}_K$. Since $\mathscr{A}_K$ is a commutative Banach algebra with unit, it follows from Tychonov's theorem that $\mathscr{A}_K$ has compact spectrum. Thus $\hat{K}$ is compact, and the $G(\mathscr{A})$-convexity is proved.

Now the Oka-Weil-Cartan theorem implies that $G(\mathscr{A})$ is a dense subalgebra of $\mathscr{O}(s\mathscr{A})$. Since $\mathscr{A}$ is complete and topologically isomorphic to $G(\mathscr{A})$, we conclude that $G(\mathscr{A}) = \mathscr{O}(s\mathscr{A})$, as desired.

3. We illustrate the proof of our theorem by three *examples*. Let $C^*: = C - \{0\}$. Then $\mathscr{A}_1: = \mathscr{O}(C)$ is a (closed) uF-subalgebra of $\mathscr{O}(C^*)$. C^* is $\mathscr{A}_1$-separable and $\mathscr{A}_1$-regular but not $\mathscr{A}_1$-convex, and indeed $\mathscr{A}_1 \neq \mathscr{O}(C^*)$.

It is easily seen that the algebra $\mathscr{A}_2$ of all holomorphic functions on the analytic set

$$\{(x, y) \in C^2 \mid x^3 - y^2 = 0\} \quad \text{(Neil's parabola)}$$

may be identified algebraically and topologically with the uniform closure of $C[z^2, z^3]$ in $\mathscr{O}(C)$. C ist $\mathscr{A}_2$-separable and $\mathscr{A}_2$-convex but not $\mathscr{A}_2$-regular. For one cannot find in $\mathscr{A}_2$ any local coordinate for a neighbourhood of $0 \in C$ which is equivalent to $z \in \mathscr{O}(C)$. In fact, the maximal ideal in $\mathscr{A}_2$ correspondent to $\mathscr{O} \in C$ is not principal, and therefore $\mathscr{A}_2$ is not a Riemann algebra.

The uniform closure of $C[z^2, 1/z^2]$ in $\mathscr{O}(C^*)$ provides an example of a uF-subalgebra $\mathscr{A}_3$ such that C^* is $\mathscr{A}_3$-convex and $\mathscr{A}_3$-regular but not $\mathscr{A}_3$-separable.

It can happen that a Riemann algebra contains a properly closed subalgebra which can be mapped topologically isomorphic onto the former algebra. For example, take the above algebras $\mathscr{A}_3 \subset \mathscr{O}(C^*)$. The map $\mathscr{O}(C^*) \to \mathscr{A}_3$ induced by $z \mapsto z^2$, is a topologic isomorphism, by the open mapping theorem for Fréchet spaces. The hence homeomorphic spectra are linked as a twosheeted covering, by the adjoint spectral map of the inclusion map $\mathscr{A}_3 \subset \mathscr{O}(C^*)$.

4. We assume in this section that all uF-algebras under consideration have locally compact and connected spectrum. "Generators" for algebras are understood to be topological generators. Call a Riemann algebra *planar*, if its associated Riemann surface can be realized as a domain in the complex plane. Then the above theorem and Runge's approximation theorem yield the

COROLLARY. *A Riemann algebra is planar if and only if it is* *a singly rationally generated uF-algebra, whose closed maximal ideals are principal.*

Finally, we note two reformulations of classic function theory in the language of uF-algebras.

PROPOSITION.

(1) *There exist two and only two singly generated uF-algebras whose closed maximal ideals are principal* (modulo topological isomorphism).

(2) *Every uF-algebra such that every closed maximal ideal is principal, can be generated by three elements.*

Proof. (1) Use Riemann's mapping theorem.

(2) Use the embedding theorem for Riemann surfaces.

ACKNOWLEDGMENT. The author is indebted to Otto Forster for a hint as to simplification, and he wishes to thank the referee for a lot of (not only) stylistic improvements.

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Received January 28, 1976 and in revised form June 15, 1976.

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