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KOROVKIN SETS FOR AN OPERATOR ON A SPACE OF CONTINUOUS FUNCTIONS

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We characterize Korovkin sets for sequences of either positive operators or contractive operators converging to an operator T. Properties of both the Korovkin sets and the operator T are given which were previously known only in the case T was the identity operator.

Let $C = C(Q)$ be the Banach space of continuous functions on a first countable, compact Hausdorff space Q. Let $\mathscr{J}$ be a subset of the bounded linear operators $\mathscr{B}(C)$ on C. A subspace X of C is said to be a *$\mathscr{J}$-Korovkin set* for an operator T in $\mathscr{J}$ if for any sequence of operators $\{T_n\}$ in $\mathscr{J}$ the convergence of $T_n f$ to Tf in the uniform norm for all f in X implies the convergence of $T_n f$ to Tf for all f in C. This paper is concerned with $\mathscr{J}$-Korovkin sets when $\mathscr{J}$ consists of either the positive ($f \geqq 0$ implies $Tf \geqq 0$) operators or the contractive ($\|T\| \leqq 1$) operators in $\mathscr{B}(C)$.

The case where T is the identity operator is now classical. See, for instance, Lorentz [3]. In this same paper the extension of the classical theory to the case of arbitrary operators T is mentioned as an open problem. This extension is the subject of the present paper. In case T is the identity operator our results reduce to the classical ones. A number of authors have considered the case where T is a lattice homomorphism between (possibly distinct) vector lattices. The present situation is different since we consider operators with the same domain and range and assume the weaker condition that T either be positive or have norm one. In addition, Cavaretta [1] and Micchelli [4, 5] have considered the case where T is a positive operator, but not necessarily a lattice homomorphism.

Many of the following results have obvious analogues in the case where the operators are assumed to be both positive and contractive.

1. General theory. Korovkin-type theorems are usually stated for either sequences of positive operators or sequences of contractive operators. Some results about Korovkin sets can be shown in a more general setting. This observation has also been made by Micchelli [5].

For a bounded linear operator T, let T^* denote the adjoint of T. For a point q in Q, let $\hat{q}$ denote the functional in C^* given by evaluation at the point q. Let $\mathscr{L}$ be a subset of C^* and $\mathscr{J}$ be the set of all bounded linear operators T on C such that $T^*\hat{q}$ is in $\mathscr{L}$ for all q in Q. For example, if $\mathscr{L}$ were the positive functionals on C, then $\mathscr{J}$ would be the positive operators.

Let λ be a functional in $\mathscr{L}$ and X be a subspace of C. We say X is an *$\mathscr{L}$-Korovkin set* for λ if for any sequence $\{\lambda_n\}$ in $\mathscr{L}$ the fact that $\lambda_n(f) \to \lambda(f)$ for all f in X implies $\lambda_n(f) \to \lambda(f)$ for all f in C. We say X is an *$\mathscr{L}$-determining set* for λ if for any μ in $\mathscr{L}$ the equality $\mu(f) = \lambda(f)$ for all f in X implies $\mu = \lambda$. The latter concept was first introduced for operators by Šaškin [8]. These two concepts are equivalent in the following sense.

THEOREM 1.1. *Let $\mathscr{L}$ be a weak* closed subset of C^* and let λ be a functional in $\mathscr{L}$. A subset X of C has the property that for any norm bounded sequence $\{\lambda_n\}$ in $\mathscr{L}$ the fact that $\lambda_n(f) \to \lambda(f)$ for all f in X implies $\lambda_n(f) \to \lambda(f)$ for all f in C* (X is a Korovkin set for norm bounded sequences in $\mathscr{L}$ converging to λ) *if and only if X is an $\mathscr{L}$-determining set for λ.*

Proof. Suppose X is a Korovkin set for norm bounded sequences in $\mathscr{L}$ converging to λ. Let μ be any functional in $\mathscr{L}$ such that $\mu(f) = \lambda(f)$ for all f in X. Define $\lambda_n = \mu$ for $n = 1, 2, \cdots$. Then $\{\lambda_n\}$ is a norm bounded sequence in $\mathscr{L}$ such that $\lambda_n(f) \to \lambda(f)$ for all f in X. By assumption $\lambda_n(f) \to \lambda(f)$ for all f in C. Therefore $\mu(f) = \lambda(f)$ for all f in C. Hence, X is an $\mathscr{L}$-determining set for λ.

On the other hand, suppose X is an $\mathscr{L}$-determining set for λ. Let $\{\lambda_n\}$ be a norm bounded sequence in $\mathscr{L}$ such that $\lambda_n(f) \to \lambda(f)$ for all f in X. By the Banach-Alaoglu theorem there exists a weak* limit point μ of $\{\lambda_n\}$. Since $\mu(f) = \lambda(f)$ for all f in X and X is an $\mathscr{L}$-determining set for λ, we have $\mu = \lambda$. The sequence $\{\lambda_n\}$ is contained in a compact set and has the unique limit point λ, hence the sequence converges to λ. Thus $\lambda_n(f) \to \lambda(f)$ for all f in C and the theorem is proved.

Another fundamental result is the relation between $\mathscr{J}$- and $\mathscr{L}$-Korovkin sets.

THEOREM 1.2. *For a convex subset $\mathscr{L}$ of C^* we define $\mathscr{J}$ as above. Let X be a subspace of C and T an operator in $\mathscr{J}$. Then X is a $\mathscr{J}$-Korovkin set for T if and only if X is an $\mathscr{L}$-Korovkin set for $T^*\hat{q}$ where q is any point in Q.*

Proof. Suppose X is a $\mathscr{J}$-Korovkin set for T. Let q be a point in Q. Suppose there is a sequence $\{\lambda_n\}$ in $\mathscr{L}$ such that $\lambda_n(f) \to (T^*\hat{q})(f)$ for f in X. We show $\lambda_n(f) \to (T^*\hat{q})(f)$ for all f in C by constructing a sequence of operators in $\mathscr{J}$. Let the sequence $\{U_n\}$

be a decreasing neighborhood base at the point q. This is possible since Q is first countable. By Urysohn's lemma there exists, for each $n \geqq 1$, a continuous function $h_n: Q \rightarrow [0, 1]$ such that $h_n(q) = 1$ and $h_n(p) = 0$ for p not in U_n. Define the operator T_n in $\mathscr{B}(C)$ by

$$(T_n f)(p) = h_n(p)\lambda_n(f) + (1 - h_n(p))(Tf)(p) .$$

It follows from the definition of $\mathscr{J}$ and the convexity of $\mathscr{L}$ that T_n is in $\mathscr{J}$ for all $n \geqq 1$. Fix $\varepsilon > 0$ and f in X. Since Tf is a continuous function, there exists a set, say U_m, in the neighborhood base of the point q such that

$$|(Tf)(q) - (Tf)(p)| < \varepsilon/2$$

for all p in U_m. Also $h_n(p) = 0$ for any p not in U_m and any $n \geqq m$. Hence, for any p in Q and any $n \geqq m$ we have

$$h_n(p)|(Tf)(p) - (Tf)(q)| < \varepsilon/2 .$$

By assumption there exists $j \geqq 1$ such that for all $n \geqq j$

$$|\lambda_n(f) - (Tf)(q)| < \varepsilon/2 .$$

Using the definition of T_n and letting k be the maximum of j and m, we have for all $n \geqq k$

$$\begin{aligned} |(T_n f)(p) - (Tf)(p)| &= |h_n(p)\lambda_n(f) - h_n(p)(Tf)(p)| \\ &= h_n(p)|\lambda_n(f) - (Tf)(p)| \\ &\leqq h_n(p)(|\lambda_n(f) - (Tf)(q)| + |(Tf)(q) - (Tf)(p)|) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon . \end{aligned}$$

Since X is a $\mathscr{J}$-Korovkin set for T, $T_n f \rightarrow Tf$ for all f in C. In particular,

$$\lambda_n(f) = (T_n f)(q) \rightarrow (Tf)(q) = (T^*\hat{q})(f) .$$

This shows that X is an $\mathscr{L}$-Korovkin set for $T^*\hat{q}$ for all q in Q.

On the other hand, suppose X is an $\mathscr{L}$-Korovkin set for $T^*\hat{q}$ for all q in Q. Let $\{T_n\}$ be a sequence of operators in $\mathscr{J}$ such that $T_n f \rightarrow Tf$ for all f in X. As in Šaškin [7] we use the well-known result that a sequence $\{f_n\}$ in C converges uniformly to f in C if and only if for any sequence $\{q_n\}$ converging to a point q in Q it follows that $f_n(q_n) \rightarrow f(q)$. Let $\{q_n\}$ be a sequence in Q converging to a point q in Q. For all f in X, $(T_n f)(q_n) \rightarrow (Tf)(q)$ or $(T_n^*\hat{q}_n)(f) \rightarrow (T^*\hat{q})(f)$. By assumption this implies $(T_n^*\hat{q}_n)(f) \rightarrow (T^*\hat{q})(f)$ for all f in C. By the same result we now have $T_n f \rightarrow Tf$ for all f in C. Hence, X is a $\mathscr{J}$-Korovkin set for T and the theorem is proved.

2. Positive operators. In this section we give two characterizations of Korovkin sets for positive operators.

Let $\mathscr{T}_+$ denote the positive bounded linear operators on C and let $\mathscr{L}_+$ denote the positive bounded linear functionals on C. For a subspace X of C and a functional μ in C^*, let $\mu|_X$ be the functional in X^* obtained by restricting μ to X. We define $M = \{\hat{p}|_X : p \in Q\}$.

In Micchelli [5] and Grossman [2], the authors assume that a $\mathscr{T}_+$-Korovkin set contains a strictly positive function. The following theorem (compare to Šaškin [7, Lemma 1]) shows this hypothesis to be unnecessary.

THEOREM 2.1. *If a subspace X of C is an $\mathscr{L}_+$-Korovkin set for a functional μ in $\mathscr{L}_+$, then X contains a strictly positive function.*

Proof. Suppose X is an $\mathscr{L}_+$-Korovkin set for a functional μ in $\mathscr{L}_+$. Let $\overline{\text{co}}(M)$ be the weak* closure of the convex hull of M. We claim $\overline{\text{co}}(M)$ does not contain $0|_X$, the zero functional restricted to X. We assume the claim is false and arrive at a contradiction. Let $\{\mu_\alpha\}$ be a net in co(M), the convex hull of M, such that $\mu_\alpha \to 0|_X$ in the weak* topology of X^*. For each α there exists a positive integer $n\beta_i \geqq 0$ and q_i in Q, $1 \leqq i \leqq n$, where $\sum_{i=1}^n \beta_i = 1$, such that

$$\mu_\alpha = \sum_{i=1}^n \beta_i \hat{q}_i|_X \,.$$

We define the natural extension $\nu_\alpha = \sum_{i=1}^n \beta_i \hat{q}_i$ in C^*. Note that ν_α is a positive functional where $\nu_\alpha(1) = 1$ and therefore $\|\nu_\alpha\| = 1$. By the Banach-Alaoglu theorem there exists a weak* limit point ν of $\{\nu_\alpha\}$. Clearly, ν is a positive functional in C^* with $\nu|_X = 0|_X$, but $\nu \neq 0$ since $\nu(1) = 1$. Then for the sequence $\{\lambda_n\}$ in $\mathscr{L}^+$ given by $\lambda_n = \nu + \mu$, we have $\lambda_n(f) \to \mu(f)$ for all f in X. However, $\lambda_n(1) = \nu(1) + \mu(1) = 1 + \mu(1)$ does not converge to $\mu(1)$. This contradicts the hypothesis of the theorem. So the claim is true. By a standard separation theorem there exists a weak closed hyperplane H separating $0|_X$ and $\overline{\text{co}}(M)$, i.e., there exists g in X and $\beta > 0$ such that $g(p) > \beta$ for all p in Q. The theorem is proved.

THEOREM 2.2. *A subspace X of C is a $\mathscr{T}_+$-Korovkin set for an operator T in $\mathscr{T}_+$ if and only if X is an $\mathscr{L}_+$-determining set for $T^*\hat{q}$ for all q in Q.*

Proof. Suppose X is a $\mathscr{T}_+$-Korovkin set for T. By Theorem 1.2, X is an $\mathscr{L}_+$-Korovkin set for $T^*\hat{q}$ for all q in Q. Therefore, as in Theorem 1.1, X is also an $\mathscr{L}_+$-determining set for $T^*\hat{q}$ for all

q in Q.

Conversely, suppose X is an $\mathscr{L}_+$-determining set for $T^*\hat{q}$ for all q in Q. By Theorem 2.1, there exists g in X such that $g(q) \geqq 1$ for all q in Q. Fix a point q in Q. We claim X is an $\mathscr{L}_+$-Korovkin set for $T^*\hat{q}$. Let $\{\lambda_n\}$ be a sequence in $\mathscr{L}_+$ such that $\lambda_n(f) \to (T^*\hat{q})(f)$ for all f in X. In particular, $\lambda_n(g) \to (T^*\hat{q})(g) = (Tg)(q)$. Since λ_n is a positive functional for each $n \geqq 1$

$$||\lambda_n|| = \lambda_n(1) \leqq \lambda_n(g) \to (Tg)(q) .$$

Therefore, the sequence $\{\lambda_n\}$ is norm bounded. By Theorem 1.1, the claim is true. The result now follows from Theorem 1.2. The theorem is proved.

If the subspace X is finite dimensional, then a more geometric condition is possible. When T is the identity operator, Corollary 2.3 is essentially Lorentz's Theorem 4 [3].

COROLLARY 2.3. *An m-dimensional subspace X is a $\mathscr{J}_+$-Korovkin set for an operator T in $\mathscr{J}_+$ if and only if X satisfies the condition that $0|_X$ is not in* co(M) *and for any subset $\{q, q_1, q_2, \cdots, q_{m+1}\}$ in Q and $\beta_i \geqq 0$ for $1 \leqq i \leqq m+1$ the equality $\sum_{i=1}^{m+1}\beta_i\hat{q}_i|_X = (T^*\hat{q})|_X$ implies $\sum_{i=1}^{m+1}\beta_i\hat{q}_i = T^*\hat{q}$.*

Proof. The necessity of the condition follows directly from Theorem 2.2.

We now show the condition is sufficient. Suppose X satisfies the condition of the theorem, but X is not a $\mathscr{J}_+$-Korovkin set for T. By Theorem 2.2 there exists a positive functional μ in C^* such that for some q in Q we have $\mu(f) = (T^*\hat{q})(f)$ for all f in X, but for some g in C we have $\mu(g) \neq (T^*\hat{q})(g)$. Let Y be the subspace of C spanned by X and g. Let M_1 be the set of point evaluations in Y^*. The functional $0|_Y$ is not in co(M_1), since $0|_X$ is not in co(M). By a known theorem there exists $\alpha_i \geqq 0$ and q_i in Q for $1 \leqq i \leqq m+1$ such that

$$\mu(f) = \sum_{i=1}^{m+1} \alpha_i\hat{q}_i(f)$$

for all f in Y. Thus,

$$\sum_{i=1}^{m+1} \alpha_i\hat{q}_i(f) = (T^*\hat{q})(f)$$

for all f in X, but not for $f = g$. This contradicts our assumption. Hence, X must be a $\mathscr{J}_+$-Korovkin set for T. The corollary is proved.

If a positive operator T has an m-dimensional $\mathscr{J}_+$-Korovkin set, then Micchelli [5] has shown that T must be *finitely defined of order* m. This means that for every q in Q there exists $\alpha_i \geqq 0$ and q_i in Q for $1 \leqq i \leqq m$ such that $T^*\hat{q} = \sum_{i=1}^{m}\alpha_i q_i$. We shall show a similar result for contractive operators in the next section.

3. **Contractive operators.** In this section we prove two characterizations of contractive operator Korovkin sets. We also establish two properties of any operator T having a contractive operator Korovkin set. Finally, we give conditions under which positive operator Korovkin sets are equivalent to contractive operator Korovkin sets. These results are all stated for an operator T of norm one. If T has norm $c > 0$, it is easy to verify the corresponding theorems for sequences of operators of norm at most c.

Let $\mathscr{T}^1$ denote the bounded linear operators on C of norm at most one and let $\mathscr{L}^1$ denote the bounded linear functionals on C of norm at most one. The following result can be compared with Šaškin [9].

THEOREM 3.1. *Let T be a norm one bounded linear operator. A subspace X of C is a $\mathscr{T}^1$-Korovkin set for T if and only if X is an $\mathscr{L}^1$-determining set for $T^*\hat{q}$ for all q in Q.*

Proof. The proof follows directly from Theorems 1.1 and 1.2.

The following is the analogue of Šaškin's Theorem 2 [9]. This condition seems to be necessarily more complicated than Šaškin's condition since for q in Q the functional $T^*\hat{q}$ is not necessarily a point evaluation.

COROLLARY 3.2. *Let T be an operator in $\mathscr{B}(C)$ of norm one. An m-dimensional subspace X of C is a $\mathscr{T}^1$-Korovkin set for T if and only if X satisfies the condition that, for any subset $\{q, q_1, \cdots, q_{m+2}\}$ in Q and for any functional $\mu = \sum_{i=1}^{m+2}\beta_i\hat{q}_i$ where β_i is in $\boldsymbol{R}$ and $\sum_{i=1}^{m+2}|\beta_i| = 1$, the equality $\mu|_X = (T^*\hat{q})|_X$ implies $\mu = T^*\hat{q}$.*

The proof of this corollary is similar to the proof of Corollary 2.3.

If an operator T has a $\mathscr{T}^1$-Korovkin set X, then $T^*\hat{q}$ for each q in Q must have certain properties. First, we note that T must be a finitely defined operator if X is finite dimensional.

COROLLARY 3.3. *If the m-dimensional subspace X of C is a $\mathscr{T}^1$-Korovkin set for the norm one operator T in $\mathscr{B}(C)$, then T is finitely defined of order $m + 1$.*

Proof. Suppose T and X satisfy the hypotheses of the theorem.

For each q in Q there exist $q_1, q_2, \cdots, q_{m+1}$ in Q and α_i in $\boldsymbol{R}$ where $\sum_{i=1}^{m+1}|\alpha_i| = 1$ such that for any f in X

$$(T^*\hat{q})(f) = \sum_{i=1}^{m+1} \alpha_i \hat{q}_i(f) .$$

Since X is a Korovkin set, by Theorem 3.1, the above equality holds for all f in C. Therefore T is finitely defined of order $m+1$. The corollary is proved.

In the following Corollary we have another condition on $(T^*\hat{q})|_X$.

COROLLARY 3.4. *Let T be a norm one operator in $\mathscr{B}(X)$. Let X be a closed proper subspace of C. If X is a $\mathscr{J}^1$-Korovkin set for T, then for all q in Q we have*

$$\| T^*\hat{q}|_X \|_{X^*} = \| T^*\hat{q} \| = 1$$

where

$$\|(T^*\hat{q})|_X\|_{X^*} = \sup \{|(T^*\hat{q})(f)| : f \in X, \|f\| \leqq 1\} .$$

Proof. Let $\alpha = \| T^*\hat{q}|_X\|_{X^*}$. Then $\alpha \leqq \| T^*\hat{q} \| \leqq 1$. Suppose $\alpha < 1$. Then by the Hahn-Banach theorem $(T^*\hat{q})|_X$ has an extension to C of norm α. By a modification (see Rusk [6]) of the proof of the Hahn-Banach theorem one shows that $(T^*\hat{q})|_X$ has an extension to C of norm β where $\alpha < \beta \leqq 1$. This contradicts Theorem 3.1. Therefore $\alpha = 1$, which proves the corollary.

A natural question arises about subspaces which are contractive Korovkin sets for any finitely defined operator of a given order. Cavaretta [1] has given such sets for the positive operators. The next corollary shows that there are no such finite dimensional subspaces for contractive operators.

COROLLARY 3.5. *Let Q also be nondiscrete. Then C has no finite dimensional $\mathscr{J}^1$-Korovkin set for all finitely defined operators T of order 2 such that $\| T^*\hat{q} \| = 1$ for all q in Q.*

Proof. Suppose the theorem is false. Then there exists a finite dimensional subspace X of C which is such a set. For any distinct points q_1 and q_2 in Q and β_1 and β_2 in $\boldsymbol{R}$ such that $|\beta_1| + |\beta_2| = 1$ there is some operator T as above such that $T^*\hat{q} = \beta_1\hat{q}_1 + \beta_2\hat{q}_2$. By Corollary 3.4, $\|(T^*\hat{q})|_X\|_{X^*} = 1$. Since the closed unit ball of X is compact, there exists f in X such that $(T^*\hat{q})(f) = 1 = \beta_1 f(q_1) + \beta_2 f(q_2)$ and $\|f\| \leqq 1$. This can happen only if $f(q_1) = \operatorname{sgn} \beta_1$ and $f(q_2) = \operatorname{sgn} \beta_2$, where $\operatorname{sgn} x = x/|x|$ if $x \neq 0$. Since Q is not discrete, there exists a sequence $\{q_n\}$ in Q such that $q_n \rightarrow q$ in Q, and such that $q_n \neq q$ for $n \geqq 1$. Then according to the previous argument there exist f_n in C of norm one such that $f_n(q_n) = 1$, but $f_n(q) = -1$ for $n \geqq 1$. Again since the closed unit ball of X is compact, there exists a subsequence $\{n_j\}$ such that $f_{n_j} \to f$ uniformly as $j \to \infty$ for some f in X. From the uniform convergence we have $1 = f_{n_j}(q_{n_j}) \to f(q)$ as $j \to \infty$. On the other hand $-1 = f_{n_j}(q) \to f(q)$ as $j \to \infty$. This contradiction implies the corollary is true.

Suppose X is a finite dimensional subspace of C containing the constants. Lorentz [3, Theorem 8] has shown that X is a $\mathscr{J}_+$-Korovkin set for the identity operator if and only if X is a $\mathscr{J}^1$-Korovkin set for the identity. We extend this result in the next theorem.

THEOREM 3.6. *Let T be a norm one positive operator in $\mathscr{B}(C)$ and let X be a subspace of C containing the constants. The subspace X is a $\mathscr{J}_+$-Korovkin set for T and $T1 = 1$ if and only if X is a $\mathscr{J}^1$-Korovkin set for T.*

Proof. Suppose X is a $\mathscr{J}_+$-Korovkin set for T and $T1 = 1$. Suppose for a point q in Q and a functional μ in $\mathscr{L}^1$ that $\mu|_X = (T^*\hat{q})|_X$. From

$$\mu(1) = (T1)(q) = 1 = \|\mu\|$$

it follows that μ is a positive functional. By Theorem 2.2 we have $\mu = T^*\hat{q}$. Therefore, by Theorem 3.1, X is a $\mathscr{J}^1$-Korovkin set for T.

Conversely, suppose X is a $\mathscr{J}^1$-Korovkin set for T. By Corollary 3.4 and since T is a positive operator $1 = \|T^*\hat{p}\| = (T1)(p)$ for all p in Q, i.e., $T1 = 1$. Suppose for a point q in Q and a functional μ in $\mathscr{L}_+$ that $\mu|_X = (T^*\hat{q})|_X$. Then

$$\|\mu\| = \mu(1) = (T1)(q) = 1 .$$

By Theorem 3.1 we have $\mu = T^*\hat{q}$. Therefore by Theorem 2.2, X is a $\mathscr{J}_+$-Korovkin set for T. The theorem is proved.

The hypothesis that $T1 = 1$ cannot be omitted in Theorem 3.6. Consider the positive norm one operator T in $\mathscr{B}(C)$ where $Q = [0, 1]$ defined by

$$(Tf)(q) = (1 + q)f(q)/3 , \quad q \in [0, 1] .$$

If X is spanned by $\{1, x, x^2\}$, then X is a $\mathscr{J}_+$-Korovkin set for T (see Cavaretta [1, Theorem 2]). However, since $\|T^*\hat{q}\| = 1/3$ when $q = 0$, by Corollary 3.4, X is not a $\mathscr{J}^1$-Korovkin set for T.

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