PACIFIC JOURNAL OF MATHEMATICS
Vol. 66, No 2, 1976

RINGS WHOSE ADDITIVE SUBGROUPS ARE SUBRINGS

JOHN D. O'NEILL

In this paper the class (subclass) of associative rings whose additive subgroups are subrings (ideals) is completely characterized by defining relations. An exact description is also given of those rings in these classes which are commutative, regular, Artinian, Noetherian, or with identity. The only integral domains in either class are the ring of integers Z and $Z/(p)$ for prime p.

0. Introduction. A ring which has the property, called S, that all its additive subgroups are subrings is called a S-ring. A ring R is a S-ring if and only if, for every x and y in R, xy is a linear combination of x and y. This fact will be used constantly in this paper. All rings herein are associative, all groups are abelian, and all constants are integers. A ring R will be primary, torsion, torsion-free, mixed, etc. exactly if its additive group, R^+, is primary, etc. By the rank of R, $r(R)$, we mean the rank of R^+. Occasionally, when the meaning is perfectly clear, distinction will not be made between R and R^+. Other terminology is essentially that of [4]. Our chief results are found in Theorems 1.4, 1.5, 1.6, 2.6, 3.5, and 3.6. Special S-rings are described in §4.

1. Torsion rings. In this section all rings are torsion. We begin with primary rings.

PROPOSITION 1.1. *If R is a p-primary S-ring, R is bounded or R is a zero-ring.*

Proof. (a) Let B be a basic subgroup of R^+ and assume B is unbounded. We prove $R^2 = 0$. Suppose $x^2 = rx \neq 0$ for some x in B and $o(x) = p^n$. Choose z independent of x such that $o(z) \geqq p^{2n}$. Then $(x + p^n z)^2 = rx + p^{2n} z = k(x + p^n z)$ for some k. This implies p^n divides k and $0 = kx = rx$, a contradiction. Thus $x^2 = 0$, for every x in B. Suppose x and y are linearly independent in B and $xy = ax + by$, $by \neq 0$. If $o(x) = p^n$ and $o(y) = p^m$, choose z in B such that z is independent of x and y and p^{n+m} divides $o(z)$. Write $xz = cx + dz$. Then $x(y + z) = (a + c)x + (by + dz)$. The right-hand term must be a multiple of $y + z$ which implies $dy = by \neq 0$. But $p^n dz = p^n(xz) = 0$ and p^m divides d which implies $dy = 0$. Therefore, $xy = 0$ and $B^2 = 0$. It follows easily that $R^2 = 0$ (see Theorem 120.1 of

[4]). (b) Assume, then, B is bounded and write $R^+ = D \oplus B$ where D is divisible. Since R is primary, $D^2 = RD = DR = 0$. It will suffice to show $B^2 = 0$ when $D \neq 0$. If B is bounded by p^n, we select z in D such that $o(z) \geqq p^{2n}$. Using this z we repeat the proof given in (a) and again deduce $B^2 = 0$. This completes the proof.

If R is a bounded primary ring, we wish to know when it has property S. The answer is given in the next proposition for which we need a lemma.

LEMMA 1.2. *Let R be a p-primary S-ring. If x, y are linearly independent in R, $o(x) \leqq o(y)$, $x^2 = rx$, and $y^2 = sy$, then $xy + yx = sx + ry + e(x+y)$ for some constant e. If $ex \neq 0$, then $p = 2$, $2ex = 0$, and $o(x) = o(y)$.*

Proof. We write $(x+y)^2 = k(x+y)$ and $(x+2y)^2 = 1(x+2y)$ where k and 1 are constants. Solving simultaneously, we obtain $2k \equiv 2(s+r) \pmod{o(x)}$. If we set $k = s + r + e$, then $xy + yx = sx + ry + e(x+y)$. If $ex \neq 0$, then $p = 2$ and $2ex = 0$. In this event, $(x+2y)^2 = (r+2s)\,(x+2y) + 2ey$. Here $2ey$ must be a multiple of $x + 2y$ which implies $o(x) = o(y)$.

PROPOSITION 1.3. *Let R be a ring on $G = \bigoplus\langle x_i \rangle$, a bounded p-group. Then R has property S iff there are constants u_i and v_i for each i such that:*

(1) $x_i x_j = v_j x_i + u_i x_j$, *for every i and j, or if $2^n G = 0$, $r(2^{n-1}G) = 2$ for some n,*

(2) $x_i x_j = v_j x_i + u_i x_j$, *for $i \neq j$ and $x_i^2 = (u_i + v_i + 2^{n-1})x_i$, for every i.*

Proof. (a) Sufficiency. Let $\alpha = \sum a_i x_i$, $\beta = \sum b_j x_j$ be elements in R. If (1) is given,

$$\alpha\beta = \Big(\sum_j b_j v_j\Big)\,\alpha + \Big(\sum_i a_i u_i\Big)\,\beta$$

and we are done. If (2) is given,

$$\alpha\beta = \Big(\sum_j b_j v_j\Big)\,\alpha + \Big(\sum_i a_i u_i\Big)\,\beta + 2^{n-1}\sum a_i b_i x_i.$$

The right-hand sum is a linear combination of α and β if and only if $a_i b_i \equiv k a_i + 1 b_i \pmod 2$ for some k and 1 and all i. This, in turn, is possible if and only if $r(2^{n-1}G) \leqq 2$, as can be shown by considering cases. (b) Necessity. If $r(G) = 1$, (1) is satisfied. We assume

$r(G) \geqq 2$. We first show that, for each i, there exist u'_i and v'_i such that, for $i \neq j$, $x_i x_j = v'_j x_i + u'_i x_j$. For fixed i choose $x_k \neq x_i$ of maximal order possible. Write $x_i x_k = cx_i + dx_k$ and, for $i \neq j \neq k$, $x_i x_j = ax_i + bx_j$. Then $x_i(x_j + x_k) = (a + c)x_i + (bx_j + dx_k)$. The right-hand term must be a multiple of $x_j + x_k$ and, hence, $dx_j = bx_j$. Similarly, if $x_k x_i = c_0 x_i + d_0 x_k$ and $x_j x_i = a_0 x_i + b_0 x_j$, an examination of $(x_j + x_k)x_i$ shows that $d_0 x_j = b_0 x_j$. We let $u'_i = d$, $v'_i = d_0$ and do this for each i. Then, $x_i x_j = v'_j x_i + u'_i x_j$ for $i \neq j$. Next, for each i, $x_i^2 = (u'_i + v'_i + e_i)x_i$ for some e_i such that $0 \leqq e_i < o(x_i)$. Write $r_i = u'_i + v'_i + e_i$. If $i \neq j$,

$$x_i x_j + x_j x_i = (u'_j + v'_j)x_i + (u'_i + v'_i)x_j = r_j x_i + r_i x_j - (e_j x_i + e_i x_j).$$

If the right-hand term equals zero for all $i \neq j$, we let $u_i = u'_i$, $v_i = v'_i + e_i$ and condition 1 of the theorem will be satisfied. Suppose, for some $i \neq j$, $e_j x_i + e_i x_j \neq 0$ where $o(x_i) \leqq (x_j)$. If $e_j x_i \neq 0$, $o(x_i) = o(x_j)$ by (1.2). If $e_i x_j \neq 0$, $e_j x_i + e_i x_j$, as a multiple of $x_i + x_j$, equals $e_i(x_i + x_j)$. Here $e_i x_i = e_j x_i \neq 0$, and again $o(x_i) = o(x_j)$. Since both terms in $e_j x_i + e_i x_j$ are nonzero, the sum would remain nonzero if, say, i or j were replaced by k where x_k has maximal order p^n in R^+. It means $o(x_i) = o(x_j) = p^n$. By (1.2), e_i and e_j may be replaced by 2^{n-1} and $r(2^{n-1}G) = 2$ (we already knew $r(2^{n-1}G) \leqq 2$). It follows that x_i^2 equals, for each i, $(u'_i + v'_i + 2^{n-1})x_i$. Let $u_i = u'_i$, $v_i = v'_i$ for every i. The proof is complete.

We now consider how a S-ring can be constructed on a bounded primary group. In general, if group $G = \bigoplus \langle x_i \rangle$, a ring (not necessarily associative) can be constructed on G in the following way (see Theorem 120.1 of **[4]**). For each i, j, let $x_i x_j$ be an element in G subject to the sole condition: $o(x_i x_j) \leqq \min\,(o(x_i), o(x_j))$ and define multiplication on the rest of G linearly. We note that these two conditions are also necessary. We may now state a theorem.

THEOREM 1.4. *Let $G = \bigoplus \langle x_i \rangle$ be a bounded p-group where $o(x_i) = n_i$. A S-ring can be constructed on G by setting*

(1) $x_i x_j = v_j x_i + u_i x_j$, *and*
(2) $(\sum a_i x_i)(\sum b_j x_j) = \sum_{i,j} a_i b_j x_i x_j$, *where*
(3) $n_i u_i G = n_i v_i G = 0$, *and*
(4) $u_i v_j x_k = 0$ *unless* $i = j = k$.

If $p \neq 2$, any p-primary S-ring, not a zero-ring, is of this type.

Proof. By (1.3) and the preceding paragraph, conditions 1–3 are sufficient for constructing a S-ring (not necessarily associative) on G. Conversely, let R be a p-primary S-ring, not a zero-ring. Its additive group has the form G by (1.1) and, by (1.2) and the paragraph above, conditions 1–3 are necessary. We now show that (4) is necessary and sufficient for associativity. Multiplication is associative if and only if, for all i, j, k, $(x_i x_j)x_k = x_i(x_j x_k)$ or, by computation, if and only if $u_j v_k x_i = v_j u_i x_k$ for every i, j, k. Clearly, (4) implies this equation. Conversely, suppose this equation is given. If $i \neq k$, x_i and x_k are linearly independent and $u_i v_j x_k = 0$. If $j \neq k$, we substitute, in the above equation, k, i, j for i, j, k respectively, and derive $u_i v_j x_k = v_i u_k x_j$, and again $u_i v_j x_k = 0$. Property (4) is established.

REMARK. A S-ring on a 2–group, however, need not satisfy condition 1 of the preceding theorem. An example is the ring-direct sum $Z/(2) \oplus Z/(2)$, which we call M. The following theorem will allow for the ring M.

THEOREM 1.5. *Let G be as in* (1.4) *where* $2^n G = 0$ *and* $r(2^{n-1}G) = 2$ *for some n. A S-ring R can be constructed on G by setting*

(1′) $x_i x_j = v_j x_i + u_i x_j$, *for* $i \neq j$ *where* $x_i^2 = (u_i + v_i + 2^{n-1})x_i$, *for every* i,

(5) *all* u_i, v_i *are even*,

and satisfying conditions 2–4 *of* (1.4). *A S-ring on a 2–group, not a zero-ring, has this construction or that of* (1.4).

Proof. (a) Let G and multiplication on G be as given. Conditions 1′, 2, and 3 define a S-ring R (not necessarily associative) on G for the same reasons as in (1.4). We claim (4) and (5) ensure $x_i(x_j x_k) = (x_i x_j)x_k$ for all i, j, k. If $i = j = k$, this is clear. If i, j, k are all distinct, these products are computed exactly as in (1.4) and the equation is true by (4). Suppose, in the equation, exactly two subscripts agree. Again, the value of each product in G is as in (1.4). This is so because each term involving the number 2^{n-1} disappears. We illustrate this with one example. In $x_i(x_i x_j) = v_j(u_i + v_i + 2^{n-1})x_i + u_i x_i x_j$, $v_j 2^{n-1} x_i = 0$ by (5). Therefore, R is associative. (b) We now verify the last sentence of the theorem. Suppose R is a S-ring on a 2–group G but is neither a zero-ring nor structured as in (1.4). By (1.3), (1′) is necessary together with (2) and (3). If all u_i and v_i are even, the products $x_i(x_j x_k)$ and $(x_i x_j)x_k$ are computed as in (1.4) unless $i = j = k$ and (4) is again needed. If (5) is not satisfied, we will prove R is isomorphic to M, the ring mentioned in the Remark above. That M has a construction satisfying the conditions of this theorem is routine to show. Therefore, suppose (5) is not satisfied and u_i or v_i is odd for fixed i. Consider $x_i^2 x_j = x_i(x_i x_j)$ for $j \neq i$ and $o(x_j) = p^n$. If we compute this equation in G and equate the two x_j-terms, we obtain $(u_i + v_i + 2^{n-1})u_i x_j = u_i^2 x_j$ or $(v_i + 2^{n-1})u_i x_j = 0$. Similarly, from $x_j x_i^2 = (x_j x_i)x_i$, we obtain $(u_i + 2^{n-1})v_i x_j = 0$. We deduce from these equations that u_i and v_i have the same parity. Thus, u_i and v_i are both odd, $(v_i + 2^{n-1})x_j = 0$, and $n = 1$. Also, $x_i^2 = x_i$ and $x_j^2 = x_j$. From the condition on rank, it follows that $R^+ = \langle x_i \rangle \oplus \langle x_j \rangle$. We claim $x_i x_j = x_j = x_j x_i$. Suppose $x_i x_j = x_i + x_j$ (the only other possibility since u_i is odd). Then $x_i + x_j = x_i^2 x_j = x_i(x_i x_j) = x_j$, a contradiction. Thus, $x_i x_j = x_j$. Similarly, $x_j x_i = x_j$. By writing $R^+ = \langle x_i + x_j \rangle \oplus \langle x_j \rangle$, we readily see that R is isomorphic to M which satisfies the conditions of the theorem.

REMARK. At first glance, the construction of S-rings might seem unduly complicated, but this is not so. Suppose G is the group given in (1.4) with least bound p^N. We can satisfy the conditions of the theorem by, for each i, letting $u_i = r_i$, $v_i = 0$ where p^N divides $r_i n_i$. Another example shows that all u_i, v_i can be nonzero. Let R be the ring on the group $\langle x \rangle \oplus \langle y \rangle$ where $o(x) = 81 = o(y)$, $x^2 = 27x$, $y^2 = 27y$, $xy = 9x + 18y$, and $yx = 18x + 9y$. This ring also has property S.

We now consider torsion S-rings in general.

THEOREM 1.6. *Let G be a torsion group with primary decomposition, $G = \bigoplus_p G_p$. A ring R on G has property S if and only if, for each p, the subring R_p on G_p has property S.*

Proof. The necessity of the condition is clear. We suppose R_p is a S-ring for each prime p and prove $R = \bigoplus R_p$ is a S-ring. Let $\alpha = \Sigma x_p$ and $\beta = \Sigma y_p$ be elements in R where $x_p, y_p \in R_p$ ($x_p = y_p = 0$ for almost all p's). Since, for each p, $x_p y_p = a_p x_p + b_p y_p$ for constants a_p and b_p, $\alpha\beta = \Sigma x_p y_p = \Sigma a_p x_p + \Sigma b_p y_p$. For each p, let $n_p = \max\ (o(x_p), o(y_p))$. By the Chinese Remainder Theorem, there exist constants k and 1 satisfying $k \equiv a_p \pmod{n_p}$ and $1 \equiv b_p \pmod{n_p}$ for each p for which x_p or y_p is nonzero. It follows that $\alpha\beta = k\alpha + 1\beta$ and that R is a S-ring.

2. Torsion-free rings. In this section all rings are assumed to be torsion-free. We first establish, in 5 lemmas, some properties of S-rings. Theorem 2.6 will be the culmination of this effort.

LEMMA 2.1. *If x and y are elements in a S-ring, $x^2 = rx$, and $y^2 = sy$, then $xy + yx = sx + ry$ and $xy = sx$ or ry.*

Proof. First, suppose x and y are linearly independent. The proof that $xy + yx = sx + ry$ is the same as in (1.2), except that k now equals $s + r$ exactly. Next, set $xy = ax + by$, $yx = cx + dy$. Then, $arx + bry = x^2 y = x(xy) = (ar + ab)x + b^2 y$ and $ab = 0 = b(b - r)$. Similarly, $cd = 0 = c(c - r)$. If $b \neq 0$, then $b = r$, $a = 0$, and $xy = ry$. If $c \neq 0$, then $c = r$, $d = 0$, and $yx = rx$ which means that $xy = sx$. If $b = c = 0$, then $xy = sx$ since $a + c = s$. Secondly, suppose x and y are linearly dependent and $mx = ny \neq 0$. By computation, $mxy = m(sx)$ and $xy = sx$. Similarly, $yx = ry$, and, as a result, $xy + yx = sx + ry$.

LEMMA 2.2. *For elements x and y in a S-ring the following are equivalent*:

(1) $x^2 = y^2 = 0$,

(2) $xy = yx = 0$,

(3) $xy + yx = 0$.

Proof. From (2.1), (1) implies (2) and (2) implies (3). We must show that (3) implies (1). If x and y are linearly independent, this follows from (2.1). Suppose $mx = ny \neq 0$. Then $0 = mn(xy + yx) = m(x(ny) + (ny)x) = m(x(mx) + (mx)x) = 2m^2x^2$ and $x^2 = 0$. Similarly, $y^2 = 0$.

LEMMA 2.3. *If K is a subring of the S-ring R and $K^2 = 0$, then K is an ideal.*

Proof. Let $x \in K$, $y \in R \backslash K$. If x and y are linearly dependent, then $xy = yx = 0 \in K$. Assume that they are linearly independent and write $xy = ax + by$. Then $0 = x^2y = x(xy) = x(ax + by) = bxy = abx + b^2y$ and $by = 0$. Thus, xy (similarly yx) is in K.

LEMMA 2.4. *If the S-ring R has rank one, it is a zero-ring or R^+ is cyclic.*

Proof. Assume the lemma to be false. Then there is an element $x \neq 0$ in R^+ of unbounded height. Since $R^2 \neq 0$, $x^2 = ax \neq 0$ for some a. Choose n and y such that $ny = x$ but n is not a factor of a. If $y^2 = by$, by computation $ax = n^2by = nbx$ and n divides a, a contradiction. Therefore, the lemma is true.

LEMMA 2.5. *If R is a S-ring, then it is a zero-ring or it contains an ideal K and an element z such that $R^+ = K^+ \oplus \langle z \rangle$, where $K^2 = 0$ and $z^2 = sz$ for some $s \neq 0$. Also, for every x in K, $xz = ux$ and $zx = (s - u)x$ where $u(s - u) = 0$.*

Proof. Suppose R is not a zero-ring and K is a maximal subring in R such that $K^2 = 0$. By (2.3), K is an ideal and we may form the quotient ring $R/K = L$. We will show that L^+ is cyclic and torsion-free. First, L has property S. For, if D is a subgroup of L^+, so is its inverse image in R^+ and closure under multiplication in R implies it in L. Secondly, L is torsion-free. Suppose $y \neq 0$ but $ny = 0$ in L. If x in R maps to y, then nx is in K and $(nx)^2 = 0$. If $x^2 = 0$, the subring generated by x and K is a zero-ring by (2.2) which contradicts the maximality of K. Thus, $x^2 \neq 0$, $n = 0$, and L is torsion-free. Thirdly, we prove that L has rank one. By the previous argument, it suffices to show that $r(L) > 1$ implies $R \backslash K$ has a nonzero element whose square is zero. Suppose, then, $r(L) > 1$ and x and y are linearly independent in $R \backslash K$. If $x^2 = rx$, $y^2 = sy$, then, by (2.1), $xy + yx = sx + ry$ and $(sx - ry)^2 = 0$. Therefore, the rank of L must be one. By the maximality of K, L is not a zero-ring. By (2.4), L^+ is cyclic and $R^+ = K^+ \oplus \langle z \rangle$ for z where $z^2 \neq 0$. Finally, we verify the last sentence of the theorem. Let x, y be elements in K. By (2.1) it will suffice to show that $yz = uy$ implies $xz = ux$. Since K is an ideal, $xz = ax$ for some a. Suppose x and y are linearly independent. Then $(x + y)z = ax + uy$ must be a multiple of $x + y$ and $xz = ux$. Suppose $mx = ny \neq 0$. Then $max = mxz = nyz = nuy = umx$ and $a = u$, as desired. The proof is complete.

We can now describe all S-rings on torsion-free groups.

THEOREM 2.6. *Let G be a torsion-free group of the form $H \oplus \langle z \rangle$. A S-ring can be constructed on G by, for x, $y \in H$, setting $(x + az)\ (y + bz) = bux + a(s - u)y + absz$, where $u(s - u) = 0$. Any torsion-free S-ring, not a zero-ring, has such a structure.*

Proof. If R is a S-ring and $R^2 \neq 0$, then the structure of R is as above by (2.5). Conversely, suppose the above structure is given. It is easy to check that multiplication is well-defined and that the distributive law holds. Let $\alpha = x + az$, $\beta = y + bz$, $\gamma = w + cz$, where x, y, $w \in H$. We may assume $u = 0$ (the proof is similar if $u = s$). Then $\alpha\beta = as\beta$ and property S is assured. Since $(\alpha\beta)\gamma = (as)\ (bs)\gamma = \alpha(\beta\gamma)$, the associative law is satisfied. Therefore, the construction defines a S-ring.

3. Mixed rings.

In this section the ring R is always mixed and T is its torsion subring. It is well-known that the torsion subring of a ring is an ideal. If R is a S-ring, we can say more.

PROPOSITION 3.1. *If R is a S-ring, $T^2 = 0$.*

Proof. We first show that $x^2 = 0$ for every x in T. Let $x \in T$, $y \in R \backslash T$, where $o(x) = n$, $x^2 = rx$, and $y^2 = sy$. Then $(x + ny)^2 = rx + n^2sy = k(x + ny)$ for some k. Hence, $k = ns$ and $x^2 = nsx = 0$. To complete the proof, it will suffice to show $xz = 0$ for linearly independent x and z in T. We write $xz = ax + bz$ and, since T is an ideal, $yz = cz$. Then $(x + y)z = ax + (b + c)z$ and ax must be a multiple of $x + y$ which implies $ax = 0$. Similarly, we can show $bz = 0$ by considering $x(z + y)$. Therefore, $xz = 0$ and, as a result, $T^2 = 0$.

If R is a mixed S-ring, not a zero-ring, then R/T may or may not be a zero-ring. We consider the second case first and describe the structure of R when $(R/T)^2 \neq 0$ in Theorem 3.5. To prove it three lemmas are required.

LEMMA 3.2. *If R is a S-ring and R/T is not a zero-ring, then R has an ideal K containing T and an element z such that $K^2 = 0$, $z^2 \neq 0$, and $R^+ = K \oplus \langle z \rangle$.*

Proof. By (2.5), $(R/T)^+ = (K/T) \oplus \langle z + T \rangle$ for some ideal K/T and element $z + T$ such that $(K/T)^2 = 0$, $(z + T)^2 \neq 0$. By the isomorphism theorems for groups, $R^+ = K^+ \oplus \langle z \rangle$, where $K \supset T$ and $z^2 \neq 0$. Since T and K/T are ideals, so is K. We must show $K^2 = 0$. Since any two elements of K are contained in a subgroup of torsion-free rank at most two, it will suffice to show $K^2 = 0$ where $K^+ = T \oplus \langle x \rangle \oplus \langle y \rangle$ for some x and y. Since K/T is a zero-ring, so is $\langle x \rangle \oplus \langle y \rangle$ as well as T. We will show $tx = 0$ for t in T and it will follow, by symmetry, that $K^2 = 0$. Since T and K are ideals, $tx = at$, $zx = bx$ for some a and b. Then, $(t + z)x = at + bx$ and at is a multiple of $t + z$ which implies $tx = at = 0$. This completes the proof.

For the rest of this section, T_i is the p_i-component of T where $\{p_i\}$, $i \in I$, is the set of prime numbers.

LEMMA 3.3. *Let R be a S-ring, $R^+ = K \oplus \langle z \rangle$, where $K \supset T$, $K^2 = 0$, and $z^2 = sz$. Then:*

(1) *if K is mixed, for every x in K, $xz = ux$, $zx = (s - u)x$ where $u(s - u) = 0$;*

(2) *if $K = \bigoplus T_i$, there exists u_i for each i such that $x_iz = u_ix_i$, $zx_i = (s - u_i)x_i$, and $u_i(s - u_i)x_i = 0$ for every x_i in T_i.*

Before proving this lemma, we establish the following.

LEMMA 3.4. *Let R be as in* (3.3). *If $x, y \in K$ and $o(y)$ is infinite or $o(x)$ divides $o(y)$, then $yz = uy$ implies $xz = ux$.*

Proof. If x and y are linearly independent or if $o(x)$ is infinite, the argument at the end of the proof of (2.5) may be used to prove the lemma. If x and y are dependent and $o(x)$ is a finite number, then $x = w + ay$ for element w and constant a such that w is independent of y and $o(w)$ divides $o(y)$. By the previous sentence applied to w and y, $wz = uw$. Again $xz = ux$.

Proof of 3.3. In general, if $x \in K$ and $xz = ux$, then $zx = (s - u)x$ and $u(s - u)x = 0$. To see this, consider $(x + z)^2 = xz + zx + sz$. Since this expression must be a multiple of $x + z$, $xz + zx = sx$ or $zx = (s - u)x$. Also, $sux = sxz = x(sz) = xz^2 = u^2x$ and $u(s - u)x = 0$. Suppose, then, K is mixed. If $y \in K \backslash T$, and $yz = uy$, then $u(s - u) = 0$ and, by (3.4), $xz = ux$, for every x in K. We have proved (1). Assume, then, $K = T$ as in (2). We will examine the following possible cases: (a) T_i is bounded, (b) T_i is not reduced, and (c) T_i is unbounded and reduced. (a) Let T_i be bounded. Let y be an element of maximal order in T_i and $yz = u_iy$. Then (2) follows from (3.4) and from what was said at the beginning of this proof. (b) Let T_i^+ contain a divisible subgroup D of rank one. We claim $Dz = 0$ or $zD = 0$. Suppose this is not so. Then D contains an element y such that neither yz nor zy is zero. For $n > 0$, there exists x such that $p^nx = y$, and for some u, $xz = ux$ and $zx = (s - u)x$ by what was said above. But, if n is large relative to $o(y)$, uy or $(s - u)y$ is zero since $u(s - u)x = 0$, although $uy = yz \neq 0 \neq zy = (s - u)y$. From this contradiction we conclude that $Dz = 0$ or $zD = 0$ and $yz = uy$ for all y in D where $u(s - u) = 0$. If $x \in T_i \backslash D$, we may select $y \in D$ such that $o(x)$ divides $o(y)$ and (3.4) completes tne proof of (2) in case (b). (c) Suppose T_i is reduced with unbounded basic subgroup B. First, assume $Bz \neq 0 \neq zB$. We show a contradiction. If $xz \neq 0 \neq zy$ for x, y in B and $o(x)$ divides $o(y)$, then $yz \neq 0$ by (3.4). In this event, we select $w \in B$ of large order relative to that of y. Then, if $wz = uw$, $zw = (s - u)w$, $uy = yz \neq 0 \neq zy = (s - u)y$. But $u(s - u)w = 0$ and the largeness of $o(w)$ imply uy or $(s - u)y$ equals zero. Therefore, $Bz = 0$ or $zB = 0$ and, for all $x \in B$, $xz = ux$ where $u(s - u) = 0$. If $g \in T_i \backslash B$, we may choose $x \in B$ such that $o(g) \leqq o(x)$ and, by (3.4), $gz = ug$. We have established (2) for case (c).

THEOREM 3.5. *Let $G = H \oplus \langle z \rangle$ be a mixed group where $H \supset T$. A S-ring R can be constructed on G by defining*:

(1) *if $H \neq T$, $(x + az)\ (y + bz) = a(s - u)y + bux + absz$ for x, $y \in H$ where $u(s - u) = 0$;*

(2) *if $H = \oplus T_i$, $(\Sigma x_i + az)(\Sigma y_i + bz) = a\,\Sigma(s - u_i)y_i + b\,\Sigma u_ix_i + absz$ for $x_i, y_i \in T_i$ where $u_i(s - u_i)T_i = 0$ for each i. If R is a mixed S-ring and R/T is not a zero-ring, R has the above structure with $s \neq 0$.*

Proof. We prove the last sentence first. If R is a mixed S-ring and R/T is not a zero-ring, R has the structure of the theorem by (3.2), (3.3), and the distributive law. Therefore, we assume G is as given and show that (1) and (2) define S-rings on G. The proof for definition (1) is the same as in (2.6). We assume (2) is given. A routine calculation shows that multiplication is well-defined and that the distributive law holds. This leaves associativity and property S to be verified. Let

$\alpha = \Sigma x_i + az$, $\beta = \Sigma y_i + bz$, $\gamma = \Sigma w_i + cz$, where x_i, y_i, $w_i \in T_i$ and let $n_i = \max(o(x_i), o(y_i), o(w_i))$. By the Chinese Remainder Theorem, we can find k such that $k \equiv u_i \pmod{n_i}$ for each i such that x_i, y_i, or w_i is nonzero. Then $\alpha\beta = bk\alpha + a(s-k)\beta$ and property S is assured. Finally, $(\alpha\beta)\gamma = bck^2\alpha + ack(s-k)\beta + abs(s-k)\gamma = \alpha(\beta\gamma)$ and the ring is associative. The proof is complete.

If R is a S-ring, it may happen that R/T is a zero-ring but R is not. In this case, as we shall see, the additive group of R/T has rank one but may not be cyclic. Hence, a few words on mixed groups of torsion-free rank one are in order. Let G be such a group and $T = \bigoplus T_i$, $i \in I$, its torsion subgroup (T_i is the p_i-component of T where p_i is a prime number). Let z be a torsion-free element in G. We partition I into subsets J and K such that i is in J or K exactly if the p_i-height of $z + T$ in G/T is infinite or finite, respectively. The latter height is designated n_i. In this case G has a generating set:

$$A = \{T_i, z_{jn}, z_k\},\ i \in I,\ j \in J,\ k \in K, \qquad n = 1, 2, \cdots$$

where

$$(*) \qquad \begin{aligned} z &= p_j^n z_{jn} + t_{jn}, \qquad t_{jn} \in T_j \\ &= p_k^{n_k} z_k + t_k, \qquad t_k \in T_k. \end{aligned}$$

Here $z_k = z$ if $n_k = 0$. We write $G = (A, *)$. Each element α in G has a unique expression of the form:

$$(**) \qquad \alpha = \sum x_i + \sum_j a_j z_{jm_j} + \sum b_k z_k + cz,$$

where $x_i \in T_i$, m_j depends on j, $(a_j, p_j) = 1$, $|a_j| < p_j^{m_j}$, and $|b_k| < p_k^{n_k}$. The last sentence can be seen by examining the image of α in G/T.

We are ready to state and prove the final theorem of this section.

THEOREM 3.6. *Let $G = (A, *)$. A S-ring R can be constructed on G in the following manner. For each i in I, choose u_i such that $u_i^2 T_i = 0$ and, if $i \in J$, $u_i = 0$. For each i and $k \neq i \neq j$, choose u_{ki} and u_{ji} such that $u_{ki} = 0 = u_{ji}$ if T_i is unbounded and $p_k u_{ki} \equiv 1 \equiv p_j u_{ji} \pmod{p_i^{N_i}}$ if T_i has least bound $p_i^{N_i}$. Multiplication on G is defined by the following rules:*

(1) *$T^2 = 0$, and $xy = -yx$ for $x, y \in A$.*

(2) *For $x_i \in T_i$, z_k, and z_{jn},*

$$x_i z_k = u_{ki}^{n_k} p_i^{n_i} u_i x_i$$

and

$$x_i z_{jn} = u_{ji}^n p_i^{n_i} u_i x_i \qquad (n_i = 0 \quad \textit{if} \quad i \in J).$$

(3) *For $i \in K$ and z_k, z_{jn},*

$$z_i z_k = u_{ik}^{n_i} u_k t_k - u_{ki}^{n_k} u_i t_i$$

and

$$z_i z_{jn} = -u_{ji}^n u_i t_i.$$

(4) *For $i \in J$ and z_{im}, z_{jn},*

$$z_{im} z_{jn} = 0.$$

(5) *Multiplication on G is linear relative to A. If R is a S-ring and R/T but not R is a zero-ring, then R has the above structure.*

Proof. (a) We prove the last sentence first. Assume S-ring R is not a zero-ring but R/T is. We claim $r(R/T) = 1$. To show this, it will suffice to prove R is a zero-ring if $R^+ = T \oplus \langle x \rangle \oplus \langle y \rangle$, for some x and y. We know $T^2 = 0$. Since R/T is a zero-ring, so is the ring on $\langle x \rangle \oplus \langle y \rangle$. If $t \in T$, $(t + y)x = tx$ is in T and must be a multiple of $t + y$. Thus, $tx = 0$. Other products in R are zero by symmetry, and $R^2 = 0$. Therefore, $r(R/T) = 1$, and we let $R^+ = G = (A, *)$. We proceed to verify properties 1–5. Throughout this proof we assume k and j are always in K and J respectively. First, since T and R/T are zero-rings, it follows that, for x, $y \in R$, $x^2 = y^2 = 0 = (x + y)^2 = xy + yx$ and, hence, (1) is necessary. If $i \in K$, consider the subgroup $T_i \oplus \langle z_i \rangle$. By (3.3) with $s = 0$, there is a constant u_i such that $u_i^2 T_i = 0$ and $x_i z_i = u_i x_i$, for every x_i in T_i. If $i \in J$ and $x_i \in T_i$, $x_i z_{in} = 0$ for all n because of the height condition on z_{in} (modulo T_i) and, as a result, $x_i z_{in} = u_i x_i$ where $u_i = 0$. We have proved (2) for $i = k$ or j. Next, for fixed i and $k \neq i \neq j$, let u_{ki} and u_{ji} be as stated in the theorem. If $i \in K$, by reason of $(*)$,

$$p_i^{n_i} u_i x_i = x_i (p_i^{n_i} z_i + t_i) = x_i (p_k^{n_k} z_k + t_k) = x_i (p_j^n z_{jn} + t_{jn}).$$

If $i \in J$, the same equations hold true when, for $m > 0$, p_i^m and z_{im} replace $p_i^{n_i}$ and z_i, respectively. These equations, then, imply (2) for the remaining possible cases. We next verify (3). By (1), any element in A squared is zero and (3) is true for $i = k$. For $i \in K$ but $i \neq k$, consider $p_i^{n_i} z_i + t_i = p_k^{n_k} z_k + t_k$. Multiplying on the right (left) by z_k (z_i), we derive:

$$p_i^{n_i} z_i z_k = u_k t_k - u_{ki}^{n_k} p_i^{n_i} u_i t_i \quad \text{and} \quad p_k^{n_k} z_i z_k = u_{ik}^{n_i} p_k^{n_k} u_k t_k - u_i t_i.$$

Since $z_i z_k \in T_k \oplus T_i$, $z_i z_k$ has the form given in (3). Similarly, considering $p_i^{n_i} z_i + t_i = p_j^n z_{jn} + t_{jn}$ for $i \in K$ and multiplying right (left) by z_{jn} (z_i), we derive the second equation of (3). We next prove (4). For $i \in J$ but $i \neq j$, consider $p_i^m z_{im} + t_{im} = p_j^n z_{jn} + t_{jn}$. If we multiply right (left) by z_{jn} (z_{im}), we obtain $p_i^m z_{im} z_{jn} = 0 = p_j^n z_{im} z_{jn}$ which implies $z_{im} z_{jn} = 0$. Suppose, however, $i = j$ and $m > n$. Then $z_{jn} = p^{m-n} z_{im} + x_i$ for some $x_i \in T_i$, and $z_{im} z_{jn} = p^{m-n} z_{im}^2 + z_{im} x_i = 0$. We have proved (4). Since (5) is obvious, the proof of the final sentence of the theorem is complete. (b) Suppose G and multiplication on G are as given in the theorem. We must prove that multiplication is well-defined and that the associative and S properties are present. We first observe that the second equation of (1) is consistent with the first equation of (3) and of (4). We now prove multiplication is well-defined. First, if $x \in A$ and y, y_1 are two representatives of z given by $(*)$, we claim $xy = xy_1$ and it is meaningful to write $xz = xy = xy_1$. To prove this, we consider cases. If $x \in T_i$, a reversal of the proof of (2) above yields the desired result. If $x = z_i$, $i \in K$, a reversal of the proof of (3) above yields the desired result. If $x = z_{jn}$, then $xy = 0$ for every representative y of z. We compute the less obvious product. Let $z = p_i^{n_i} z_i + t_i$, $i \in K$. Then

$$\begin{aligned} z_{jn}(p_i^{n_i} z_i + t_i) &= -p_i^{n_i} z_i z_{jn} - t_i z_{jn} \\ &= p_i^{n_i} u_{ji}^n u_i t_i - u_{ji}^n p_i^{n_i} u_i t_i = 0. \end{aligned}$$

Secondly, if $x \in A$ and $z_{jn} = p_j z_{jn+1} + x_{jn}$ where $x_{jn} \in T_j$, we claim $xz_{jn} = x(p_j z_{jn+1} + x_{jn})$. Again we consider cases. If $x \in T_i$, (2) yields the desired result. If $x = z_i$, $i \in K$, then, by (3),

$$\begin{aligned} z_i z_{jn} &= -u_{ji}^n u_i t_i = p_j(-u_{ji}^{n+1} u_i t_i) = p_j(z_i z_{jn+1}) \\ &= z_i(p_j z_{jn+1} + x_{jn}), \end{aligned}$$

as desired. If $x = z_{im}$, $i \in J$, then $z_{im} z_{jn} = 0 = z_{im}(p_j z_{jn+1} + x_{jn})$, as desired. Thirdly, multiplication on all of G is well-defined. By reason of (1) and (5) it will suffice to show that $\alpha\beta = \alpha'\beta$ where $\alpha = \alpha'$, α is the representative of $(**)$, and β is arbitrary in G. Suppose, then, $\alpha' = \Sigma_i y_i + \Sigma_j(\Sigma_n d_{jn} z_{jn}) + \Sigma_k e_k z_k + fz$ where $y_i \in T_i$. For fixed j and $n < m_j$, by what was said above, $d_{jn} z_{jn}$ may be replaced by $d_{jn} p_j z_{jn+1} + d_{jn} x_{jn}$ in computing $\alpha'\beta$. If $n > m_j$, by the uniqueness of the form of α, $d_{jn} = p_j d'_{jn}$ for some d'_{jn} and $d_{jn} z_{jn} = d'_{jn} p_j z_{jn} = d_{jn}(z_{jn-1} - x_{jn-1})$. We replace the left term by the right term in computing $\alpha'\beta$. Continuing in this manner and simplifying, for each j, we may replace $y_j + \Sigma_n d_{jn} z_{jn}$ by $x_j + a_j z_{jm_j} + f_j z$ for some f_j. In like manner, for each k, we may replace $y_k + e_k z$ by $x_k + b_k z_k + f_k z$ for some f_k. The sum of the f_j's, f, and the f_k's must equal c and it follows that $\alpha'\beta = \alpha\beta$. Multiplication is well-defined. Finally, we establish the associative law and property S. It will suffice to show them for a subring R_1 of R where R_1 is finitely generated over T. Since any finitely generated subgroup of the group G/T is cyclic, we may assume $R_1^+ = T \oplus \langle y \rangle$ where y is a linear combination of members of $A \setminus T$. By reason of (1) and (2) and linearity, there exists a constant v_i for each i such that $x_i y = v_i x_i = -yx_i$ for every x_i in T_i and $v_i^2 T_i = 0$. By the proof of (3.5), R_1 is associative and possesses property S. The proof of the theorem is complete.

4. Special properties.

The additive group of an Artinian, Noetherian, or regular ring belongs to a narrowly defined and well-known class of groups (see chapter XVII of **[4]**). This knowledge together with the theorems of the preceding sections enable us to decide the nature of particular classes of S-rings. We describe the classes that seem most important in the following corollaries.

COROLLARY 4.1. *A S-ring R has an identity if and only if R is isomorphic to Z, to $Z/(2) \oplus Z/(2)$ (ring-direct sum), or to $\oplus Z/(p^{n_p})$ summed over a finite number of distinct primes p where n_p depends on p.*

COROLLARY 4.2. *The rings Z and $Z/(p)$ are the only integral domains with property S.*

COROLLARY 4.3. *A nontrivial S-ring R is regular if and only if it has the form $\oplus Z/(p)$ (ring-direct sum) where each $p \neq 2$ ($p = 2$) occurs at most once (twice).*

COROLLARY 4.4. *A S-ring R is Artinian if and only if it is a torsion ring of finite rank.*

COROLLARY 4.5. *A S-ring R is Noetherian if and only if R^+ is a direct sum of a finite number of cyclic groups.*

In the following let T be the torsion subring of R and T_i be the p_i-component of T for prime number p_i. Let $\oplus$ signify ring-direct sum and let (z) and (z_i) signify the subring generated by z and z_i, respectively.

COROLLARY 4.6. *A ring R has the property that every additive subgroup is an ideal if and only if*

(1) *R is a zero-ring,*

(2) *$R = T \oplus (z)$ and $T^2 = sT$ where $z^2 = sz$, or*

(3) *$R = \bigoplus T_i$ and, for $T_i^2 \neq 0$, $T_i = H_i \bigoplus (z_i)$ and $H_i^2 = 0 = s_i H_i$ where $z_i^2 = s_i z_i$.*

The last corollary is not new and may be found in **[5]**. We include it here because it follows readily from the theorems of the paper.

References

1. R. A. Beaumont, *Rings with additive group which is the direct sum of cyclic groups*, Duke Math. J., **15** (1948), 367–369.
2. R. A. Beaumont and R. S. Pierce, *Torsion-free rings*, Illinois J. Math., **5** (1961), 61–98.
3. L. Fuchs, *Abelian Groups*, Publ. House Hungar. Acad. Science, Budapest, 1958.
4. L. Fuchs, *Infinite Abelian Groups*, Academic Press, N. Y., Vol. I (1970), Vol. II (1973).
5. L. Redei, *Die Vollidealringe*, Monatshefte für Mathematik, **56** (1952), 89–95.

Received February 23, 1976.

University of Detroit