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WEAK HOMOMORPHISMS AND INVARIANTS: AN EXAMPLE

ANDREW ADLER

Notions of weak isomorphism, weak epimorphism, and weak embedding are defined. For countable algebras, these specialize to the ordinary notions. Certain invariants for superatomic Boolean algebras are described. It is shown that the existence or non-existence of weak isomorphisms, weak epimorphisms, and weak embeddings between two such algebras A and B can be decided from the invariants of A and B.

I. Introduction. In [4], Day described certain invariants for superatomic Boolean algebras that refine invariants first introduced by Mazurkiewicz and Sierpinski [6]. Day showed using topological methods that any two countable superatomic Boolean algebras with the same invariants are isomorphic. In [3], Cramer described a partial order $\leqq$ on the Day invariants. He showed, again using topological methods, that the countable algebra A is embeddable in B if and only if the Day invariant of A is $\leqq$ the Day invariant of B, and that the countable algebra B is a homomorphic image of the countable algebra A if and only if the invariant of A is $\geqq$ the invariant of B. Day and Cramer give examples that show the countability assumptions cannot be dropped.

In this paper, we describe notions of weak isomorphism, weak embedding and weak epimorphism that have already been used with success in the study of Abelian torsion groups [2]. We then show that for any two superatomic Boolean algebras A and B, A is weakly isomorphic to B iff A and B have the same Mazurkiewicz-Sierpinski invariant, A is weakly embeddable in B iff the invariant of A is $\leqq$ the invariant of B, and B is a weak image of A iff the invariant of A is $\geqq$ the invariant of B. From these results it is in particular easy to derive the results of Day and Cramer mentioned above.

The motivation for looking at the subject came from infinitary logic, and our first proof of the main result used a certain amount of machinery from that subject. The proof we present here, however, uses only a little elementary algebra. There is a good deal of evidence (see Barwise [1]) that the notion of weak isomorphism is algebraically more natural and better behaved that the notion of isomorphism. Our main result will add a little to that evidence.

II. Weak homomorphisms. Let A, B be algebraic structures of the same type (so A and B are both groups, or both ordered fields, or both R-modules, $\cdots$).

DEFINITION. A weak homomorphism from A into B is a nonempty collection Φ of maps such that: (i) For any $\phi \in \Phi$, the domain of ϕ is a substructure of A, the range of $\operatorname{rng}(\phi)$ is a substructure of B, and ϕ is a homomorphism from $\operatorname{dom}(\phi)$. (ii) (The extendability property.) For any $a \in A$ and any $\phi \in \Phi$, there exists $\phi' \in \Phi$ such that ϕ' is an extension of ϕ and $a \in \operatorname{dom}(\phi')$. If in addition for every $b \in B$ and $\phi \in \Phi$ there $\phi' \in \Phi$ such that is ϕ' extends ϕ and $b \in \operatorname{rng}(\phi')$, Φ is a weak epimorphism. If every ϕ in the weak epimorphism Φ is one-to-one, Φ is a weak embedding. If Φ is at once a weak epimorphism and a weak embedding, Φ is a weak isomorphism.

The notion of weak isomorphism goes back to Karp [5]. Notions very close to our notion of weak epimorphism and weak embedding have been used in [2]. At the cost of complicating the definition somewhat, we could make weak homomorphisms into genuine morphisms in the sense of category theory. Since this would not change the mathematical content of the main result, we will stay with the simple definitions given above.

The following result in principle goes back to Cantor. Part (ii) is done by a simple "back and forth" or "zipper" argument. Part (i) is even simpler.

LEMMA 1. (i) *If A is countably generated, and A is weakly embeddable in B, then A is embeddable in B.* (ii) *If A and B are countably generated, and A is weakly isomorphic to B* (respectively: is a weak homomorphic image of B) *then A and B are isomorphic* (respectively: A is a homomorphic image of B).

III. Superatomic Boolean algebras—the invariants. Let B be a superatomic Boolean algebra, that is, a Boolean algebra B such that every homormophic image of B is atomic. Define a sequence $I_0, I_1, \cdots$ of ideals of B by the following rules:

(i) $I_0 = (0)$. (ii) If $\beta = \alpha + 1$, let I be the ideal of B/I_α generated by the atoms of B/I_α. Let I_β be the set of preimages in B of elements of I. (iii) If β is a limit ordinal, let $I_\beta = \bigcup_{\alpha<\beta} I_\alpha$.

Because B is superatomic, there is a first ordinal α such that $I_\alpha = B$ (see Day [4]). α is necessarily a successor ordinal. Let $\rho = \rho(B)$ be the greatest ordinal such that $I_\rho \neq B$. Then B/I_ρ has a finite number $n(B)$ of atoms.

For any nonzero $b \in B$, let $\rho(b)$ (the rank of b) be the greatest ordinal such that $b \notin I_\rho$. So $\rho(B) = \rho(1)$, where 1 is the unit of the algebra B. If $b \neq 0$, b is the preimage in B of an object which is a finite join of atoms of B/I_ρ. Let $n(b)$ be the number of atoms used in this representation. So $n(b)$ is a positive integer, and $n(B) = u(1)$. Let $s(b)$ be the ordered pair $(\rho(b), n(b))$. For completeness, let $s(0) = (0, 0)$. Let $s(B) = s(1)$. Let $\leqq$ be the lexicographic order on pairs of the form (ρ, n). The following is an easy consequence of the definitions:

LEMMA 2. (i) $\rho(a \vee b) = \max(\rho(a), \rho(b))$. (ii) *If* $\rho(a) = \rho(b)$ *and* $\rho(a \wedge b) < \rho(a)$, *then* $n(a \vee b) = n(a) + n(b)$. *If* $\rho(a) < \rho(b)$ *then* $n(a \vee b) = n(b)$.

So in particular, if B_f is a finite subalgebra of B, then $s(b)$ can be easily computed for every $b \in B_f$ if we know $s(a)$ for every atom a of B_f. In what follows, we will make use also of the following observation:

LEMMA 3. *Let* $b \in B$, *and suppose* $(\rho, n) \leqq s(b)$. *Then there exists* $a \subseteq b$ *such that* $s(a) = (\rho, n)$.

LEMMA 4. *Let* A, B *be superatomic Boolean algebras. Let* $u \in A$, $v \in B$, *with* $s(u) \leqq s(v)$. (i) *Let* $u = u_1 \vee u_2$, *with* $u_1 \wedge u_2 = 0$. *Then there exist* $v_1, v_2 \in B$, *with* $v = v_1 \vee v_2$, $v_1 \wedge v_2 = 0$, *and* $s(u_1) \leqq s(v_1)$, $s(u_2) \leqq s(v_2)$. *If* $s(u) = s(v)$, *then* v_1, v_2 *may be chosen so that* $s(u_1) = s(v_1)$, $s(u_2) = s(v_2)$. (ii) *Let* $v = v_1 \vee v_2$, *with* $v_1 \wedge v_2 = 0$. *Then there exist* $u_1, u_2 \in A$, *with* $u = u_1 \vee u_2$, $u_1 \wedge u_2 = 0$, *and* $s(u_1) \leqq s(v_1)$, $s(u_2) \leqq s(v_2)$.

Proof. We prove (i). Suppose $s(u_1) \leqq s(u_2)$. Choose $v_1 \subseteq v$ so that $s(u_1) = s(v_1)$. Let $v_2 = v - v_1$. By Lemma 2, $s(u_2) \leqq s(v_2)$, and if $s(u) = s(v)$, $s(u_2) = s(v_2)$.

IV. Weak homomorphisms and the invariants. In this section we show that the existence of weak isomorphisms (weak epimorphisms, weak embeddings) between superatomic algebras A and B depends only on the invariants of A and B.

LEMMA 5. *Let* Φ *be a weak homomorphism of* A *into* B. *Let* $\phi \in \Phi$, *and let* $a \in \operatorname{dom}(\phi)$. *If* Φ *is a weak embedding*, $s(a) \leqq s(\phi a)$. *If* Φ *is a weak epimorphism*, $s(a) \geqq s(\phi a)$. *In particular, if* Φ *is a weak isomorphism*, $s(a) = s(\phi a)$.

Proof. We deal with the case that Φ is a weak embedding. For weak epimorphisms essentially the same argument will do. We proceed by induction on the well-ordering $\leqq$. Now $s(a) = (0, 1)$ iff

a is an atom of A. But then $\phi a \neq 0$, so $s(a) \leqq s(\phi a)$. Suppose now $\rho(a) = \rho$ and $n(a) > 1$. Then by Lemma 3 one can find disjoint elements $c, d \in A$ such that $a = c \vee d$, $\rho(c) = \rho(d) = \rho$, $n(c) = n(a) - 1$, $n(d) = 1$. Let $\phi' \in \Phi$ be an extension of ϕ that has c, d in its domain. By induction hypothesis, $s(c) \leqq s(\phi' c)$, $s(d) \leqq s(\phi' d)$, and so easily $s(a) \leqq s(\phi' a) = s(\phi a)$. If $n(a) = 1$, then for every $(\rho, n) < s(a)$ there exists $c \leqq a$ such that $s(c) = (\rho, n)$. Let $\phi' \in \Phi$ extend ϕ to c. Then $(\rho, n) \leqq s(\phi' c) \leqq s(\phi' a) = s(\phi a)$. So for any $(\rho, n) < s(a)$, $(\rho, n) < s(\phi a)$. Hence $s(a) \leqq s(\phi a)$.

THEOREM. *Let A, B be superatomic Boolean algebras. Then* (i) *A is a weakly embeddable in B iff $s(A) \leqq s(B)$.* (ii) *B is a weak homomorphic image of A iff $s(A) \geqq s(B)$.* (iii) *A and B are weakly isomorphic iff $s(A) = s(B)$.*

Proof. In one direction, everything is settled by Lemma 5. We prove now that if $s(A) = s(B)$, A and B are weakly isomorphic. The arguments for (i) and (ii) are essentially the same as those for (iii).

So suppose $s(A) = s(B)$. Let Φ be the set of all maps ϕ such that:

(a) dom (ϕ) is a finite subalgebra of A, rng (ϕ) is a finite subalgebra of B, and ϕ is an isomorphism of dom (ϕ) and rng (ϕ).

(b) For any $a \in \operatorname{dom}(\phi)$, $s(a) = s(\phi a)$.

We prove that Φ is a weak isomorphism from A to B. Φ is nonempty for since $s(A) = s(B)$, the map that sends 0 to 0 and 1 to 1 belongs to ϕ. Let $\phi \in \Phi$ and let $a \in A$. We wish to find $\phi' \in \Phi$ such that ϕ' extends ϕ and $a \in \operatorname{dom}(\phi')$. Let $A_0 = \operatorname{dom}(\phi)$, $B_0 = \operatorname{rng}(\phi)$. Using Lemma 4, choose v_j^1, v_j^2 so that $v_j = v_j^1 \vee u_j^2$, $u_j^1 \wedge v_j^2 = 0$, and $s(u_j^i) = s(v_j^i)$. Let B_1 be the subalgebra of B generated by the v_j^i. Let $u_1, \cdots, u_k$ be the atoms of A_0, and let $v_j = \phi u_j$. Let $u_j^1 = u_j \wedge a$, $u_j^2 = u_j \wedge a'$. Let A_1 be the subalgebra of A generated by the u_j^i. The map that sends u_j^i into v_j^i extends to an isomorphism ϕ' of A_1 to B_1 which extends ϕ. Lemma 2 easily yields that $\phi' \in \Phi$. In exactly the same way, if $b \in B$ one can find an extension ϕ' of ϕ such that $b \in \operatorname{rng}(\phi')$.

COROLLARY. (i) (Day [4]). *If A and B are countable and $s(A) = s(B)$, A and B are isomorphic.* (ii) (Cramer [3]). *If A is countable and $s(A) \leqq s(B)$, A is embeddable in B. If A and B are countable and $s(A) \leqq s(B)$, A is a homomorphic image of B.*

Proof. The result follows immediately from the previous theorem and Lemma 1.

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UNIVERSITY OF BRITISH COLUMBIA