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ON CAMERON AND STORVICK'S OPERATOR VALUED FUNCTION SPACE INTEGRAL

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In this paper "the probability density of path space" is introduced by the formula

$$p_\lambda^\alpha(t,u)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-iu\eta-\frac{t}{\lambda}|\eta|^\alpha)d\eta,\qquad (\alpha>0).$$

If $\alpha=2$ and $\lambda>0$, $p_\lambda^\alpha(t,u)$ is the normal probability density. But if $\alpha>2$ this density can not be considered as a probability density. By this generalization, one can generalize the operator valued function space integral based on the Wiener integral.

Introduction. Cameron and Storvick introduced the operator valued function space integral in **[1]**. Johnson and Skoug **[9]** developed Cameron and Storvick's theory and improved the results obtained in **[1]**. To make the arguments in the following sections comprehensible, we will quote the operators $I_\lambda^\sigma(F)$ and $I_\lambda^{\text{sec}}(F)$ from **[1]**, which played an important role in **[1]**, **[9]**. Let $B[a,b]$ denote the space of real valued functions on an interval $[a,b]$ which are continuous except for a finite number of finite jump discontinuities. Let $F(x)$ be a functional on $B[a,b]$ and $\psi\in L_2(-\infty,\infty)$, $\xi\in(-\infty,\infty)$. Then for $\text{Re}\,\lambda>0$ and any partition σ: $a=t_0<t_1<\cdots<t_n=b$, the operator $I_\lambda^\sigma(F)$ is defined by the formula

$$(I_\lambda^\sigma(F)\psi)(\xi)=\lambda^{n/2}[(2\pi)^n(t_1-a)\cdots(t_n-t_{n-1})]^{-1/2}\int_{-\infty}^{\infty}(n)\int_{-\infty}^{\infty}\psi(v_n)$$

$$\cdot f_\sigma(\xi,v_1,\cdots,v_n)\exp\left(-\sum_{j=1}^{n}\frac{\lambda(v_j-v_{j-1})^2}{2(t_j-t_{j-1})}\right)dv_1\cdots dv_n \tag{0.1}$$

where $v_0=\xi$, $f_\sigma(\xi,v_1,\cdots,v_n)=F[z(\sigma,\xi,v_1,\cdots,v_n,\cdot)]$,

$$z(\sigma,\xi,v_1,\cdots,v_n,t)=\begin{cases}v_j & \text{if}\quad t_j\leqq t<t_{j+1},\qquad j=0,1,\cdots,n-1,\\ v_n & \text{if}\quad t=b,\end{cases}$$

and where if n is odd we always choose $\lambda^{n/2}$ with nonnegative real part. Here $\int(n)\int$ means the n-fold integral. If $\lambda>0$, by using the Wiener integral, this can be written as

$$(I_\lambda^\sigma(F)\psi)(\xi)=\int_{C_0[a,b]} F(\lambda^{-1/2}x_\sigma+\xi)\psi(\lambda^{-1/2}x(b)+\xi)dx$$

where

$$x_\sigma(t)=\begin{cases} x(t_{j-1}) & \text{if} \quad t_{j-1}\leqq t<t_j, \\ x(b) & \text{if} \quad t=b. \end{cases}$$

As an example important in quantum theory, the functional

$$F(x)=\exp\left\{\int_a^b \theta(s,x(s))ds\right\}$$

is discussed in **[1]**, **[9]**. In this case $f_\sigma(\xi,v_1,\cdots,v_n)$ is given by

$$\exp\left\{\sum_{j=1}^n \int_{t_{j-1}}^{t_j} \theta(s,v_{j-1})ds\right\}.$$

Let us denote $\max\{t_j-t_{j-1}\}$ by norm σ or $\|\sigma\|$. For $\operatorname{Re}\lambda>0$ the operator $I_\lambda^{\text{sec}}(F)$ is defined by

$$(0.2)\qquad I_\lambda^{\text{sec}}(F)=w\lim_{\|\sigma\|\to 0} I_\lambda^\sigma(F),$$

where w lim refers to the limit with respect to the weak operator topology. **[1]** proved that for a class of functionals F, $I_\lambda^{\text{sec}}(F)$ exists by using the Wiener integral, and furthermore that $I_\lambda^{\text{sec}}(F)$ converges in the weak operator topology as $\lambda=p-iq\to+0-iq$ for almost all $q\neq 0$. **[9]** proved that for a class of functionals F, $I_\lambda^{\text{sec}}(F)$ exists as the strong operator limit, and furthermore that $I_\lambda^{\text{sec}}(F)$ converges in the strong operator topology as $\lambda=p-iq\to+0-iq$ for all $q\neq 0$ by using the analytic continuation of the Wiener integral. In this paper we introduce the following operator from L_2 to L_2 in §2 corresponding to (0.1),

$$(\mathscr{F}_\lambda^{\alpha,\sigma}(F)\psi)(\xi)=\int_{-\infty}^{\infty} p_\lambda^\alpha(t_1-a,v_1-\xi)dv_1\int_{-\infty}^{\infty} p_\lambda^\alpha(t_2-t_1,v_2-v_1)dv_2\Big|$$

$$\cdots\int_{-\infty}^{\infty} p_\lambda^\alpha(t_n-t_{n-1},v_n-v_{n-1})\psi(v_n)f_\sigma(v_1,v_2,\cdots,v_n)dv_n,$$

for $\operatorname{Re}\lambda\geqq 0$, $\lambda\neq 0$. We note that if $\operatorname{Re}\lambda>0$, $p_\lambda^1(t,u)$ and $p_\lambda^2(t,u)$ are especially given by

$$p_\lambda^1(t,u)=\frac{1}{\pi}\frac{t\lambda}{t^2+(\lambda u)^2},$$

$$(0.3)\qquad p_\lambda^2(t,u)=\left(\frac{\lambda}{4\pi t}\right)^{1/2}\exp\left(-\frac{\lambda u^2}{4t}\right),$$

respectively.

In general $p_\lambda^\alpha(t,u)$ is given by the formula

$$(0.4)\qquad p_\lambda^\alpha(t,u)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(-iu\eta-\frac{t}{\lambda}|\eta|^\alpha\right)d\eta$$

for $\alpha>0$, $t>0$ and $\operatorname{Re}\lambda>0$. **[1]–[3]** and **[6]–[10]** used only (0.3). Even by this generalization we can also show that for a class of functionals F, $\mathcal{F}_\lambda^{\alpha,\sigma}(F)$ converges in the strong operator topology as norm $\sigma\to 0$. Furthermore we can show that the same integral equation as in **[1]** **[9]** holds.

1. Stable density and semigroup.

The stable density of exponent α is

$$(1.1)\qquad p_1^\alpha(t,u)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-iu\eta-t|\eta|^\alpha)d\eta$$

where $0<\alpha\leqq 2$, $0<t<\infty$. cf. **[12]**.

It is well-known that (1.1) satisfies the Chapman and Kolmogorov equation, that is, when $\lambda=1$

$$(1.2)\qquad p_\lambda^\alpha(t+s,u)=\int_{-\infty}^{\infty}p_\lambda^\alpha(s,u-y)p_\lambda^\alpha(t,y)dy.$$

If we consider $p_\lambda^\alpha(t,u)$ in the operator's sense in the Hilbert space L_2, (1.2) holds for $\operatorname{Re}\lambda>0$. We shall denote the Fourier transform of $f\in L_2$ by Uf and the inverse Fourier transform by U^*f, that is,

$$\text{(a)}\qquad (Uf)(y)=(2\pi)^{-1/2(y)}\int_{-\infty}^{\infty}\exp(-iyx)f(x)dx,$$

$$\text{(b)}\qquad (U^*f)(y)=(2\pi)^{-1/2(y)}\int_{-\infty}^{\infty}\exp(iyx)f(x)dx,$$

where (y) denotes the so-called limit in the mean.

In what follows, let us assume that $\alpha>0$.

Let

$$(1.3)\qquad (P_\lambda^\alpha(t)f)(y) = U^*\left(\exp\left(-\frac{t}{\lambda}|\xi|^\alpha\right)(Uf)(\xi)\right)(y),$$

where $f \in L_2$ and $\lambda \in D \equiv \{\lambda : \operatorname{Re}\lambda \geqq 0, \lambda \neq 0\}$.

THEOREM 1.1. *$\{P_\lambda^\alpha(t): t \geqq 0\}$ is a strongly continuous semigroup of contraction operators on L_2. Furthermore $P_\lambda^\alpha(t)$ is also strongly continuous with respect to λ on D.*

This follows from the fact that the Fourier transformation is a unitary operator on L_2.

LEMMA 1.1. *Let $\operatorname{Re}\lambda > 0$ and $t > 0$. Then*

$p_\lambda^\alpha(t, \cdot)$ is L_2-integrable,
$|p_\lambda^\alpha(t,u)| \leqq M(\alpha, t, \lambda) < \infty$,

and $p_\lambda^\alpha(t,u)$ is continuous in t, λ and u.

Proof. Let $\operatorname{Re}\lambda > 0$. Then

$$|p_\lambda^\alpha(t,u)| \leqq \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\exp\left(-\frac{t}{\lambda}|\eta|^\alpha\right)\right| d\eta.$$

Let $t/\lambda = \delta - i\gamma$, $(\delta > 0)$. We obtain

$$|p_\lambda^\alpha(t,u)| \leqq \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-\delta|\eta|^\alpha)d\eta = \frac{1}{\pi}\int_0^\infty \exp(-\delta\eta^\alpha)d\eta$$
$$= \frac{1}{\pi}\beta\delta^{-\beta}\Gamma(\beta) < \infty,$$

where $\beta = 1/\alpha$ and $\Gamma(\beta)$ is the gamma function. On using the Dominated Convergence Theorem, we can show that $p_\lambda^\alpha(t,u)$ is continuous in t, λ and u. From the above proof, clearly it holds that

$$(1.4)\qquad (2\pi)^{-1/2}\exp\left(-\frac{t}{\lambda}|\eta|^\alpha\right) \in L_p, \qquad (p = 1, 2, \cdots).$$

We may write (0.4) as

$$(1.5)\qquad p_\lambda^\alpha(t,u) = U\left[(2\pi)^{-1/2}\exp\left(-\frac{t}{\lambda}|\cdot|^\alpha\right)\right](u).$$

Since the Fourier transformation is a unitary operator on L_2, $p_\lambda^\alpha(t, u)$ is L_2-integrable in u.

LEMMA 1.2. *Let* $\mathrm{Re}\,\lambda > 0$ *and* $t > 0$. *Then* $p_\lambda^\alpha(t, \cdot)$ *is* L_1-*integrable.*

Proof. Let

$$f(\xi) = (2\pi)^{-1/2} \exp\left(-\frac{t}{\lambda}|\xi|^\alpha\right).$$

The derivative of $f(\xi)$ is

$$f'(\xi) = (2\pi)^{-1/2} \exp\left(-\frac{t}{\lambda}|\xi|^\alpha\right)\left(-\frac{t}{\lambda}|\xi|^{\alpha-1}\right)\alpha \operatorname{sgn} \xi,$$

where

$$\operatorname{sgn} \xi = \begin{cases} -1 & \text{if} \quad \xi < 0, \\ 1 & \text{if} \quad \xi > 0. \end{cases}$$

Then since f is absolutely continuous on each bounded interval and f' is absolutely integrable, it holds that $(Uf')(\xi) = i\xi(Uf)(\xi)$. Therefore in order to show that Uf is L_1-integrable, by [**13**. 12.42. p. 382], it is sufficient to show that $Uf' \in L_q(1, +\infty)$ and $(Uf')(-\cdot) \in L_q(1, +\infty)$ since $1/\xi \in L_p(1, +\infty)$, where $1/p + 1/q = 1$, $p, q > 1$. With respect to f', it holds that

$$\int_{-\infty}^{\infty} |f'(\xi)|^p d\xi = 2(2\pi)^{-p/2}\alpha^{p-1}\left|\frac{t}{\lambda}\right|^p \int_0^\infty \tau^{(\beta-1)(1-p)} \exp(-\delta p\tau)d\tau,$$

where $1/\alpha = \beta$ and $\mathrm{Re}(t/\lambda) = \delta$. Hence f' is L_p-integrable if $(\beta - 1)(1 - p) > -1$ holds. For a fixed α, there always exists a number p in the interval $(1, 2]$ such that $(\beta - 1)(1 - p) > -1$. Therefore we can consider that f' is L_p-integrable for some $p (> 1)$. When f' is L_p-integrable, by using [**14**. Theorem (3.2) p. 254], we obtain that

$$F(\xi, a) = (2\pi)^{-1/2} \int_{-a}^{a} \exp(-i\xi x) f'(x) dx$$

converges in mean with exponent q as $a \to \infty$. Since f' is L_1-integrable, $F(\xi, a)$ also converges for all ξ as $a \to \infty$. Then it follows from [**13**. 12.5 12.51] that the pointwise limit is equal to the limit in mean with exponent

q for almost all ξ. Therefore Uf' is L_q-integrable, hence $(Uf')(\xi)$ and $(Uf')(-\xi)$ are in $L_q(1,+\infty)$.

LEMMA 1.3. *Let* $\operatorname{Re}\lambda>0$ *and* $t>0$. *Then for* $f\in L_2$

$$\int_{-\infty}^{\infty} p_\lambda^\alpha(t,u-y)f(u)du$$

is L_2-integrable and continuous in y.

Proof. Let us put

$$g(y)=\int_{-\infty}^{\infty} p_\lambda^\alpha(t,u-y)f(u)du.$$

By Lemma 1.1 and the Schwartz inequality, we have

$$|g(y)-g(y+h)|\leqq\|f\|\left[\int_{-\infty}^{\infty} |p_\lambda^\alpha(t,u-y)-p_\lambda^\alpha(t,u-y-h)|^2du\right]^{1/2}.$$

Hence from [**13**. 19. p. 397], it follows that $g(y)$ is continuous in y.
It holds that

$$\|g\|^2\leqq\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|p_\lambda^\alpha(t,x-y)||f(x)|dx\right)\left(\int_{-\infty}^{\infty}|p_\lambda^\alpha(t,u-y)||f(u)|du\right)dy$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|p_\lambda^\alpha(t,x-y)||p_\lambda^\alpha(t,u-y)||f(x)||f(u)|dudxdy.$$

Let us make a change of variables, $x=x$, $x-y=z$, $u-y=v$. Then we have

$$\frac{\partial(x,y,u)}{\partial(x,v,z)}=1.$$

Hence we obtain by Lemma 1.2 that

$$\begin{aligned}\|g\|^2&\leqq\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|p_\lambda^\alpha(t,z)||p_\lambda^\alpha(t,v)||f(x)||f(v+x-z)|dxdvdz\\&\leqq\|f\|^2\|p_\lambda^\alpha(t,\cdot)\|_1^2,\end{aligned}\tag{1.6}$$

where $\|\cdot\|_1$ denotes L_1-norm.

THEOREM 1.2. *Let* $\operatorname{Re}\lambda>0$ *and* $t>0$. *Then for* $f\in L_2$

$$(1.7)\qquad (P_\lambda^\alpha(t)f)(y)=\int_{-\infty}^{\infty} p_\lambda^\alpha(t,u-y)f(u)du.$$

Proof. Case I. Let $f\in L_1\cap L_2$. Then $Uf\in L_2$. By this and (1.4), we can change the order of integration of (1.3) by the Fubini Theorem, therefore it follows from this fact that

$$(P_\lambda^\alpha(t)f)(y)=\int_{-\infty}^{\infty} p_\lambda^\alpha(t,u-y)f(u)du.$$

Case II. Let $f\in L_2$. Then we put

$$f_n(x)=\begin{cases} f(x) & \text{if } |x|\leqq n,\\ 0 & \text{if } |x|>n.\end{cases}$$

Since $f_n\in L_1\cap L_2$, it holds that

$$(P_\lambda^\alpha(t)f_n)(y)=\int_{-\infty}^{\infty} p_\lambda^\alpha(t,u-y)f_n(u)du,$$

and

$$\|P_\lambda^\alpha(t)f-P_\lambda^\alpha(t)f_n\|\leqq\|f_n-f\|\to 0\quad\text{as}\quad n\to\infty.$$

Let us put

$$g(y)=\int_{-\infty}^{\infty} p_\lambda^\alpha(t,u-y)f(u)du$$

and

$$g_n(y)=\int_{-\infty}^{\infty} p_\lambda^\alpha(t,u-y)f_n(u)du.$$

We obtain the following by using the inequality (1.6),

$$\|g-g_n\|\leqq d\|f-f_n\|\to 0\quad\text{as}\quad n\to\infty.$$

Therefore (1.7) holds.

COROLLARY 1.1. *Let q be a nonzero real number and let*

$$p_{iq}^2(t,x)=\left(\frac{iq}{4\pi t}\right)^{1/2}\exp\left(-\frac{iq}{4t}x^2\right).$$

Then for $f \in L_2$

$$(P^2_{iq}(t)f)(y) = {}^{(y)}\int_{-\infty}^{\infty} p^2_{iq}(t, u-y)f(u)du.$$

Proof. Let $\operatorname{Re}\lambda > 0$ and $f \in L_2$. We see that

$$\begin{aligned}&\left\| {}^{(\cdot)}\int_{-\infty}^{\infty} p^2_{iq}(t, u-\cdot)f(u)du - (P^2_{iq}(t)f)(\cdot)\right\| \\ &\leqq \left\| {}^{(\cdot)}\int_{-\infty}^{\infty} p^2_{iq}(t, u-\cdot)f(u)du - \int_{-\infty}^{\infty} p^2_{\lambda}(t, u-\cdot)f(u)du\right\| \\ &\quad + \|(P^2_{\lambda}(t)f)(\cdot) - (P^2_{iq}(t)f)(\cdot)\| \to 0 \quad \text{as} \quad \lambda \to iq,\end{aligned}$$

by Theorem 1.1, Theorem 1.2 and the proof of Theorem in [**6**. p. 778].

NOTE 1.1. From the assertion of Theorem 1.2, for convenience, we write $(P^{\alpha}_{\lambda}(t)f)(y)$ as

$$\int_{-\infty}^{\infty} p^{\alpha}_{\lambda}(t, u-y)f(u)\,du \tag{1.8}$$

even if $\lambda = iq$, where q is a nonzero real number. In what follows, we always use the notation (1.8) instead of $P^{\alpha}_{\lambda}(t)f$ for $\lambda \in D$.

LEMMA 1.4. *Let* $t > 0$. *Then* $\|p^{\alpha}_{\lambda}(t,\cdot)\|$ *and* $\|p^{\alpha}_{\lambda}(t,\cdot)\|_1$ *are continuous functions of* λ *on* $C^+ \equiv \{\lambda : \operatorname{Re}\lambda > 0\}$.

Proof. By (1.5) we have

$$\int_{-\infty}^{\infty} |p^{\alpha}_{\lambda}(t,\xi)|^2 d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left| \exp\left(-\frac{t}{\lambda}|\eta|^{\alpha}\right)\right|^2 d\eta.$$

Therefore $\|p^{\alpha}_{\lambda}(t,\cdot)\|$ is continuous in λ. From (1.5) and the proof of Lemma 1.2, and using f as defined there we have

$$\begin{aligned}\int_{-\infty}^{\infty} |p^{\alpha}_{\lambda}(t,\xi)|\,d\xi &= \int_{-1}^{1} |(Uf)(\xi)|\,d\xi + \int_1^{\infty} \frac{1}{\xi}|(Uf')(\xi)|\,d\xi \\ &\quad + \int_1^{\infty} \frac{1}{\xi}|(Uf')(-\xi)|\,d\xi.\end{aligned} \tag{1.9}$$

The first term is continuous in λ. By [**14**. Theorem (3.2), p. 254] it holds that

$$\int_1^\infty |(Uf')(\xi)|\,\xi^{-1}d\xi \leqq \left(\int_1^\infty \xi^{-p}d\xi\right)^{1/p}\cdot \|Uf'\|_q \leqq K\|f'\|_p$$

where let p satisfy the inequalities $(\beta-1)(1-p) > -1$ and $1 < p \leqq 2$, and K is constant. By replacing f in the above by

$$h(\xi,\lambda,\Delta\lambda) = (2\pi)^{-1/2}\exp\left(-\frac{t}{\lambda+\Delta\lambda}|\xi|^\alpha\right) - \exp\left(-\frac{t}{\lambda}|\xi|^\alpha\right),$$

we can show that the second term of (1.9) is continuous in λ. The above argument can be applicable to the third term of (1.9). Therefore the left side of (1.9) is continuous in λ.

2. Definition of operator valued function space integral. Let $C[a,b]$ denote the space of real valued right continuous functions defined on the interval $[a,b]$ and $C_0[a,b]$ denote those $x \in C[a,b]$ such that $x(a)=0$. Let σ be any partition of $[a,b]$, σ: $a=t_0<t_1<\cdots<t_n=b$. For any $x \in C[a,b]$, let $x_\sigma(t)=x(t_j)$ if $t_{j-1}< t \leqq t_j$, and $x_\sigma(a)=x(a)$. Let $F(x)$ be a bounded functional defined on $B[a,b]$. We suppose that $F(x_\sigma)$ has the form

$$F(x_\sigma) = f_\sigma(x(t_1), x(t_2), \cdots, x(t_n)), \qquad (x \in C[a,b]) \tag{2.1}$$

where $f_\sigma(v_1, v_2, \cdots, v_n)$ is a bounded Borel function on R^n, and that for each $\lambda \in D$ the operator $\mathscr{F}_\lambda^{\alpha,\sigma}(F)$ on L_2 can be defined by

$$\begin{aligned}(\mathscr{F}_\lambda^{\alpha,\sigma}(F)\psi)(\xi) &= \int_{-\infty}^\infty p_\lambda^\alpha(t_1-a, v_1-\xi)dv_1\int_{-\infty}^\infty p_\lambda^\alpha(t_2-t_1, v_2-v_1)dv_2 \\ &\cdots\int_{-\infty}^\infty p_\lambda^\alpha(b-t_{n-1}, v_n-v_{n-1})f_\sigma(v_1, v_2, \cdots, v_n)\psi(v_n)dv_n,\end{aligned} \tag{2.2}$$

where $\psi \in L_2$.

DEFINITION 2.1. If $\mathscr{F}_\lambda^{\alpha,\sigma}(F)$ converges in the weak operator topology as norm σ tends to 0 for $\lambda \in D$, we denote its limit by $\mathscr{F}_\lambda^\alpha(F)$ and for the moment we call $\mathscr{F}_\lambda^\alpha(F)$ the operator valued function space integral of F.

We should note from (0.3) that if $\mathrm{Re}\,\lambda > 0$, (2.2) corresponds to (0.2) except the fact that f_σ in (0.2) has the variable v_0. To compare the operator valued function space integral with the Wiener integral, it is convenient to use the following notations;

$$(\mathscr{F}_\lambda^{\alpha,\sigma}(F)\psi)(\xi)=\int_{C_0[a,b]} F(x_\sigma+\xi)\psi(x(b)+\xi)dp_\lambda^\alpha(x)$$

for (2.2) and

$$(\mathscr{F}_\lambda^{\alpha}(F)\psi)(\xi)=\int_{C_0[a,b]} F(x+\xi)\psi(x(b)+\xi)dp_\lambda^\alpha(x).$$

LEMMA 2.1. *Let $F(x)$ be a factorable functional given by*

$$F(x)=f_1(x(s_1))f_2(x(s_2))\cdots f_m(x(s_m)),\qquad a<s_1<\cdots<s_m=b,$$

where $f_i(v)$ are bounded Borel functions. Then for $\lambda\in D$, $\mathscr{F}_\lambda^{\alpha,\sigma}(F)\psi$ converges to

$$(2.3)\qquad \begin{aligned}(\mathscr{F}_\lambda^{\alpha}(F)\psi)(\xi)=\int_{-\infty}^{\infty} & p_\lambda^\alpha(s_1-a,v_1-\xi)f_1(v_1)dv_1\cdots\\ &\cdot\int_{-\infty}^{\infty} p_\lambda^\alpha(b-s_{m-1},v_m-v_{m-1})f_m(v_m)\psi(v_m)dv_m\end{aligned}$$

in the norm topology as norm $\sigma\to 0$, where $\psi\in L_2$.

Proof. Let σ be any partition of $[a,b]$, σ: $a=t_0<t_1<\cdots<t_n=b$. Clearly $F(x_\sigma)$ can be expressed as

$$F(x_\sigma)=f_1(x(t_{r(1)}))f_2(x(t_{r(2)}))\cdots f_m(x(b))$$

where $t_{r(j)-1}<s_j\leqq t_{r(j)}$. Then

$$(2.4)\qquad \begin{aligned}(\mathscr{F}_\lambda^{\alpha,\sigma}(F)\psi)(\xi)=\int_{-\infty}^{\infty} & p_\lambda^\alpha(t_1-a,v_1-\xi)dv_1\int_{-\infty}^{\infty} p_\lambda^\alpha(t_2-t_1,v_2-v_1)dv_2\\ &\cdots\int_{-\infty}^{\infty} p_\lambda^\alpha(b-t_{n-1},v_n-v_{n-1})f_1(v_{r(1)})f_2(v_{r(2)})\\ &\cdots f_m(v_n)\psi(v_n)dv_n.\end{aligned}$$

Here $\mathscr{F}_\lambda^{\alpha,\sigma}(F)$ is a well-defined operator on L_2 for each $\lambda\in D$. By the semigroup property of Theorem 1.1 and Note 1.1, we obtain when norm $\sigma<\mathrm{Min}\{s_i-s_{i-1}\}$

$$\begin{aligned}(\mathscr{F}_\lambda^{\alpha,\sigma}(F)\psi)(\xi)=\int_{-\infty}^{\infty} & p_\lambda^\alpha(t_{r(1)}-a,v_1-\xi)f_1(v_1)dv_1\cdots\\ &\cdot\int_{-\infty}^{\infty} p_\lambda^\alpha(b-t_{r(m-1)},v_m-v_{m-1})f_m(v_m)\psi(v_m)dv_m.\end{aligned}$$

Since $t_{r(j)}$ converge to s_j, $(j = 1, 2, \cdots, m)$ as norm $\sigma \to 0$, by using the strong continuity of semigroup of Theorem 1.1 and the boundedness of f_j, it follows that $\mathcal{F}_\lambda^{\alpha,\sigma}(F)\psi \to \mathcal{F}_\lambda^\alpha(F)\psi$ in the L_2-norm topology as norm $\sigma \to 0$ for $\lambda \in D$.

LEMMA 2.2. *If F is a functional to which $\mathcal{F}_\lambda^\alpha$ applies, then $\mathcal{F}_\lambda^\alpha(F)$ is a linear operator defined on a linear manifold $\mathfrak{M}$ of functions ψ. Moreover the operator valued integral $\mathcal{F}_\lambda^\alpha$ is linear in the sense that if the linear operators $\mathcal{F}_\lambda^\alpha(F)$ and $\mathcal{F}_\lambda^\alpha(G)$ are defined on the same manifold $\mathfrak{M}$, then $\mathcal{F}_\lambda^\alpha(c_1F + c_2G)$ is defined on $\mathfrak{M}$ for each pair of complex numbers c_1, c_2, and*

$$\mathcal{F}_\lambda^\alpha(c_1F + c_2G) = c_1\mathcal{F}_\lambda^\alpha(F) + c_2\mathcal{F}_\lambda^\alpha(G). \tag{2.5}$$

In particular, if F and G satisfy the hypotheses of Lemma 2.1, $\mathcal{F}_\lambda^\alpha(c_1F + c_2G)$ maps L_2 into L_2 and satisfies (2.5).

Proof. For each fixed σ, we note from (2.1) that f_σ depends linearly on F, and hence it follows from (2.2) that $(\mathcal{F}_\lambda^{\alpha,\sigma}(F)\psi)(\xi)$ depends linearly on F as well as on ψ. Hence the lemma follows from Definition 2.1 and the linearity of the weak limit and Lemma 2.1.

NOTE 2.1. If $\operatorname{Re}\lambda > 0$, by Lemma 1.1, Lemma 1.2 and the Fubini Theorem, we can write (2.3), (2.4) for all $\xi \in R$ as follows

$$(\mathcal{F}_\lambda^\alpha(F)\psi)(\xi) = \int_{-\infty}^{\infty} (m) \int_{-\infty}^{\infty} f_1(v_1) \cdots f_m(v_m)\psi(v_m) p_\lambda^\alpha(s_1 - a, v_1 - \xi) \cdots p_\lambda^\alpha(b - s_{m-1}, v_m - v_{m-1}) dv_1 \cdots dv_m,$$

$$(\mathcal{F}_\lambda^{\alpha,\sigma}(F)\psi)(\xi) = \int_{-\infty}^{\infty} (n) \int_{-\infty}^{\infty} f_1(v_{r(1)}) \cdots f_m(v_n)\psi(v_n) \cdot p_\lambda^\alpha(t_1 - a, v_1 - \xi) \cdots p_\lambda^\alpha(b - t_{n-1}, v_n - v_{n-1}) dv_1 \cdots dv_n.$$

3. The operator valued function space integral of a product integral.

Let us consider the following functional,

$$F(x) = \prod_{j=1}^{m} \int_a^b \theta_j(s, x(s)) ds, \tag{3.1}$$

where

(3.2) $\theta_j(s, u)$ are Borel measurable functions on $[a, b] \times R$, and

$$|\theta_j(s, u)| \leqq M_j < \infty, \qquad (j = 1, 2, \cdots, m). \tag{3.3}$$

LEMMA 3.1. *Let $F(x)$ be a functional given by* (3.1), (3.2) *and* (3.3). *Then for each ψ in L_2, $\mathscr{F}_\lambda^{\alpha,\sigma}(F)\psi$ is a strongly continuous function of λ on D and analytic in C^+.*

Proof. For a partition σ: $t_0 = a < t_1 < \cdots < t_n = b$ and any $x \in C[a, b]$, we have

$$F(x_\sigma) = \int_a^b (m) \int_a^b \theta_1(s_1, x_\sigma(s_1)) \cdots \theta_m(s_m, x_\sigma(s_m)) ds_1 \cdots ds_m.$$

Since $x_\sigma(s) = x(t_j)$ if $t_{j-1} < s \leqq t_j$, and $x_\sigma(a) = x(a)$, it holds that

$$\int_a^b \theta_j(s, x_\sigma(s))ds = \int_a^{t_1} \theta_j(s, x(t_1))ds + \cdots + \int_{t_{n-1}}^{t_n} \theta_j(s, x(t_n))ds.$$

Let us denote

$$\int_{t_{i-1}}^{t_i} \theta_j(s, v)ds$$

by $\phi_i^j(v)$. Then we have

$$F(x_\sigma) = \sum_{i(1)=1}^{n} \cdots \sum_{i(m)=1}^{n} \phi_{i(1)}^1(x(t_{i(1)})) \cdots \phi_{i(m)}^m(x(t_{i(m)})). \tag{3.4}$$

Since $\phi_i^j(v)$ are bounded Borel measurable functions on R, we can define the following operator on L_2 for each $\lambda \in D$,

$$(K_\lambda^\sigma(F)\psi)(\xi) = \int_{-\infty}^{\infty} p_\lambda^\alpha(t_1 - a, v_1 - \xi)dv_1 \int_{-\infty}^{\infty} p_\lambda^\alpha(t_2 - t_1, v_2 - v_1)dv_2 \cdots$$
$$\cdot \int_{-\infty}^{\infty} p_\lambda^\alpha(t_n - t_{n-1}, v_n - v_{n-1})\phi_{i(1)}^1(v_{i(1)}) \cdots \phi_{i(m)}^m(v_{i(m)})\psi(v_n)dv_n,$$

where $\psi \in L_2$. As we have stated in the proof of Lemma 2.2, it holds that

$$(\mathscr{F}_\lambda^{\alpha,\sigma}(F)\psi)(\xi) = \sum_{i(1)=1}^{n} \cdots \sum_{i(m)=1}^{n} (K_\lambda^\sigma(F)\psi)(\xi), \tag{3.5}$$

and by the boundedness of θ_j, it holds that

$$\| K_\lambda^\sigma(F)\psi \| \leqq M_1(t_{i(1)} - t_{i(1)-1}) \cdots M_m(t_{i(m)} - t_{i(m)-1}) \| \psi \|. \tag{3.6}$$

By using the boundedness of $\phi_l^j(v)$ and Theorem 1.1, we can show that $K_\lambda^\sigma(F)\psi$ is strongly continuous in λ on D. Therefore $\mathcal{F}_\lambda^{\alpha,\sigma}(F)\psi$ is a continuous function of λ on D. Next we wish to show that $K_\lambda^\sigma(F)\psi$ is an analytic function of λ in C^+ for a fixed $\psi \in L_2$. Let $g(\lambda) = (K_\lambda^\sigma(F)\psi, \varphi)$, $\varphi \in L_2$. As we have shown above, since $g(\lambda)$ is a continuous function of λ on D, we show that

$$\int_\Gamma g(\lambda)d\lambda = 0$$

for triangular path Γ in C^+. Then by Morera's Theorem, $g(\lambda)$ is the analytic function on C^+. We can consider the ordered integration of $K_\lambda^\sigma(F)\psi$ as an n-fold Lebesgue integral by Lemma 1.1 and Lemma 1.2 since λ is in C^+. It holds that

$$\int_\Gamma g(\lambda)d\lambda = \int_{-\infty}^{\infty} \overline{\varphi(\xi)} \left[\int_\Gamma (K_\lambda^\sigma(F)\psi)(\xi)\, d\lambda \right] d\xi,$$

since $\overline{\varphi(\xi)} \cdot (K_\lambda^\sigma(F)\psi)(\xi)$ is integrable with respect to λ, ξ over $\Gamma \times R$ by the Schwartz inequality and (3.6). Moreover by Lemma 1.1, Lemma 1.2 and Lemma 1.4

$$|\psi(v_n)|\,|p_\lambda^\alpha(t_1 - a, v_1 - \xi) \cdots p_\lambda^\alpha(t_n - t_{n-1}, v_n - v_{n-1})| \tag{3.7}$$

is integrable with respect to $v_1, \cdots, v_n, \lambda$ over $R^n \times \Gamma$, therefore by using the Fubini Theorem,

$$\begin{aligned}\int_\Gamma (K_\lambda^\sigma(F)\psi)d\lambda = \int_{-\infty}^{\infty} (n) \int_{-\infty}^{\infty} \Big[\int_\Gamma & p_\lambda^\alpha(t_1 - a, v_1 - \xi) \\ & \cdots p_\lambda^\alpha(t_n - t_{n-1}, v_n - v_{n-1})d\lambda \Big] \\ & \cdot \phi_{\iota(1)}^1(v_{\iota(1)}) \cdots \phi_{\iota(m)}^m(v_{\iota(m)})\psi(v_n)dv_1 \cdots dv_n.\end{aligned}$$

By (0.4) it holds that

$$\int_\Gamma \prod_{j=1}^n p_\lambda^\alpha(t_j - t_{j-1}, v_j - v_{j-1})d\lambda = 0.$$

Therefore we obtain

$$\int_\Gamma g(\lambda)d\lambda = 0.$$

Let us put

$$S(\tau)=\{(s_1,\cdots,s_m)\in(a,b)^m:\ a<s_{\tau(1)}<\cdots<s_{\tau(m)}<b\}.$$

Here τ means a permutation of $\{1,2,\cdots,m\}$ and $(s_1,\cdots,s_m)\in S(\tau)$ means that $(s_1,\cdots,s_m)$ satisfies the order relation $a<s_{\tau(1)}<\cdots<s_{\tau(m)}<b$.

THEOREM 3.1. *Let $F(x)$ be a functional given by* (3.1), (3.2) *and* (3.3). *Then for $\lambda\in D$, $\mathcal{F}_\lambda^{\alpha,\sigma}(F)\psi$ converges to*

$$(3.8)\quad \begin{aligned}&(\mathcal{F}_\lambda^{\alpha}(F)\psi)(\xi)\\ &=\sum_\tau (B)\int_{S(\tau)}(m)\int\left[\int_{-\infty}^{\infty}p_\lambda^\alpha(s_{\tau(1)}-a,v_1-\xi)\theta_{\tau(1)}(s_{\tau(1)},v_1)dv_1\right.\\ &\cdots\int_{-\infty}^{\infty}p_\lambda^\alpha(s_{\tau(m)}-s_{\tau(m-1)},v_m-v_{m-1})\theta_{\tau(m)}(s_{\tau(m)},v_m)dv_m\\ &\left.\cdot\int_{-\infty}^{\infty}p_\lambda^\alpha(b-s_{\tau(m)},v_{m+1}-v_m)\psi(v_{m+1})dv_{m+1}\right]ds_1\cdots ds_m\end{aligned}$$

in the norm topology as norm $\sigma\to0$, where $\psi\in L_2$ and the sum is taken over all $m!$ permutations τ of $\{1,2,\cdots,m\}$ and (B) *denotes the Bochner integral with respect to Lebesgue measure on $S(\tau)$. Furthermore*

$$(3.9)\qquad \|\mathcal{F}_\lambda^\alpha(F)\|\leqq(b-a)^m M_1M_2\cdots M_m.$$

Proof. Let λ be in C^+. For all $\xi\in R$, (3.7) is integrable with respect to $v_1,\cdots,v_n$ over R^n. By this fact, we can change the order of integration of $K_\lambda^\sigma(F)\psi$ by the Fubini Theorem, hence we obtain

$$(3.10)\quad \begin{aligned}&(K_\lambda^\sigma(F)\psi)(\xi)\\ &=\int_{t_{i(1)-1}}^{t_{i(1)}}(m)\int_{t_{i(m)-1}}^{t_{i(m)}}(H_\lambda^\sigma(s_1,\cdots,s_m)\psi)(\xi)ds_1\cdots ds_m,\end{aligned}$$

where

$$\begin{aligned}(H_\lambda^\sigma(s_1,\cdots,s_m)\psi)(\xi)&=\int_{-\infty}^{\infty}p_\lambda^\alpha(t_1-a,v_1-\xi)dv_1\int_{-\infty}^{\infty}p_\lambda^\alpha(t_2-t_1,v_2-v_1)dv_2\\ &\cdots\int_{-\infty}^{\infty}p_\lambda^\alpha(t_n-t_{n-1},v_n-v_{n-1})\theta_1(s_1,v_{i(1)})\cdots\theta_m(s_m,v_{i(m)})\psi(v_n)dv_n.\end{aligned}$$

From (3.5) and (3.10) we see that

$$
\begin{aligned}
&(\mathscr{F}_\lambda^{\alpha,\sigma}(F)\psi(\xi) \\
&= \int_a^b (m) \int_a^b ds_1 \cdots ds_m \left[\int_{-\infty}^{\infty} p_\lambda^\alpha(t_1 - a, v_1 - \xi) dv_1 \right. \\
&\quad \cdot \int_{-\infty}^{\infty} p_\lambda^\alpha(t_2 - t_1, v_2 - v_1) dv_2 \\
(3.11) \quad &\quad \left. \cdots \int_{-\infty}^{\infty} p_\lambda^\alpha(t_n - t_{n-1}, v_n - v_{n-1}) \prod_{j=1}^m \theta_j(s_j, V_\sigma(s_j))\psi(v_n) dv_n \right] \\
&= \sum_\tau \int_{S(\tau)} (m) \int \left[\int_{-\infty}^{\infty} p_\lambda^\alpha(t_1 - a, v_1 - \xi) dv_1 \int_{-\infty}^{\infty} p_\lambda^\alpha(t_2 - t_1, v_2 - v_1) dv_2 \right. \\
&\quad \left. \cdots \int_{-\infty}^{\infty} p_\lambda^\alpha(t_n - t_{n-1}, v_n - v_{n-1}) \prod_{j=1}^m \theta_j(s_j, V_\sigma(s_j))\psi(v_n) dv_n \right] ds_1 \cdots ds_m,
\end{aligned}
$$

where $V_\sigma(s) = v_i$ if $t_{i-1} < s \leqq t_i$.

By rearranging the product $\theta_1(s_1, V_\sigma(s_1)) \cdots \theta_m(s_m, V_\sigma(s_m))$ of the last member of (3.11) as

$$
\theta_{\tau(1)}(s_{\tau(1)}, V_\sigma(s_{\tau(1)})) \cdot \theta_{\tau(2)}(s_{\tau(2)}, V_\sigma(s_{\tau(2)})) \cdots \theta_{\tau(m)}(s_{\tau(m)}, V_\sigma(s_{\tau(m)}))
$$

since $(s_1, \cdots, s_m) \in S(\tau)$, (3.11) can be expressed in the following form,

$$
\begin{aligned}
&(\mathscr{F}_\lambda^{\alpha,\sigma}(F)\psi)(\xi) \\
(3.12) \quad &= \sum_\tau \int_{S(\tau)} (m) \int (H_{\lambda,\tau}^\sigma(s_1, \cdots, s_m)\psi)(\xi) ds_1 \cdots ds_m
\end{aligned}
$$

where

$$
\begin{aligned}
&(H_{\lambda,\tau}^\sigma(s_1, \cdots, s_m)\psi)(\xi) \\
&\equiv \int_{-\infty}^{\infty} p_\lambda^\alpha(t_1 - a, v_1 - \xi) dv_1 \int_{-\infty}^{\infty} p_\lambda^\alpha(t_2 - t_1, v_2 - v_1) dv_2 \cdots \\
(3.13) \quad &\quad \cdot \int_{-\infty}^{\infty} p_\lambda^\alpha(t_n - t_{n-1}, v_n - v_{n-1}) \theta_{\tau(1)}(s_{\tau(1)}, V_\sigma(s_{\tau(1)})) \cdots \\
&\quad \cdot \theta_{\tau(m)}(s_{\tau(m)}, V_\sigma(s_{\tau(m)}))\psi(v_n) dv_n, \ (s_1, \cdots, s_m) \in S(\tau).
\end{aligned}
$$

Now we prove that (3.13) is Bochner integrable over $S(\tau)$. In order to show this, it suffices to show that

(1) $H^{\sigma}_{\lambda,\tau}(s_1, \cdots, s_m)\psi$ is strongly measurable in $(s_1, \cdots, s_m)$ on $S(\tau)$,

(2) $\displaystyle\int_{S(\tau)} (m) \int \| H^{\sigma}_{\lambda,\tau}(s_1, \cdots, s_m)\psi \| \, ds_1 \cdots ds_m < \infty.$

Clearly it follows from Theorem 1.1 and (3.3) that

(a) $\| H^{\sigma}_{\lambda,\tau}(s_1, \cdots, s_m)\psi \| \leqq M_1 \cdots M_m \| \psi \|$,

for almost all $(s_1, \cdots, s_m) \in S(\tau)$.

In order to prove (1), it suffices to prove by [**5**. Corollary 2. p. 73] that $H^{\sigma}_{\lambda,\tau}(s_1, \cdots, s_m)\psi$ is weakly measurable on $S(\tau)$, that is $(H^{\sigma}_{\lambda,\tau}(s_1, \cdots, s_m)\psi, \varphi)$ is measurable in $(s_1, \cdots, s_m)$ for $\varphi \in L_2$. If $\varphi \in L_1 \cap L_2$, from Lemma 1.1, Lemma 1.2 and the boundedness of θ_j,

$$\overline{\varphi(\xi)} \left[\prod_{j=1}^{m} \theta_{\tau(j)}(s_{\tau(j)}, V_\sigma(s_{\tau(j)})) \right] \psi(v_n) p^{\alpha}_{\lambda}(t_1 - a, v_1 - \xi) \cdots p^{\alpha}_{\lambda}(t_n - t_{n-1}, v_n - v_{n-1})$$

is integrable with respect to the variables $s_1, \cdots, s_m, \xi, v_1, \cdots, v_n$ over $S(\tau) \times R^{n+1}$. Therefore it follows from the Fubini Theorem that $(H^{\sigma}_{\lambda,\tau}(s_1, \cdots, s_m)\psi, \varphi)$ is measurable in $(s_1, \cdots, s_m)$ on $S(\tau)$. If $\varphi \in L_2$, let us put

$$\varphi_N(x) = \begin{cases} \varphi(x) & \text{if } |x| \leqq N, \\ 0 & \text{if } |x| > N. \end{cases}$$

Then since $\| \varphi_N - \varphi \| \to 0$ as $N \to \infty$, it holds by (a) that for almost all $(s_1, \cdots, s_m) \in S(\tau)$,

$$(H^{\sigma}_{\lambda,\tau}(s_1, \cdots, s_m)\psi, \varphi_N) \to (H^{\sigma}_{\lambda,\tau}(s_1, \cdots, s_m)\psi, \varphi) \quad \text{as} \quad N \to \infty.$$

By the fact that $(H^{\sigma}_{\lambda,\tau}(s_1, \cdots, s_m)\psi, \varphi_N)$ is measurable in $(s_1, \cdots, s_m)$, $(H^{\sigma}_{\lambda,\tau}(s_1, \cdots, s_m)\psi, \varphi)$ is also measurable with respect to the variables $s_1, \cdots, s_m$ on $S(\tau)$. Furthermore $\| H^{\sigma}_{\lambda,\tau}(s_1, \cdots, s_m)\psi \|$ is measurable on $S(\tau)$ by [**5**. Theorem 3.5.2] since L_2 is separable. Therefore by (a), (2) holds. Hence we see that

(b) $H^{\sigma}_{\lambda,\tau}(s_1, \cdots, s_m)\psi$ is Bochner integrable over $S(\tau)$ if $\lambda \in C^+$.

Next we wish to prove that $H^{\sigma}_{\lambda,\tau}(s_1, \cdots, s_m)\psi$ is also Bochner integrable over $S(\tau)$ even if $\lambda = -iq$ (q is a nonzero real number). Here we should note that for $\lambda = -iq$, (3.13) is well-defined for almost all $(s_1, \cdots, s_m) \in S(\tau)$ by the boundedness of θ_j. In order to prove the Bochner integrability, by (a), (b) and [**5**. Theorem 3.7.9] it suffices to prove that for almost all $(s_1,\cdots,s_m)\in S(\tau)$,

$$\|H^{\sigma}_{\lambda,\tau}(s_1,\cdots,s_m)\psi - H^{\sigma}_{-iq,\tau}(s_1,\cdots,s_m)\psi\|\to 0 \tag{3.15}$$

as $\lambda\to -iq$. By using the boundedness of θ_j and Theorem 1.1, we can see that (3.15) holds for almost all $(s_1,\cdots,s_m)\in S(\tau)$. Hence we have

(c) $H^{\sigma}_{-iq,\tau}(s_1,\cdots,s_m)\psi$ is Bochner integrable over $S(\tau)$,

(d) $\|H^{\sigma}_{-iq,\tau}(s_1,\cdots,s_m)\psi\|\leqq M_1\cdots M_m\|\psi\|$, for almost all $(s_1,\cdots,s_m)\in S(\tau)$.

Furthermore, by using the Dominated Convergence Theorem [**5**. Theorem 3.7.9] we obtain

$$\begin{aligned}&\lim_{\lambda\to -iq}(B)\int_{S(\tau)}(m)\int H^{\sigma}_{\lambda,\tau}(s_1,\cdots,s_m)\psi ds_1\cdots ds_m\\ &=(B)\int_{S(\tau)}(m)\int H^{\sigma}_{-iq,\tau}(s_1,\cdots,s_m)\psi ds_1\cdots ds_m.\end{aligned}\tag{3.16}$$

Let $\varphi\in L_2$ and $\lambda\in C^+$. Then $|\overline{\varphi(\xi)}(H^{\sigma}_{\lambda,\tau}(s_1,\cdots,s_m)\psi)(\xi)|$ is integrable over $S(\tau)\times R$ by (a). Hence by using the Fubini Theorem and [**5**. Theorem 3.7.12 and the remark following], we have

$$\begin{aligned}&\int_{-\infty}^{\infty}\overline{\varphi(\xi)}\left[\int_{S(\tau)}(m)\int (H^{\sigma}_{\lambda,\tau}(s_1,\cdots,s_m)\psi)(\xi)\,ds_1\cdots ds_m\right]d\xi\\ &=\int_{-\infty}^{\infty}\overline{\varphi(\xi)}\left[(B)\int_{S(\tau)}(m)\int (H^{\sigma}_{\lambda,\tau}(s_1,\cdots,s_m)\psi)(\xi)\,ds_1\cdots ds_m\right]d\xi.\end{aligned}$$

From this fact, it follows that for $\lambda\in C^+$

$$\begin{aligned}&\int_{S(\tau)}(m)\int H^{\sigma}_{\lambda,\tau}(s_1,\cdots,s_m)\psi ds_1\cdots ds_m\\ &=(B)\int_{S(\tau)}(m)\int H^{\sigma}_{\lambda,\tau}(s_1,\cdots,s_m)\psi ds_1\cdots ds_m.\end{aligned}\tag{3.17}$$

From (3.12) and (3.17) we have

$$\mathscr{F}^{\alpha,\sigma}_{\lambda}(F)\psi=\sum_{\tau}(B)\int_{S(\tau)}(m)\int H^{\sigma}_{\lambda,\tau}(s_1,\cdots,s_m)\psi ds_1\cdots ds_m$$

for $\lambda \in C^+$. From this equality, by using (3.16) and Lemma 3.1, we obtain

$$(3.18)\quad \mathscr{F}_\lambda^{\alpha,\sigma}(F)\psi = \sum_\tau (B)\int_{S(\tau)} (m)\int H_{\lambda,\tau}^\sigma(s_1, \cdots, s_m)\psi ds_1 \cdots ds_m$$

for every $\lambda \in D$.

Now we show that (3.18) converges to (3.8) in the L_2-norm topology as norm $\sigma \to 0$. For almost all $(s_1, \cdots, s_m) \in S(\tau)$, it follows from (3.13) that

$$(3.19)\quad \begin{aligned} (H_{\lambda,\tau}^\sigma(s_1, \cdots, s_m)\psi)(\xi) &= \int_{-\infty}^\infty p_\lambda^\alpha(t_1 - a, v_1 - \xi)dv_1 \cdots \\ &\cdot \int_{-\infty}^\infty p_\lambda^\alpha(t_n - t_{n-1}, v_n - v_{n-1})\theta_{\tau(1)}(s_{\tau(1)}, v_{r(1)}) \cdots \\ &\cdot \theta_{\tau(m)}(s_{\tau(m)}, v_{r(m)})\psi(v_n)dv_n, \end{aligned}$$

where let $t_{r(j)-1} < s_{\tau(j)} \leqq t_{r(j)}$. By the same argument as in the proof of Lemma 2.1, for almost all $(s_1, \cdots, s_m) \in S(\tau)$, (3.19) converges to

$$(3.20)\quad \begin{aligned} &I_\tau(s_1, \cdots, s_m)\psi)(\xi) \\ &\equiv \int_{-\infty}^\infty p_\lambda^\alpha(s_{\tau(1)} - a, v_1 - \xi)\theta_{\tau(1)}(s_{\tau(1)}, v_1)dv_1 \cdots \\ &\cdot \int_{-\infty}^\infty p_\lambda^\alpha(s_{\tau(m)} - s_{\tau(m-1)}, v_m - v_{m-1})\theta_{\tau(m)}(s_{\tau(m)}, v_m)dv_m \\ &\cdot \int_{-\infty}^\infty p_\lambda^\alpha(b - s_{\tau(m)}, v_{m+1} - v_m)\psi(v_{m+1})dv_{m+1} \end{aligned}$$

in the norm topology as norm $\sigma \to 0$. From this fact and from (a), (b), (c), (d) and from [**5**. Theorem 3.7.9], it follows that

(e) $\quad \| I_\tau(s_1, \cdots, s_m)\psi \| \leqq M_1 \cdots M_m \|\psi\|$,

for almost all $(s_1, \cdots, s_m) \in S(\tau)$,

(f) $I_\tau(s_1, \cdots, s_m)\psi$ is also Bochner integrable over $S(\tau)$, furthermore

$$\mathscr{F}_\lambda^{\alpha,\sigma}(F)\psi \to \sum_\tau (B)\int_{S(\tau)} (m)\int I_\tau(s_1, \cdots, s_m)\psi ds_1 \cdots ds_m$$

in the norm topology as norm $\sigma \to 0$ for each $\lambda \in D$.

Now we prove (3.9). From (e) and [**5**. Theorem 3.7.6], it follows that

$$\|\mathscr{F}_\lambda^\alpha(F)\psi\| \leqq \sum_\tau \int_{S(\tau)} (m) \int \|I_\tau(s_1, \cdots, s_m)\psi\| ds_1 \cdots ds_m$$
$$= \sum_\tau \frac{(b-a)^m}{m!} M_1 \cdots M_m \|\psi\| = (b-a)^m M_1 \cdots M_m \|\psi\|.$$

Therefore we have

$$\|\mathscr{F}_\lambda^\alpha(F)\| \leqq (b-a)^m M_1 \cdots M_n.$$

COROLLARY 3.1. *Let $\theta(t,u) = \theta_1(t,u) = \cdots = \theta_m(t,u)$ in Theorem 3.1. Then for $\lambda \in D$*

$$(\mathscr{F}_\lambda^\alpha(F)\psi)(\xi)$$
$$(3.21) \quad = m!(B)\int_{S_m(a,b)} (m) \int \left[\int_{-\infty}^{\infty} p_\lambda^\alpha(s_1 - a, v_1 - \xi)\theta(s_1, v_1)dv_1 \cdots \int_{-\infty}^{\infty} p_\lambda^\alpha(s_m - s_{m-1}, v_m - v_{m-1})\theta(s_m, v_m)dv_m \cdot \int_{-\infty}^{\infty} p_\lambda^\alpha(b - s_m, v_{m+1} - v_m)\psi(v_{m+1})dv_{m+1}\right] ds_1 \cdots ds_m,$$

where

$$S_m(a,b) = \{(s_1, \cdots, s_m) \in (a,b)^m : a < s_1 < s_2 < \cdots s_m < b\}$$

and $\psi \in L_2$.

NOTE 3.1. Let us denote the right side of (3.21) by $m!\mathscr{J}_\lambda^\alpha(F)\psi$. For convenience, we use a notation $\mathscr{J}_\lambda^\alpha(F)$ in place of $\mathscr{F}_\lambda^\alpha(F)$ when $F(x) \equiv 1$.

4. The operator valued function space integral of $F(x) = \sum_{m=1}^\infty F_m(x)$.

Let us denote the set of all the functionals satisfying (3.1), (3.2) and (3.3) by A_0 and let us introduce the quantity

$$(4.1) \quad N_0(F) = (b-a)^m M_1 \cdots M_m,$$

where let F satisfy (3.1), (3.2) and (3.3).

Let

$$A \equiv \left\{ F(x) = \sum_{m=1}^{\infty} F_m(x) \colon F_m \in A_0 \quad \text{and} \quad \sum_{m=1}^{\infty} N_0(F_m) < \infty \right\}.$$

THEOREM 4.1. *Let F be in A and let $F(x) = \Sigma_{m=1}^{\infty} F_m(x)$. Then for $\lambda \in D$, $\mathscr{F}_\lambda^{\alpha,\sigma}(F)\psi$ converges to $\mathscr{F}_\lambda^{\alpha}(F)\psi = \Sigma_{m=1}^{\infty} \mathscr{F}_\lambda^{\alpha}(F_m)\psi$ in the norm topology as norm $\sigma \to 0$, where $\psi \in L_2$.*

Furthermore

$$\| \mathscr{F}_\lambda^{\alpha}(F)\psi \| \leqq \sum_{m=1}^{\infty} N_0(F_m) \| \psi \|. \tag{4.2}$$

NOTE 4.1. We do not assume that F determines $F_1, F_2, \cdots$ uniquely or that F_m determines $N_0(F_m)$ uniquely; but we assert that (4.2) holds for any choice of $F_1, F_2, \cdots$ whose sum is F, where each F_m satisfies (3.1), (3.2), (3.3), (4.1) with the m in those equations corresponding to the subscript of F_m.

Proof. Let

$$F_m(x) = \prod_{j=1}^{m} \int_a^b \theta_j^m(s, x(s))ds,$$

where each θ_j^m is bounded and Borel measurable in $[a, b] \times R$; let σ be any partition $a = t_0 < t_1 < \cdots < t_n = b$; and let

$$\phi_i^{m,j}(v) = \int_{t_{i-1}}^{t_i} \theta_j^m(s, v)ds.$$

Then we have formally that

$$(\mathscr{F}_\lambda^{\alpha,\sigma}(F)\psi)(\xi) = \int_{-\infty}^{\infty} p_\lambda^\alpha(t_1 - a, v_1 - \xi)dv_1 \cdots \int_{-\infty}^{\infty} p_\lambda^\alpha(t_n - t_{n-1}, v_n - v_{n-1}) \cdot \psi(v_n) \sum_{m=1}^{\infty} F_m(V_\sigma)dv_n$$

where $\psi \in L_2$ and by (3.4)

$$F_m(V_\sigma) = \sum_{i(1)=1}^{n} \cdots \sum_{i(m)=1}^{n} \phi_{i(1)}^{m,1}(v_{i(1)}) \cdots \phi_{i(m)}^{m,m}(v_{i(m)}).$$

If necessary, by multiplying by additional functions which are identically 1, we can write $F_m(V_\sigma)$ as

$$F_m(V_\sigma) = \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} \varphi_1^m(i_1, \cdots, i_m : v_1) \cdots \varphi_n^m(i_1, \cdots, i_m : v_n), \tag{4.3}$$

where notation i_j is made use of instead of $i(j)$.

From the fact that $F_m(x)$ is in A_0, it is clear that for almost all $v_1, \cdots, v_n$

$$\sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} |\varphi_1^m(i_1, \cdots, i_m : v_1) \cdots \varphi_n^m(i_1, \cdots, i_m : v_n)| \leqq N_0(F_m), \qquad (m = 1, 2, \cdots). \tag{4.4}$$

From (4.4), we have for almost all $v_1, \cdots, v_n$,

$$\sum_{m=1}^{\infty} \left(\sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} |\varphi_1^m(i_1, \cdots, i_m : v_1) \cdots \varphi_n^m(i_1, \cdots, i_m : v_n)| \right) \leqq \sum_{m=1}^{\infty} N_0(F_m) < \infty. \tag{4.5}$$

Now we will prove that

$$\int_{-\infty}^{\infty} p_\lambda^\alpha(b - t_{n-1}, v_n - v_{n-1}) \psi(v_n) \sum_{m=1}^{\infty} F_m(V_\sigma) dv_n \tag{4.6}$$

is L_2-integrable with respect to v_{n-1} for almost all $v_1, \cdots, v_{n-2}$. By Note 1.1 and (4.3), (4.6) is

$$U_{v_n}^* \left\{ \exp\left(-\frac{b - t_{n-1}}{\lambda} |v_n|^\alpha \right) U_\eta \left[\psi(\eta) \sum_{m=1}^{\infty} \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} \varphi_1^m(i_1, \cdots, i_m : v_1) \cdots \varphi_{n-1}^m(i_1, \cdots, i_m : v_{n-1}) \varphi_n^m(i_1, \cdots, i_m : \eta) \right] (v_n) \right\} (v_{n-1}), \tag{4.7}$$

where we use the notations $(U_x f)(y)$ and $(U_x^* f)(y)$ in place of (a), (b) in §1. We wish to prove that

$$U_\eta \left[\psi(\eta) \sum_{m=1}^{\infty} \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} \varphi_1^m(i_1, \cdots, i_m : v_1) \cdots \varphi_{n-1}^m(i_1, \cdots, i_m : v_{n-1}) \varphi_n^m(i_1, \cdots, i_m : \eta) \right] (v_n) \tag{4.8}$$

is L_2-integrable with respect to v_n for almost every $v_1, \cdots, v_{n-1}$, that is, (4.8) is well-defined.

Let

$$\psi_N(x) = \begin{cases} \psi(x) & \text{if} \quad |x| \leqq N, \\ 0 & \text{if} \quad |x| > N. \end{cases}$$

Since

$$\psi_N(\eta) \sum_{m=1}^{\infty} \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} \varphi_1^m(i_1, \cdots, i_m : v_1) \cdots \varphi_{n-1}^m(i_1, \cdots, i_m : v_{n-1}) \cdot \varphi_n^m(i_1, \cdots, i_m : \eta)$$

is L_1-integrable with respect to η by (4.5) for almost all $v_1, \cdots, v_{n-1}$, we have for each v_n and for almost every $v_1, \cdots, v_{n-1}$

$$(4.9) \quad \begin{aligned} Q_N(v_n) &= U_\eta \Big[\psi_N(\eta) \sum_{m=1}^{\infty} \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} \varphi_1^m(i_1, \cdots, i_m : v_1) \\ &\qquad \cdots \varphi_{n-1}^m(i_1, \cdots, i_m : v_{n-1}) \varphi_n^m(i_1, \cdots, i_m : \eta) \Big] (v_n) \\ &= \sum_{m=1}^{\infty} \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} \varphi_1^m(i_1, \cdots, i_m : v_1) \cdots \\ &\qquad \cdot \varphi_{n-1}^m(i_1, \cdots, i_m : v_{n-1}) U_\eta [\psi_N(\eta) \varphi_n^m(i_1, \cdots, i_m : \eta)](v_n). \end{aligned}$$

From (4.4), it follows that for almost all $v_1, \cdots, v_{n-1}$

$$(4.10) \quad \begin{aligned} &\sum_{m=1}^{\infty} \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} |\varphi_1^m(i_1, \cdots, i_m : v_1) \cdots \varphi_{n-1}^m(i_1, \cdots, i_m : v_{n-1})| \\ &\qquad \| \varphi_n^m(i_1, \cdots, i_m : \cdot) \|_\infty \cdot \| \psi_N \| \leqq \| \psi_N \| \sum_{m=1}^{\infty} N_0(F_m), \end{aligned}$$

where $\| \cdot \|_\infty$ means the essential supremum norm.

By using (4.10) we can prove that the last member of (4.9) also converges in the L_2-norm topology with respect to v_n for almost every $v_1, \cdots, v_{n-1}$. By [**13**. 12.53. Example (iii)], both limits with the infinite sum of the last member of (4.9) are equal to each other except on a null set. Therefore we see that $Q_N(v_n)$ is L_2-integrable with respect to v_n for almost every $v_1, \cdots, v_{n-1}$. Next we wish to prove that $Q_N(v_n)$ converges to a L_2-integrable function $Q(v_n)$ in the L_2-norm topology with respect to v_n as $N \to \infty$ for almost all $v_1, \cdots, v_{n-1}$. From (4.4), it follows that

$$(4.11) \quad \begin{aligned} &\Big\| \sum_{m=1}^{\infty} \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} \varphi_1^m(i_1, \cdots, i_m : v_1) \cdots \varphi_{n-1}^m(i_1, \cdots, i_m : v_{n-1}) \\ &\qquad \cdot \{ U_\eta [\psi_N(\eta) \varphi_n^m(i_1, \cdots, i_m : \eta)](\cdot) \\ &\qquad\qquad - U_\eta [\psi(\eta) \varphi_n^m(i_1, \cdots, i_m : \eta)](\cdot) \} \Big\| \\ &\leqq \| \psi_N - \psi \| \sum_{m=1}^{\infty} N_0(F_m) \to 0 \quad \text{as} \quad N \to \infty \end{aligned}$$

for almost all $v_1, \cdots, v_{n-1}$, where the infinite sum is taken in the L_2-norm topology. Hence we obtain from (4.11) that

$$\begin{aligned}
&\underset{N\to\infty}{\mathrm{l.i.m}}\ Q_N(v_n) \\
&= \underset{N\to\infty}{\mathrm{l.i.m}} \sum_{m=1}^{\infty}\sum_{i_1=1}^{n}\cdots\sum_{i_m=1}^{n} \varphi_1^m(i_1,\cdots,i_m : v_1)\cdots \\
&\quad \cdot \varphi_{n-1}^m(i_1,\cdots,i_m : v_{n-1})U_\eta[\psi_N(\eta)\varphi_n^m(i_1,\cdots,i_m:\eta)]'(v_n) \\
(4.12)\qquad &= \sum_{m=1}^{\infty}\sum_{i_1=1}^{n}\cdots\sum_{i_m=1}^{n} \varphi_1^m(i_1,\cdots,i_m : v_1)\cdots \\
&\quad \cdot \varphi_{n-1}^m(i_1,\cdots,i_m : v_{n-1})U_\eta[\psi(\eta)\varphi_n^m(i_1,\cdots,i_m:\eta)](v_n) \\
&= Q(v_n)
\end{aligned}$$

for almost every $v_1, \cdots, v_{n-1}$. Here the outer sum in the third member converges in the L_2-norm topology. Therefore we obtain the facts that (4.8) is L_2-integrable with respect to v_n for almost every $v_1, \cdots, v_{n-1}$ and in the sense of L_2-equivalence

$$\begin{aligned}
Q(\cdot) = U_\eta\Bigg[\psi(\eta)\sum_{m=1}^{\infty}\sum_{i_1=1}^{n}\cdots\sum_{i_m=1}^{n} \varphi_1^m(i_1,\cdots,i_m : v_1)\cdots \\
\cdot \varphi_{n-1}^m(i_1,\cdots,i_m : v_{n-1})\varphi_n^m(i_1,\cdots,i_m:\eta)\Bigg](\cdot)
\end{aligned}$$

for almost every $v_1, \cdots, v_{n-1}$.

Next we wish to prove that (4.7) is L_2-integrable with respect to v_{n-1} for almost all $v_1, \cdots, v_{n-2}$.

Let

$$X_N(x) = \begin{cases} 1 & \text{if } |x| \leqq N, \\ 0 & \text{if } |x| > N. \end{cases}$$

From (4.12) and from the fact that Q is in L_2, it follows that

$$\begin{aligned}
&X_N(v_n)Q(v_n)\exp\left(-\frac{b-t_{n-1}}{\lambda}|v_n|^\alpha\right) \\
(4.13)\quad &= X_N(v_n)\exp\left(-\frac{b-t_{n-1}}{\lambda}|v_n|^\alpha\right)\sum_{m=1}^{\infty}\sum_{i_1=1}^{n}\cdots\sum_{i_m=1}^{n} \varphi_1^m(i_1,\cdots,i_m : v_1) \\
&\quad \cdots \varphi_{n-1}^m(i_1,\cdots,i_m : v_{n-1})U_\eta[\psi(\eta)\varphi_n^m(i_1,\cdots,i_m:\eta)](v_n)
\end{aligned}$$

is L_1-integrable in v_n for almost all $v_1, \cdots, v_{n-1}$. Since the infinite sum of the third member of (4.12) converges to $Q(v_n)$ in the L_2-norm topology, we have for almost all v_n and for almost every $v_1, \cdots, v_{n-1}$,

$$X_N(v_n)Q(v_n)\exp\left(-\frac{b-t_{n-1}}{\lambda}|v_n|^\alpha\right)$$

$$(4.14)\qquad =\sum_{m=1}^{\infty}\sum_{i_1=1}^{n}\cdots\sum_{i_m=1}^{n}\varphi_1^m(i_1,\cdots,i_m:v_1)\cdots\varphi_{n-1}^m(i_1,\cdots,i_m:v_{n-1})$$
$$\cdot X_N(v_n)\exp\left(-\frac{b-t_{n-1}}{\lambda}|v_n|^\alpha\right)U_\eta[\psi(\eta)\varphi_n^m(i_1,\cdots,i_m:\eta)](v_n)$$

where the infinite sum of the right side is taken in the L_2-norm topology. With the right side of (4.14), from the Schwartz inequality and (4.4), it follows that

$$\sum_{m=1}^{\infty}\sum_{i_1=1}^{n}\cdots\sum_{i_m=1}^{n}|\varphi_1^m(i_1,\cdots,i_m:v_1)\cdots\varphi_{n-2}^m(i_1,\cdots,i_m:v_{n-2})|$$
$$\|\varphi_{n-1}^m(i_1,\cdots,i_m:\cdot)\|_\infty$$
$$\cdot\left\|X_N\exp\left(-\frac{b-t_{n-1}}{\lambda}|\cdot|^\alpha\right)U_\eta[\psi(\eta)\varphi_n^m(i_1,\cdots,i_m:\eta)](\cdot)\right\|_1$$
$$\leqq(2N)^{1/2}\|\psi\|\sum_{m=1}^{\infty}N_0(F_m)<\infty.$$

From this fact and from [**13**. 10.83], it follows that for almost every $v_1,\cdots,v_{n-1}$, the infinite sum of the right side of (4.14) converges pointwisely for almost all v_n. Hence by [**13**. 12.53. Example (iii)], both limits with the infinite sum of the right side of (4.14) are equal to each other except on a null set. By the Dominated Convergence Theorem, we have

$$(2\pi)^{-1/2}\int_{-\infty}^{\infty}\exp(iv_{n-1}v_n)X_N(v_n)Q(v_n)\exp\left(-\frac{b-t_{n-1}}{\lambda}|v_n|^\alpha\right)dv_n$$
$$=\lim_{k\to\infty}(2\pi)^{-1/2}\int_{-\infty}^{\infty}\exp(iv_{n-1}v_n)\sum_{m=1}^{k}\sum_{i_1=1}^{n}\cdots\sum_{i_m=1}^{n}\varphi_1^m(i_1,\cdots,i_m:v_1)$$
$$\cdots\varphi_{n-1}^m(i_1,\cdots,i_m:v_{n-1})X_N(v_n)\exp\left(-\frac{b-t_{n-1}}{\lambda}|v_n|^\alpha\right)$$
$$\cdot U_\eta[\psi(\eta)\varphi_n^m(i_1,\cdots,i_m:\eta)](v_n)dv_n$$

(4.15)

$$=\lim_{k\to\infty}\sum_{m=1}^{k}\sum_{i_1=1}^{n}\cdots\sum_{i_m=1}^{n}\varphi_1^m(i_1,\cdots,i_m:v_1)\cdots$$
$$\cdot\varphi_{n-1}^m(i_1,\cdots,i_m:v_{n-1})U_{v_n}^*\left\{X_N(v_n)\exp\left(-\frac{b-t_{n-1}}{\lambda}|v_n|^\alpha\right)\right.$$
$$\left.\cdot U_\eta[\psi(\eta)\varphi_n^m(i_1,\cdots,i_m:\eta)](v_n)\right\}(v_{n-1})$$

$$= \sum_{m=1}^{\infty} \sum_{\iota_1=1}^{n} \cdots \sum_{\iota_m=1}^{n} \varphi_1^m(i_1, \cdots, i_m : v_1) \cdots \varphi_{n-1}^m(i_1, \cdots, i_m : v_{n-1})$$
$$\cdot U_{v_n}^* \left\{ X_N(v_n) \exp\left(-\frac{b - t_{n-1}}{\lambda} | v_n |^\alpha \right) U_\eta [\psi(\eta) \varphi_n^m(i_1, \cdots, i_m : \eta)](v_n) \right\} \times (v_{n-1})$$

in the sense of the pointwise convergence with respect to v_{n-1} for almost every $v_1, \cdots, v_{n-2}$. From (4.4), it follows that for almost all $v_1, \cdots, v_{n-2}$

$$\sum_{m=1}^{\infty} \sum_{i_1=1}^{n} \cdots \sum_{\iota_m=1}^{n} | \varphi_1^m(i_1, \cdots, i_m : v_1) \cdots \varphi_{n-2}^m(i_1, \cdots, i_m : v_{n-2}) |$$
$$\| \varphi_{n-1}^m(i_1, \cdots, i_m : \cdot) \|_\infty \cdot \left\| U_{v_n}^* \left\{ X_N(v_n) \exp\left(-\frac{b - t_{n-1}}{\lambda} | v_n |^\alpha \right) \right.\right.$$
$$\left.\left. \cdot U_\eta [\psi(\eta) \varphi_n^m(i_1, \cdots, i_m : \eta)](v_n) \right\} (\cdot) \right\| \leqq \sum_{m=1}^{\infty} N_0(F_m) \| \psi \|.$$

By this fact, the last member of (4.15) also converges in the L_2-norm topology with respect to v_{n-1} for almost all $v_1, \cdots, v_{n-2}$. From the fact that both limits of the last member of (4.15) are equal to each other except on a null set, by an argument similar to that used in obtaining (4.12), we can see that for almost every $v_1, \cdots, v_{n-2}$, (4.15) converges to

$$\sum_{m=1}^{\infty} \sum_{\iota_1=1}^{n} \cdots \sum_{\iota_m=1}^{n} \varphi_1^m(i_1, \cdots, i_m : v_1) \cdots$$
$$\cdot \varphi_{n-1}^m(i_1, \cdots, i_m : v_{n-1}) U_{v_n}^* \left\{ \exp\left(-\frac{b - t_{n-1}}{\lambda} | v_n |^\alpha \right) \right. \tag{4.16}$$
$$\left. \cdot U_\eta [\psi(\eta) \varphi_n^m(i_1, \cdots, i_m : \eta)](v_n) \right\} (v_{n-1})$$

in the L_2-norm with respect to v_{n-1}, where the infinite sum of (4.16) is taken in the L_2-norm topology with respect to v_{n-1}. From the above argument, it follows that (4.6) is L_2-integrable in v_{n-1} for almost every $v_1, \cdots, v_{n-2}$ and

$$(4.6) = \sum_{m=1}^{\infty} \sum_{\iota_1=1}^{n} \cdots \sum_{\iota_m=1}^{n} \varphi_1^m(i_1, \cdots, i_m : v_1) \cdots \varphi_{n-1}^m(i_1, \cdots, i_m : v_{n-1})$$
$$\cdot \int_{-\infty}^{\infty} p_\lambda^\alpha(b - t_{n-1}, v_n - v_{n-1}) \psi(v_n) \varphi_n^m(i_1, \cdots, i_m : v_n) dv_n$$

in the sense of the L_2-equivalence. By repeating the above argument, we can prove that

$$(\mathscr{F}_\lambda^{\alpha,\sigma}(F)\psi)(\xi) = \sum_{m=1}^{\infty} (\mathscr{F}_\lambda^{\alpha,\sigma}(F_m)\psi)(\xi),$$

where the sum is taken in the L_2-norm topology with respect to ξ. By using the fact that

$$\sum_{m=1}^{\infty} \| \mathscr{F}^{\alpha,\sigma}(F_m) \| \leqq \sum_{m=1}^{\infty} N_0(F_m) < \infty, \tag{4.17}$$

we have that

$$\begin{aligned} \lim_{\|\sigma\|\to 0} \mathscr{F}_\lambda^{\alpha,\sigma}(F)\psi &= \lim_{\|\sigma\|\to 0} \lim_{N\to\infty} \sum_{m=1}^{N} \mathscr{F}_\lambda^{\alpha,\sigma}(F_m)\psi \\ &= \lim_{N\to\infty} \lim_{\|\sigma\|\to 0} \sum_{m=1}^{N} \mathscr{F}_\lambda^{\alpha,\sigma}(F_m)\psi = \lim_{N\to\infty} \sum_{m=1}^{N} \mathscr{F}_\lambda^{\alpha}(F_m)\psi \\ &= \sum_{m=1}^{\infty} \mathscr{F}_\lambda^{\alpha}(F_m)\psi \end{aligned}$$

in the L_2-norm. From (4.17), (4.2) follows.

COROLLARY 4.1. *Let*

$$F(x) = \exp\left\{\int_a^b \theta(s, x(s))ds\right\},$$

where $\theta(s, u)$ is bounded by M and measurable in $[a, b] \times R$. Then for $\lambda \in D$

$$\mathscr{F}_\lambda^{\alpha}(F)\psi = \sum_{m=0}^{\infty} \mathscr{F}_\lambda^{\alpha}(F_m)\psi$$

where $\psi \in L_2$ and

$$F_m(x) = \left(\int_a^b \theta(s, x(s))ds\right)^m,$$

and where the sum is taken in the norm topology. Furthermore

$$\| \mathscr{F}_\lambda^{\alpha}(F) \| \leqq e^{(b-a)M}.$$

Proof. By expanding $F(x)$ into a series of the functionals in A_0 and by applying Theorem 4.1 and Corollary 3.1, we obtain Corollary.

5. Integral equation. Let $\theta(t, u)$ be a Borel function on $[0, t_0] \times R$ and bounded by M. For each $t \in (0, t_0]$, let $\theta_t(s, u)$ be defined on $[0, t] \times R$ by $\theta_t(s, u) \equiv \theta(t - s, u)$. In what follows we shall consider the functional

$$F^t(x) = \exp\left\{\int_0^t \theta_t(s, x(s))ds\right\}.$$

For convenience, let us put

$$G(t, \xi, \lambda) = (\mathcal{F}_\lambda^\alpha(F^t)\psi)(\xi),$$
$$g_m(t, \xi, \lambda) = (\mathcal{J}_\lambda^\alpha(F_m^t)\psi)(\xi),$$

where

$$F_m^t(x) = \left\{\int_0^t \theta_t(s, x(s))ds\right\}^m, \qquad m = 0, 1, \cdots.$$

THEOREM 5.1. *$G(t, \xi, \lambda)$ satisfies the integral equation*

$$(5.1)\qquad \begin{aligned} G(t, \xi, \lambda) &= \int_{-\infty}^{\infty} p_\lambda^\alpha(t, u - \xi)\psi(u)du \\ &+ (B)\int_0^t \left[\int_{-\infty}^{\infty} p_\lambda^\alpha(t - s, u - \xi)\theta(s, u)G(s, u, \lambda)du\right] ds \end{aligned}$$

where $t_0 \geqq t > 0$, $\lambda \in D$ and $\psi \in L_2$.

Proof. Let $\lambda \in D$ and $\psi \in L_2$. From Corollary 4.1, we have

$$G(t, \xi, \lambda) = \sum_{m=0}^{\infty} g_m(t, \xi, \lambda)$$

where the sum is taken in the L_2-norm topology.

At first we will prove that

$$(5.2)\qquad \begin{aligned} \sum_{m=0}^{\infty} g_m(t, \xi, \lambda) &= \int_{-\infty}^{\infty} p_\lambda^\alpha(t, u - \xi)\psi(u)du \\ &+ \sum_{m=0}^{\infty} (B)\int_0^t ds \left[\int_{-\infty}^{\infty} p_\lambda^\alpha(t - s, u - \xi)\theta(s, u)g_m(s, u, \lambda)du\right] \end{aligned}$$

where both sums are taken in the L_2-norm topology. In order to prove this, since

$$g_0(t, \xi, \lambda) = \int_{-\infty}^{\infty} p_\lambda^\alpha(t, u - \xi)\psi(u)du,$$

it suffices to prove that

$$(5.3)\quad g_{m+1}(t,\xi,\lambda)=(B)\int_0^t ds\left[\int_{-\infty}^{\infty} p_\lambda^\alpha(t-s,u-\xi)\theta(s,u)g_m(s,u,\lambda)du\right].$$

We prove that the integrand of (5.3) is Bochner integrable with respect to the variable s over $[0,t]$. By [**5**. Theorem 3.7.12 and the remark following], it holds that

$$\begin{aligned}Y(s,\xi)&=\int_{-\infty}^{\infty} p_\lambda^\alpha(t-s,u-\xi)\theta(s,u)g_m(s,u,\lambda)du\\&=(B)\int_{S_m(0,s)}(m)\int ds_1\cdots ds_m\left\{\int_{-\infty}^{\infty} p_\lambda^\alpha(t-s,u-\xi)\theta(s,u)du\right.\\&\quad\cdot\int_{-\infty}^{\infty} p_\lambda^\alpha(s_1,u_1-u)\theta_s(s_1,u_1)du_1\cdots\int_{-\infty}^{\infty} p_\lambda^\alpha(s_m-s_{m-1},u_m-u_{m-1})\\&\quad\left.\cdot\,\theta_s(s_m,u_m)du_m\int_{-\infty}^{\infty} p_\lambda^\alpha(s-s_m,u_{m+1}-u_m)\psi(u_{m+1})du_{m+1}\right\}\end{aligned}$$

for almost all $s\in[0,t]$ and hence by using [**5**. Theorem 3.7.6], we obtain for almost all $s\in[0,t]$,

$$\text{(a)}\quad \|Y\|\leqq M^{m+1}\frac{t^m}{m!}\|\psi\|.$$

Furthermore it is necessary to show that Y is strongly measurable in s. To show this, it is sufficient to show that Y is weakly measurable in s since L_2 is separable. Let φ be in L_2. By a change of variables, we have

$$\begin{aligned}&\int_{S_m(0,s)}(m)\int ds_1\cdots ds_m\left\{\int_{-\infty}^{\infty}\overline{\varphi(\xi)}\left[\int_{-\infty}^{\infty} p_\lambda^\alpha(t-s,u-\xi)\theta(s,u)du\right.\right.\\&\quad\cdot\int_{-\infty}^{\infty} p_\lambda^\alpha(s_1,u_1-u)\theta_s(s_1,u_1)du_1\cdots\int_{-\infty}^{\infty} p_\lambda^\alpha(s_m-s_{m-1},u_m-u_{m-1})\\&\quad\left.\left.\cdot\,\theta_s(s_m,u_m)du_m\int_{-\infty}^{\infty} p_\lambda^\alpha(s-s_m,u_{m+1}-u_m)\psi(u_{m+1})du_{m+1}\right]d\xi\right\}\end{aligned}$$

(5.4)

$$\begin{aligned}&=\int_{S_m(\tau_1,t)}(m)\int d\tau_2\cdots d\tau_{m+1}\left\{\int_{-\infty}^{\infty}\overline{\varphi(\xi)}\left[\int_{-\infty}^{\infty} p_\lambda^\alpha(\tau_1,u-\xi)\theta_t(\tau_1,u)du\right.\right.\\&\quad\cdot\int_{-\infty}^{\infty} p_\lambda^\alpha(\tau_2-\tau_1,u_1-u)\theta_t(\tau_2,u_1)du_1\cdots\int_{-\infty}^{\infty} p_\lambda^\alpha(\tau_{m+1}-\tau_m,u_m-u_{m-1})\\&\quad\left.\left.\cdot\,\theta_t(\tau_{m+1},u_m)du_m\int_{-\infty}^{\infty} p_\lambda^\alpha(t-\tau_{m+1},u_{m+1}-u_m)\psi(u_{m+1})du_{m+1}\right]d\xi\right\}\end{aligned}$$

where let $t-s=\tau_1$, $s_1=\tau_2-\tau_1$, $s_2=\tau_3-\tau_1,\cdots,s_m=\tau_{m+1}-\tau_1$. The integrand with the variables $\tau_1,\tau_2,\cdots,\tau_{m+1}$ of the right side of (5.4) is measurable in $(\tau_1,\cdots,\tau_{m+1})$ on $S_{m+1}(0,t)$ as shown in the proof of Theorem 3.1 and integrable with respect to $\tau_1,\cdots,\tau_{m+1}$ over $S_{m+1}(0,t)$. Therefore by the Fubini Theorem, the right side of (5.4) is measurable in τ_1, hence we can say that

(b) $Y(s,\xi)$ is strongly measurable in s.

From (a) and (b), it follows that

(c) $Y(s,\xi)$ is Bochner integrable with respect to s over $[0,t]$.

Therefore by [**5**. Theorem 3.7.12 and the remark following] and the Fubini Theorem,

$$\int_{-\infty}^{\infty}\overline{\varphi(\xi)}\left\{(B)\int_0^t ds\left[\int_{-\infty}^{\infty}p_\lambda^\alpha(t-s,u-\xi)\theta(s,u)g_m(s,u,\lambda)du\right]\right\}d\xi$$

$$=\int_0^t ds\int_{-\infty}^{\infty}\overline{\varphi(\xi)}\left[\int_{-\infty}^{\infty}p_\lambda^\alpha(t-s,u-\xi)\theta(s,u)g_m(s,u,\lambda)du\right]d\xi$$

$$=\int_0^t ds\left\{\int_{S_m(0,s)}(m)\int ds_1\cdots ds_m\left[\int_{-\infty}^{\infty}\overline{\varphi(\xi)}\left[\int_{-\infty}^{\infty}p_\lambda^\alpha(t-s,u-\xi)\theta(s,u)du\right.\right.\right.$$

$$\cdot\int_{-\infty}^{\infty}p_\lambda^\alpha(s_1,u_1-u)\theta_s(s_1,u_1)du_1\cdots$$

$$\left.\left.\left.\cdot\int_{-\infty}^{\infty}p_\lambda^\alpha(s-s_m,u_{m+1}-u_m)\psi(u_{m+1})du_{m+1}\right]d\xi\right]\right\}$$

$$=\int_{S_{m+1}(0,t)}(m+1)\int d\tau_1\cdots d\tau_{m+1}\left\{\int_{-\infty}^{\infty}\overline{\varphi(\xi)}\left[\int_{-\infty}^{\infty}p_\lambda^\alpha(\tau_1,u-\xi)\theta_t(\tau_1,u)du\right.\right.$$

$$\cdot\int_{-\infty}^{\infty}p_\lambda^\alpha(\tau_2-\tau_1,u_1-u)\theta_t(\tau_2,u_1)du_1\cdots$$

$$\left.\left.\cdot\int_{-\infty}^{\infty}p_\lambda^\alpha(t-\tau_{m+1},u_{m+1}-u_m)\psi(u_{m+1})du_{m+1}\right]d\xi\right\}$$

$$=\int_{-\infty}^{\infty}\overline{\varphi(\xi)}g_{m+1}(t,\xi,\lambda)d\xi.$$

From this fact, (5.3) follows. Therefore (5.2) is valid.

From Corollary 4.1, Theorem 1.1 and from the boundedness of θ, we obtain that

$$\sum_{m=0}^{n}\int_{-\infty}^{\infty} p_\lambda^\alpha(t-s,u-\xi)\theta(s,u)g_m(s,u,\lambda)du \tag{5.5}$$
$$\to\int_{-\infty}^{\infty} p_\lambda^\alpha(t-s,u-\xi)\theta(s,u)G(s,u,\lambda)du$$

in the L_2-norm topology as $n\to\infty$ for almost all $s\in[0,t]$. Since the left side of (5.5) is also Bochner integrable in s over $[0,t]$ by (c), it follows from (a) and [**5**. Theorem 3.7.9] that the right side of (5.5) is Bochner integrable in s over $[0,t]$. Furthermore, we have

$$\begin{aligned}\sum_{m=0}^{\infty}(B)\int_0^t ds\int_{-\infty}^{\infty} p_\lambda^\alpha(t-s,u-\xi)\theta(s,u)g_m(s,u,\lambda)du\\ =\lim_{n\to\infty}(B)\int_0^t ds\left[\sum_{m=0}^{n}\int_{-\infty}^{\infty} p_\lambda^\alpha(t-s,u-\xi)\theta(s,u)g_m(s,u,\lambda)du\right]\\ =(B)\int_0^t ds\left[\sum_{m=0}^{\infty}\int_{-\infty}^{\infty} p_\lambda^\alpha(t-s,u-\xi)\theta(s,u)g_m(s,u,\lambda)du\right]\\ =(B)\int_0^t ds\left[\int_{-\infty}^{\infty} p_\lambda^\alpha(t-s,u-\xi)\theta(s,u)G(s,u,\lambda)du\right],\end{aligned}$$

where both infinite sums are to be taken in the sense of the norm topology. By this fact, we see that $G(t,\xi,\lambda)$ is a solution of the integral equation (5.1).

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