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UNBOUNDED COMPLETELY POSITIVE LINEAR MAPS ON C^*-ALGEBRAS

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We define unbounded, completely positive, operator valued linear maps on C^*-algebras, and investigate their natural order structure. Following F. Combes, J. Math. Pure et Appl., we study the quasi equivalence, equivalence and type of the Stinespring representations associated with unbounded completely positive maps. Following A. van Daele, Pacific J. Math., we study an unbounded completely positive map α with dense domain which is invariant under a group G of *-automorphisms and construct a G-invariant projection map ϕ' of the set $\mathscr{F}$ of continuous completely positive maps dominated by α, onto the set $\mathscr{F}_0$ of G-invariant elements of $\mathscr{F}_0$. This is used to derive various properties of the upper envelope of $\mathscr{F}_0$.

1. Introduction. We investigate the structure of unbounded completely positive linear maps on a C^*-algebra A. Our work generalises that of Combes [**3**] and van Daele [**15**] from scalar valued weights to operator valued ones. Haagerup [**9**] has also introduced a notion of an operator valued weight, which can be described as an unbounded conditional expectation. We note that an operator valued weight in the sense of Haagerup is automatically completely positive by an extension of [**9**, Lemma 4.5].

Recently, various authors [**10, 11, 12, 13**] with different applications in mind, have considered Stinespring-like constructions for certain positive definite operator-valued functions on involutive algebras. Rieffel [**13**] generalised the notion of a conditional expectation on C^*-algebras, and used their Stinespring representations to formulate a theory of induced representations of C^*-algebras. In [**11, 12**] Powers defined an unbounded *-representation of an involutive algebra, and obtained a Stinespring-decomposition with an unbounded *-representation for a completely positive linear map on a *-algebra with identity. Paschke [**10**] has also studied completely positive maps on *-algebras, and obtained the Stinespring decomposition for such a map on a unital *-algebra which is linearly spanned by its unitaries.

In §2 we construct the Stinespring representation for an unbounded completely positive linear map α on A, and begin an analysis of the natural order structure for such maps. In particular we study the family $\mathscr{F}$ of bounded, completely positive, linear maps majorised by α. In §3, when α has dense domain, we are concerned with the construction of a largest operator valued weight α_0 majorised by α, and with the property that it is the upper envelope of continuous completely positive linear maps. The fourth section deals with the quasi-equivalence, equivalence and type of the Stinespring representation associated with operator valued weights, whilst in §5 we see how this study carries over to the enveloping von Neumann algebra. We then study in §6 an unbounded completely positive linear map α with dense domain which is left invariant by a group G of *-automorphisms of A, parallel to **[15]** for scalar valued weights. We construct a G-invariant projection map ϕ' of the set $\mathcal{F}$ of continuous completely positive linear maps dominated by α, onto the set $\mathcal{F}_0$ of G-invariant elements of $\mathcal{F}$. This is used to derive various properties of the upper envelope of $\mathcal{F}_0$.

We have left Hahn-Banach type considerations as in [**3**, Lemma 1.5] for further study.

H will always denote a hilbert space, $B(H)$ the W^*-algebra of all bounded linear maps on H, and $T(H)$ its predual, the Banach space of all trace class operators on H.

2. The order structure for unbounded completely positive linear maps. We first define an unbounded completely positive map on a C^*-algebra, which we wish to regard as an operator valued weight.

Let A be a C^*-algebra. If S is any subset of A, we denote $S \cap A^+$ by S^+. We recall that a face of A^+ is a convex hereditary subcone of A^+, where a subset S of A^+ is said to be hereditary if $0 \leqq x \leqq y$, $y \in S$, $x \in A$ implies that $x \in S$. If F is a face of A^+, then lin F is a *-subalgebra $\mathfrak{M}$ of A, and $\mathfrak{M}^+ = F$. The set $\{x \in A : x^*x \in F\} = L(F)$ is a left ideal of A such that $\mathfrak{M} = L(F)^* L(F)$. A subalgebra $\mathfrak{M}$ of A such that $\mathfrak{M} = \operatorname{lin} \mathfrak{M}^+$ and $\mathfrak{M}^+$ is a face of A^+, is said to be a facial subalgebra of A. For details on these matters see **[4, 6]**.

It is convenient to introduce some more notation. We let $\mathcal{M}_n$ denote the C^*-algebra of all $n \times n$ matrices over the complex numbers. If S is a subset of a C^*-algebra A, let S_n, for each $n \geqq 1$, denote the subset of $A \otimes \mathcal{M}_n$ consisting of all $n \times n$ matrices over S; i.e.

$$S_n = \{[a_{ij}]: a_{ij} \in S,\ 1 \leqq i, j \leqq n\}.$$

If α is a linear map with domain $\mathfrak{M} = \mathfrak{M}_\alpha$ on a C^*-algebra A into another C^*-algebra B, let α_n be the induced linear map with domain $\mathfrak{M}_n$, by applying α elementwise to each matrix over $\mathfrak{M}$. We then say that α is completely positive if

i) $\mathfrak{M}$ is a facial-subalgebra of A.

ii) Given any $n \geqq 1$, $a_i \in \mathfrak{N}_\alpha \equiv L(\mathfrak{M}_\alpha)$ for $i = 1, 2, \cdots, n$ then $\alpha_n[a_i^* a_j] = [\alpha(a_i^* a_j)] \geqq 0$ in B_n.

We denote by $cp(A, B)$ all (possibly unbounded) completely positive linear maps from the C^*-algebra A into B. We let $cp(A; H)$ denote $cp(A, B(H))$. As usual, **[1]**, $CP(A; H)$ denotes the everywhere defined, completely positive (CP) linear maps from A into $B(H)$, i.e. $\{\alpha \in cp(A; H): \mathfrak{M}_\alpha = A\}$.

We are grateful to M. D. Choi **[2]** for providing us with a proof of the first part of the following proposition. We give an alternative proof which allows us to deduce the second part simultaneously.

PROPOSITION 2.1. *Let A be a C^*-algebra and $\mathfrak{M}$ a facial subalgebra of A. Then for each n in* $\mathbf{N}$, *$\mathfrak{M}_n$ is a facial subalgebra of A_n, and $L(\mathfrak{M}_n) = L(\mathfrak{M})_n$.*

Proof. Suppose $a = [a_{ij}] \in A_n^+$, and $b = [b_{ij}] \in \mathfrak{M}_n^+$ with $a \leqq b$. Then $a = x^*x$, where $x \in A_n$. Thus $a_{ij} = \Sigma_k x_{ki}^* x_{kj}$, and for each k,

$$0 \leqq [x_{ki}^* x_{kj}]_{i,j} \leqq [b_{ij}].$$

Hence $0 \leqq x_{ki}^* x_{ki} \leqq b_{ii}$, which shows that $x_{ki} \in \mathfrak{N}$ for each pair (k, i), i.e. $x \in \mathfrak{N}_n$, and $a \in \mathfrak{M}_n$.

This leads to the following characterisation:

PROPOSITION 2.2. *Let A, B be C^*-algebras, and α a linear map with domain $\mathfrak{M} = \mathfrak{M}_\alpha \subseteq A$ into B. Then α is completely positive iff α_n maps $(\mathfrak{M}_{\alpha_n})^+$ into $(B_n)^+$ for all n, i.e. α_n is positive for all n.*

Proof. Suppose $x \in (\mathfrak{M}_n)^+$. Then $x = x^{\frac{1}{2}} x^{\frac{1}{2}}$, where $x^{\frac{1}{2}} \in L(\mathfrak{M}_n) = L(\mathfrak{M})_n$. Hence x is a finite sum of at most n terms of the form $[a_i^* a_j]$ where $a_i \in L(\mathfrak{M}) = \mathfrak{N}_\alpha$.

Note that if α is as in the above proposition, then $(\alpha_n)_m = \alpha_{nm}$. Thus α is completely positive iff α_n is completely positive for all n.

REMARK 2.3. We now show how operator valued maps on facial subalgebras of C^*-algebras naturally arise. Consider the following definition of Haagerup:

DEFINITION 2.4. **[9]**. Let M be a von Neumann algebra and M_* its predual. The extended positive part of M is defined as the set of functions $m: M_*^+ \to [0, \infty)$ satisfying

1) $m(\lambda\varphi) = \lambda m(\varphi)$, $\forall \varphi \in M_*^+$, $\lambda \geqq 0$.
2) $m(\varphi + \psi) = m(\varphi) + m(\psi)$, $\forall \varphi$, $\psi \in M_*^+$.
3) m is lower semicontinuous.

The extended positive part of M is denoted $\hat{M}_+$. Note that each x in M^+

defines an element in $\hat{M}_+$ by $\varphi \mapsto \varphi(x)$, $\varphi \in M_*^+$. Hence we can regard M^+ as a subset of $\hat{M}_+$.

Let A be a C^*-algebra, M a von Neumann algebra and β an additive, positive homogeneous map from A^+ into $\hat{M}_+$; i.e.

(1) $\beta(x+y) = \beta(x) + \beta(y)$, $\forall x, y \in A^+$

(2) $\beta(\lambda x) = \lambda \beta(x)$, $\forall x \in A^+$, $\lambda \geqq 0$.

Then $\mathfrak{M}^+ = \{x \in A^+ : \beta(x) \in M^+\}$ is a face in A^+, and hence is the positive part of a facial subalgebra $\mathfrak{M}$ in A. $\beta|_{M^+}$ then has a unique linear extension to $\mathfrak{M}$, which we denote by $\dot{\beta}$. We then say that β is completely positive if $\dot{\beta}$ is completely positive.

Conversely, suppose $\mathfrak{M}$ is a facial subalgebra of A, and α a linear map with domain $\mathfrak{M}$ into M, satisfying $\alpha(\mathfrak{M}^+) \subseteq M^+$. Then $\alpha|_{\mathfrak{M}^+}$ can certainly be extended in at least one way into an additive, positive homogeneous map β from A^+ into $\hat{M}_+$ as follows:
If $x \in A^+ \backslash \mathfrak{M}^+$, $\varphi \in M_*^+$, $\varphi \neq 0$, define $\hat{\alpha} = \beta$ by

$$\beta(x)(\varphi) = \infty.$$

It is then clear that $\dot{\beta} = \alpha$.

However if H is a hilbert space, and $M = B(H)$, the cases $H = \mathbf{C}$ and $H \neq \mathbf{C}$ are strikingly different. When $H = \mathbf{C}$, the map $\varphi \mapsto (\mathfrak{M}_\varphi, \dot{\varphi})$ gives a bijection between the family of $[0, \infty]$ valued weights on A^+, and the family of pairs $(\mathfrak{M}, \tau)$ where $\mathfrak{M}$ is a facial subalgebra of A and τ a positive linear functional on $\mathfrak{M}$. But if $H \neq \mathbf{C}$, and $\mathfrak{M}$ a facial subalgebra of A, with α a positive linear map of $\mathfrak{M}$ into $B(H)$, then by considering simple examples, (e.g. direct sums of scalar valued weights), $\alpha|_{\mathfrak{M}^+}$ may have more than one extension to an additive positive homogeneous map from A^+ into $B(H)_+^n$, even if α is completely positive and $\mathfrak{M}$ is norm dense in A.

Having made these preliminary remarks, we now sketch our construction of the Stinespring representation. Let A be a C^*-algebra, and $\alpha \in cp(A; H)$. Define $\mathfrak{N}_\alpha = L(\mathfrak{M}_\alpha)$. We define a bilinear form $\langle \ , \ \rangle$ on the algebraic tensor product $\mathfrak{N}_\alpha \odot H$ as follows:
If $x_i, y_j \in \mathfrak{N}_\alpha$, $\gamma_i, \eta_j \in H$, $i = 1, \cdots, m$ $j = 1, \cdots, n$ put

$$\left\langle \sum_i x_i \otimes \gamma_i, \sum_j y_j \otimes \eta_j \right\rangle = \sum_{i,j} \langle \alpha(y_j^* x_i)\gamma_i, \eta_j \rangle$$

This bilinear form is positive semidefinite as α is completely positive. For each x in A, define a linear transformation $\pi_0(x)$ on $\mathfrak{N}_\alpha \odot H$ by

$$\pi_0(x) \colon \sum_i x_i \otimes \gamma_i \mapsto \sum_i x x_i \otimes \gamma_i$$

Then π_0 is an algebra homomorphism for which

$$\langle \pi_0(x)u, v\rangle = \langle u, \pi_0(x^*)v\rangle \qquad \forall u, v \in \mathfrak{N}_\alpha \odot H.$$

Now suppose $u = \sum x_i \otimes \gamma_i \in \mathfrak{N}_\alpha \odot H$, where $x_i \in \mathfrak{N}_\alpha$, $\gamma_i \in H$, $i = 1, \cdots, n$, and $x \in A$. Define $y_i = [\|x\|^2 - x^*x]^{\frac{1}{2}}x_i$ for $i = 1, \cdots, n$. Since α is completely positive, we have

$$\sum_{i,j=1}^{n} \langle \alpha(y_j^* y_i)\gamma_i, \gamma_j\rangle \geqq 0.$$

This means that

$$\langle \pi_0(x)u, \pi_0(x)u\rangle \leqq \|x\|^2\langle u, u\rangle.$$

Let $N_\alpha = \{u \in \mathfrak{N}_\alpha \odot H : \langle u, u\rangle = 0\}$. Then N_α is a linear subspace of $\mathfrak{N}_\alpha \odot H$, invariant under $\pi_0(x)$ for each x in A. $\langle\ ,\ \rangle$ determines an inner product on $\mathfrak{N}_\alpha \odot H/N_\alpha$, and let $\mathfrak{K}_\alpha$ be its hilbert space completion. If $\Lambda_\alpha : \mathfrak{N}_\alpha \odot H \to N_\alpha \odot H/N_\alpha$ denotes the canonical projection, there exists an unique representation π_α of A on $\mathfrak{K}_\alpha$ such that

$$\pi_\alpha(x)\Lambda_\alpha y \otimes \gamma = \Lambda_\alpha xy \otimes \gamma \qquad \forall x \in A,\ y \in \mathfrak{N}_\alpha,\ \gamma \in H.$$

Moreover

$$\langle \pi_\alpha(x)\Lambda_\alpha y \otimes \gamma, \Lambda_\alpha z \otimes \eta\rangle = \langle \alpha(z^*xy)\gamma, \eta\rangle \ \forall x \in A,\ y, z \in \mathfrak{N}_\alpha,\ \gamma, \eta \in H.$$

If $\alpha \in cp(A; H)$, we will use as standard notation the objects $\mathfrak{N}_\alpha$, N_α, Λ_α, $\mathfrak{K}_\alpha$, π_α constructed above.

We define a partial ordering $\geqslant$ on the cone $cp(A; H)$, by $\alpha \geqslant \beta$, $\alpha, \beta \in cp(A; H)$ if

(1) $\mathfrak{M}_\alpha \subseteq \mathfrak{M}_\beta$.

(2) The linear map $\alpha - \beta$ with domain $\mathfrak{M}_\alpha$ lies in $cp(A; H)$.

With this order structure on $cp(A; H)$ we can show the following theorem, which generalises [**15**, Lemma 2.3] for scalar valued weights, and [**1**] for bounded completely positive maps on C^*-algebras. It gives one side of a Radon Nikodym story. As we shall see in §6, the converse is trickier.

THEOREM 2.5. *Let A be a C^*-algebra and $\alpha, \beta \in cp(A; H)$ such that $\beta \leqslant \alpha$. The identity map on $\mathfrak{N}_\alpha$ defines on passing to the quotient a continuous linear map λ from $\Lambda_\alpha \mathfrak{N}_\alpha \odot H$ onto $\Lambda_\beta \mathfrak{N}_\alpha \odot H$, which extends to a linear continuous map λ from $\mathfrak{K}_\alpha$ into $\mathfrak{K}_\beta$. If $T = \lambda^*\lambda$ and $\lambda = WT^{\frac{1}{2}}$ is the polar decomposition of λ, then:*

(i) *T is an operator in the commutant $\pi_\alpha(A)'$ of $\pi_\alpha(A)$ satisfying $0 \leqq T \leqq 1$ and*

$$\langle \beta(y^*x)\gamma, \eta\rangle = \langle T\Lambda_\alpha(x \otimes \gamma), \Lambda_\alpha(y \otimes \eta)\rangle \qquad \forall x, y \in \mathfrak{N}_\alpha,\ \gamma, \eta \in H.$$

(ii) *The partial isometry W gives an equivalence between the subrepresentation π_α^0 of π_α induced by the support of T, which lies in $\pi_\alpha(A)'$, and the subrepresentation π_β^0 of π_β defined by the stable subspace $[\Lambda_\beta(\mathfrak{N}_\alpha \odot H)]^-$ of $\mathfrak{K}_\beta$.*

Proof. $\beta \leqslant \alpha$ implies that $\mathfrak{M}_\alpha \subseteq \mathfrak{M}_\beta$, $\mathfrak{N}_\alpha \subseteq \mathfrak{N}_\beta$, $N_\alpha \subseteq N_\beta$. Hence the identity map on $\mathfrak{N}_\alpha$ induces a linear map λ from $\mathfrak{N}_\alpha \odot H/N_\alpha$ onto

$$\mathfrak{N}_\alpha \odot H/N_\beta \subseteq \mathfrak{N}_\beta \odot H/N_\beta \quad \text{i.e.} \quad \lambda : \Lambda_\alpha \mathfrak{N}_\alpha \odot H \to \Lambda_\beta \mathfrak{N}_\alpha \odot H.$$

Let $x_i \in \mathfrak{N}_\alpha$, $\gamma_i \in H$, $i = 1, \cdots, m$. Then

$$\left\| \lambda \left(\Lambda_\alpha \sum x_i \otimes \gamma_i \right) \right\|^2 = \left\| \Lambda_\beta \left(\sum x_i \otimes \gamma_i \right) \right\|^2 = \sum_{i,j} \langle \beta(x_i^* x_j)\gamma_j, \gamma_i\rangle$$
$$\leqq \sum_{i,j} \langle \alpha(x_i^* x_j)\gamma_j, \gamma_i\rangle = \left\| \Lambda_\alpha \left(\sum x_i \otimes \gamma_i \right) \right\|^2$$

Since λ is bounded, it can be extended to a linear continuous map of $\mathfrak{K}_\alpha$ into $\mathfrak{K}_\beta$ which we also write as λ, with norm $\leqq 1$. If $T = \lambda^*\lambda$, then $0 \leqq T \leqq 1$. If $x, y \in \mathfrak{N}_\alpha$, $\gamma, \eta \in H$ then

$$\begin{aligned}\langle T\Lambda_\alpha x \otimes \gamma, \Lambda_\alpha y \otimes \eta\rangle &= \langle \lambda^*\lambda\, \Lambda_\alpha x \otimes \gamma, \Lambda_\alpha y \otimes \eta\rangle \\ &= \langle \lambda\, \Lambda_\alpha x \otimes \gamma, \lambda\, \Lambda_\alpha y \otimes \eta\rangle = \langle \Lambda_\beta x \otimes \gamma, \Lambda_\beta y \otimes \eta\rangle \\ &= \langle \beta(y^*x)\gamma, \eta\rangle.\end{aligned}$$

If also $z \in A$, then

$$\begin{aligned}\langle T\pi_\alpha(z)\Lambda_\alpha x \otimes \gamma, \Lambda_\alpha y \otimes \eta\rangle &= \langle \beta(y^*zx)\gamma, \eta\rangle \\ &= \langle T\Lambda_\alpha x \otimes \gamma, \pi_\alpha(z)^*\Lambda_\alpha y \otimes \eta\rangle.\end{aligned}$$

Thus $T \in \pi_\alpha(A)'$.

If $z \in A$, $x \in \mathfrak{N}_\alpha$, $\gamma \in H$,

$$\pi_\beta(z)\Lambda_\beta x \otimes \gamma = \Lambda_\beta zx \otimes \gamma \in \Lambda_\beta \mathfrak{N}_\alpha \odot H.$$

Hence $\Lambda_\beta \mathfrak{N}_\alpha \odot H$ is stable for π_β and defines a projection P in $\pi_\beta(A)'$, and a subrepresentation π_β^0 of π_β. We now show that the partial isometry W gives an equivalence between P and the support of $T^{\frac{1}{2}}$, (which is the same as that of T). If $z \in A$, $x \in \mathfrak{N}_\alpha$, $\gamma \in H$ then

$$\begin{aligned} W\pi^0_\alpha(z)T^{\frac{1}{2}}\Lambda_\alpha x \otimes \gamma &= W\pi_\alpha(z)T^{\frac{1}{2}}\Lambda_\alpha x \otimes \gamma = WT^{\frac{1}{2}}\pi_\alpha(z)\Lambda_\alpha x \otimes \gamma \\ &= \lambda\Lambda_\alpha zx \otimes \gamma = \Lambda_\beta zx \otimes \gamma = \pi_\beta(z)\Lambda_\beta x \otimes \gamma \\ &= \pi^0_\beta(z)\lambda\Lambda_\alpha x \otimes \gamma = \pi^0_\beta(z)WT^{\frac{1}{2}}\Lambda_\alpha x \otimes \gamma. \end{aligned}$$

Since $T^{\frac{1}{2}}\Lambda_\alpha \mathfrak{N}_\alpha \odot H$ is dense in the support of T; W gives the equivalence between π^0_α and π^0_β.

In the theory of scalar valued weights, progress has been made by studying the bounded positive linear functionals dominated by a given weight [**3**, **15**]. In order to develop our theory of operator valued weights we find it convenient to introduce some more definitions.

DEFINITION 2.6. Let A be a C^*-algebra, and $\alpha \in cp(A\,;H)$. We denote by $\mathcal{F} = \mathcal{F}^\alpha$, the family

$$\{w \in CP(A\,;H) \colon w \leqslant \alpha\}$$

and by $\mathcal{K} = \mathcal{K}^\alpha$, the set of operators S in $\pi_\alpha(A)'$ such that there is a positive real number λ such that

$$\|S\Lambda_\alpha x \otimes \gamma\| \leqq \lambda \|x\|\|\gamma\|, \qquad \forall x \in \mathfrak{N}_\alpha,\ \gamma \in H.$$

The next lemma can be proved by developing the arguments in [**15**, Lemma 2.3], to which it reduces if $H = \mathbf{C}$.

LEMMA 2.7. *A is a C^*-algebra, and $\alpha \in cp(A\,;H)$. Then $\mathcal{K}$ is a left ideal in $\pi_\alpha(A)'$. For any S in $\mathcal{K}$, there exists a bounded linear map $V = V^S$ from H into $[\pi_\alpha(\mathfrak{N}^*_\alpha)\mathfrak{K}_\alpha]^-$ such that $S\Lambda_\alpha x \otimes \gamma = \pi_\alpha(x)V\gamma$ for all x in $\mathfrak{N}_\alpha$, γ in H. Moreover if $\gamma \in H$ is fixed, $V\gamma$ is the unique element of $[\pi_\alpha(\mathfrak{N}^*_\alpha)\mathfrak{K}_\alpha]^-$ such that $S\Lambda_\alpha x \otimes \gamma = \pi_\alpha(x)V\gamma$ for all x in $\mathfrak{N}_\alpha$.*

Parallel to the theory for weights [**15**, Lemma 2.6], we can now describe the relation between $\mathcal{F}$ and $\mathcal{K}$.

PROPOSITION 2.8. *Let A be a C^*-algebra, and $\alpha \in cp(A\,;H)$. For any w in $\mathcal{F}$, there is a unique S in $\mathcal{K}$ such that $0 \leqq S \leqq 1$, and*

$$\langle w(x^*x)\gamma, \gamma\rangle = \|S\Lambda_\alpha x \otimes \gamma\|^2 \qquad \forall x \in \mathfrak{N}_\alpha,\ \gamma \in H.$$

Conversely for any S in $\mathcal{K}$ such that $\|S\| \leqq 1$, there is a w in $\mathcal{F}$, such that

$$\langle w(x^*x)\gamma, \gamma\rangle = \| S\Lambda_\alpha x \otimes \gamma \|^2, \qquad \forall x \in \mathfrak{N}_\alpha, \ \gamma \in H.$$

Proof. Let $w \in \mathcal{F}$. Then there is a T_w in $\pi_\alpha(A)'$ by Theorem 2.5 such that $0 \leqq T_w \leqq 1$ and

$$\langle w(x^*x)\gamma, \gamma\rangle = \langle T_w \Lambda_\alpha x \otimes \gamma, \Lambda_\alpha x \otimes \gamma\rangle, \qquad \forall x \in \mathfrak{N}_\alpha, \ \gamma \in H.$$

Define $S = T_w^{\frac{1}{2}}$. Then

$$\| S\Lambda_\alpha x \otimes \gamma \|^2 = \langle w(x^*x)\gamma, \gamma\rangle \leqq \| w \| \| x \|^2 \| \gamma \|^2$$

so that $S \in \mathcal{K}$. The uniqueness is clear.

Conversely, let $S \in \mathcal{K}$ such that $\| S \| \leqq 1$. By Lemma 2.7 there exists V in $B(H, \mathfrak{K}_\alpha)$ such that

$$S\Lambda_\alpha x \otimes \gamma = \pi_\alpha(x)V\gamma \qquad \forall x \in \mathfrak{N}_\alpha, \ \gamma \in H.$$

Define $w(z) = V^*\pi_\alpha(z)V$, $\forall z \in A$. If $x_i \in \mathfrak{N}_\alpha$, $\gamma_i \in H$ for $i = 1, \cdots, n$ then

$$\begin{aligned}\sum_{i,j} \langle w(x_i^*x_j)\gamma_j, \gamma_i\rangle &= \left\| \sum \pi_\alpha(x_i)V\gamma_i \right\|^2 = \left\| S\Lambda_\alpha \sum x_i \otimes \gamma_i \right\|^2 \\ &\leqq \left\| \Lambda_\alpha \sum x_i \otimes \gamma_i \right\|^2 = \sum_{i,j} \langle \alpha(x_i^*x_j)\gamma_j, \gamma_i\rangle\end{aligned}$$

i.e. $w \leqslant \alpha$, and $w \in \mathcal{F}$.

REMARK 2.9. In the situation of Proposition 2.8, we note the following:

(a) $\mathcal{K} = \pi_\alpha(A)'$ iff α coincides on $\mathfrak{M}_\alpha$ with an element of $CP(A; H)$.

(b) If also $\mathfrak{M}_\alpha$ is norm dense in A, and $w \in \mathcal{F}$, then $\Lambda_w \mathfrak{N}_\alpha \odot H$ is dense in $\mathfrak{K}_w$. Thus by applying 2.5 to the pair (α, w), and if T is the operator defined there, π_w is a representation of A equivalent with a subrepresentation of π_α defined by the support of T.

We also observe the following, which is well known in the bounded case **[1]** and for weights [**15**, Proposition 2.5].

PROPOSITION 2.10. *Let A be a C^*-algebra and $\alpha \in cp(A; H)$ such that $\mathfrak{M}_\alpha$ is norm dense in A, and $\mathcal{F}$ contains a nonzero element. Then π_α is irreducible iff α is the restriction to $\mathfrak{M}_\alpha$ of a pure element w of $CP(A; H)$. If there exists an additive, positive homogenous map β: $A^+ \to B(H)_+^\wedge$ such that $\dot{\beta} = \alpha$ and $x \to \beta(x)\varphi$ is lower semicontinuous on A^+ for each φ in $T(H)^+$, then $\alpha = w$.*

3. Upper envelopes and ϵ-filtering families of bounded CP linear maps. Throughout this section, A denotes a C^*-algebra, and H a hilbert space. If $\alpha \in cp(A;H)$, we will be concerned with the construction of a largest operator valued weight α_0 majorised by α, and with the property that it is the upper envelope of continuous CP linear maps. There arises a natural property of $\mathcal{F}$ called "ϵ-filtrating" by Combes **[3]** in the scalar case.

DEFINITION 3.1. A family $\mathcal{G}$ in $CP(A;H)$ is said to be ϵ-filtering if given $\epsilon > 0$, w_1, w_2 in $\mathcal{G}$, there exists w in $\mathcal{G}$, such that

$$(1-\epsilon)w_i \preccurlyeq w \qquad i = 1, 2.$$

When $H = \mathbf{C}$, the following reduces partly to [**3**, Lemma 1.9] and is proved but not stated by **[15]**.

PROPOSITION 3.2. *If $\alpha \in cp(A;H)$ such that $\mathfrak{M}_\alpha$ is dense in A, then $\mathcal{F}$ is ϵ-filtering.*

Proof. Given $\epsilon > 0$, w_1, $w_2 \in \mathcal{F}$, there exist S_1, $S_2 \in \mathcal{K}$, such that $0 \leqq S_i \leqq 1$ and

$$\langle w_i(x^*x)\gamma, \gamma\rangle = \|S_i \Lambda_\alpha x \otimes \gamma\|^2 \qquad \forall x \in \mathfrak{N}_\alpha, \ \gamma \in H.$$

Since $\mathcal{K}$ is a left ideal in $\pi_\alpha(A)'$, we can get $S \in \mathcal{K}$ by [**7**, Lemma 3.1] such that $(1-\epsilon)S_i^* S_i \leqq S^*S \leqq 1$ for $i = 1, 2$.

Let $w \in \mathcal{F}$ be the element determined by S according to 2.8.

Then for all $x_j \in \mathfrak{N}_\alpha$, $\gamma_j \in H$, $j = 1, \cdots, n$

$$\sum_{j,k} (1-\epsilon)\langle w_i(x_j^* x_k)\gamma_k, \gamma_j\rangle = (1-\epsilon)\left\| S_i\left(\Lambda \sum_j x_j \otimes \gamma_j\right)\right\|^2$$
$$\leqq \left\| S\left(\Lambda \sum_j x_j \otimes \gamma_j\right)\right\|^2 = \sum_{j,k} \langle w(x_j^* x_k)\gamma_k, \gamma_j\rangle$$

Since $\mathfrak{N}_\alpha$ is dense in A, we deduce $(1-\epsilon)w_i \preccurlyeq w$ $i = 1, 2$.

PROPOSITION 3.3. *If a family $\mathcal{G}$ in $CP(A;H)$ is ϵ-filtering, then*

$$\beta(x)(\varphi) = \sup\langle w(x), \varphi\rangle \qquad \varphi \in T(H)^+, \ x \in A^+$$

defines a completely positive, additive, positive homogenous map β from A^+ into $B(H)_+^\wedge$. If $\alpha_0 = \dot{\beta}$ (Remark 2.3) then:

(i) $\mathfrak{M}_{\alpha_0}^+ = \{x \in A^+ : \sup\|w(x)\| < \infty\}$

(ii) $\alpha_0 \geqslant w \quad \forall w \in \mathcal{G}$.

(iii) *If* $x_i \in \mathfrak{N}_{\alpha_0}$, $\gamma_i \in H$ $i = 1, 2, \cdots, n$ *then*

$$\sum_{i,j=1}^{n} \langle \alpha_0(x_i^* x_j)\gamma_j, \gamma_i \rangle = \sup_{w \in \mathcal{G}} \sum_{i,j} \langle w(x_i^* x_j)\gamma_j, \gamma_i \rangle$$

Proof. It is clear by ϵ-filtering that

$$\beta(x)(\varphi) = \sup_{w \in \mathcal{G}} \langle w(x), \varphi \rangle \qquad \varphi \in T(H)^+, \ x \in A^+$$

defines an additive, positive homogeneous map from A^+ into $B(H)_+^\wedge$. Moreover if $\alpha_0 = \dot{\beta}$, it is trivial that

$$\mathfrak{M}_{\alpha_0}^+ \subseteq \{x \in A^+ : \sup_{w \in \mathcal{G}} \| w(x) \| < \infty\}.$$

Now suppose $x_1 \in A^+$ and $\sup_{w \in \mathcal{G}} \| w(x_1) \| \leqq c < \infty$, for some c. Take $\varphi \in T(H)$, then

$$\varphi = \varphi_1 - \varphi_2 + i(\varphi_3 - \varphi_4) \qquad \varphi_i \in T(H)^+$$

and $\|\varphi_1 - \varphi_2\| = \|\varphi_1\| + \|\varphi_2\|$, $\|\varphi_3 - \varphi_4\| = \|\varphi_3\| + \|\varphi_4\|$ **[14]**. Then the linear extension of $\beta(x_1)(\varphi)$ to $T(H)$ satisfies

$$|\beta(x_1)(\varphi)| \leqq 2c \|\varphi\| \qquad \forall \varphi \in T(H).$$

Hence $\beta(x_1) \in B(H) = T(H)^*$, and $x_1 \in \mathfrak{M}_{\alpha_0}^+$.

In order to prove the remainder, it is enough by polarisation to show that if $y_i \in \mathfrak{N}_{\alpha_0}$, $\gamma_i \in H$, $i = 1, 2, \cdots, m$; $\epsilon > 0$, $w \in \mathcal{G}$, that there exists w in $\mathcal{G}$, satisfying

$$|\langle \alpha_0(y_i^* y_i)\gamma_i, \gamma_i \rangle - \langle w(y_i^* y_i)\gamma_i, \gamma_i \rangle| < \epsilon, \qquad i = 1, \cdots, m$$

and $w \geqslant (1 - \epsilon) w_1$.
This follows by ϵ-filtering.

DEFINITION 3.4. $\gamma \in cp(A; H)$ is said to be the upper envelope of a family $\mathcal{G}$ in $CP(A; H)$ if

(i) If $x \in A^+$, then $x \in \mathfrak{M}_\gamma$ iff $\sup_{w \in \mathcal{G}} \| w(x) \| < \infty$.

(ii) For all x_i in $\mathfrak{N}_\gamma$, η_i in H, $i = 1, \cdots, n$

$$\sum_{i,j=1}^{n} \langle \gamma(x_i^* x_j)\eta_j, \eta_i \rangle = \sup_{w \in \mathcal{G}} \left\{ \sum \langle w(x_i^* x_j)\eta_j, \eta_i \rangle \right\}$$

We now state our decomposition theorem. If we restrict ourselves to scalar valued weights, Theorem 3.5 reduces to [**15**, 2.9, 2.11].

THEOREM 3.5. *Let $\alpha \in cp(A;H)$ such that $\mathfrak{M}_\alpha$ is dense in A. Then there exist α_0, α_1 in $cp(A;H)$ such that*

(i) $\alpha = \alpha_0 + \alpha_1$

(ii) *$\alpha_0 \leqslant \alpha$ and α_0 is the upper envelope of $\mathcal{F}^\alpha$. α_0 majorises any other map $\gamma \in cp(A;H)$ which is majorised by α, and such that γ is the upper envelope of a family in $CP(A;H)$.*

(iii) *$\alpha_1 \leqslant \alpha$ and α_1 majorises no nonzero element of $CP(A;H)$.*

Proof. It is clear from Propositions 3.2 and 3.3 that α_0, the upper envelope of $\mathcal{F}$ exists, and $\alpha_0 \leqslant \alpha$. Let γ be any other element of $cp(A;H)$ majorised by α, which is the upper envelope of a family $\mathcal{G}$ in $CP(A;H)$. Let $x \in \mathfrak{M}^+_{\alpha_0}$. Then $\sup_{w\in\mathcal{F}} \|w(x)\| < \infty$. However if $w \in \mathcal{G}$, $w \leqslant \gamma$, and $\gamma \leqslant \alpha$ show $w \in \mathcal{F}$. Hence $\sup_{w\in\mathcal{G}} \|w(x)\| < \infty$, i.e. $x \in \mathfrak{M}^+_\gamma$. Thus $\mathfrak{M}_{\alpha_0} \subseteq \mathfrak{M}_\gamma$. Take $a_i \in \mathfrak{N}_{\alpha_0} \subseteq \mathfrak{N}_\gamma$, $\eta_i \in H$, $i = 1, \cdots, n$. Then there exists w in $\mathcal{G} \subseteq \mathcal{F}$ such that

$$\sum_{i,j} \langle \gamma(a_i^* a_j)\eta_j, \eta_i\rangle \leqq \sum_{i,j} \langle w(a_i^* a_j)\eta_j, \eta_i\rangle + 1$$

$$\leqq \sum_{i,j} \langle \alpha_0(a_i^* a_j)\eta_j, \eta_i\rangle + 1.$$

Hence

$$\sum_{i,j} \langle \gamma(a_i^* a_j)\eta_j, \eta_i\rangle \leqq \sum_{i,j} \langle \alpha_0(a_i^* a_j)\eta_j, \eta_i\rangle$$

and $\gamma \leqslant \alpha_0$.

Now define $\mathfrak{M}_{\alpha_1} = \mathfrak{M}$ and $\alpha_1(x) = \alpha(x) - \alpha_0(x)$ for x in $\mathfrak{M}$. Then $\alpha \geqslant \alpha_1 \geqslant 0$ and $\alpha = \alpha_0 + \alpha_1$. Let $w \in CP(A;H)$ such that $w \leqslant \alpha_1$. Then $w \leqslant \alpha$, hence $w \in \mathcal{F}$ so that $w \leqslant \alpha_0$, and $2w \leqslant \alpha$. Similarly $nw \leqslant \alpha$, $\forall n$. Hence $w = 0$, since $\mathfrak{M} = \operatorname{lin} \mathfrak{M}^+$ and $\mathfrak{M}$ is dense in A.

4. The Stinespring representation.

We will now study quasi-equivalence, equivalence and the type of representations associated with various operator valued weights. All the results of this section are operator valued versions of those in [**3**, 2.6–2.17] for the case of $[0, \infty]$ valued weights. Again A denotes a C^*-algebra.

DEFINITION 4.1. Let $\alpha \in cp(A;H)$ with domain $\mathfrak{M}$, and F a family in $cp(A;H)$ majorised by α. We say α belongs to the closure of F for the topology of simple convergence if any one of the following equivalent conditions are satisfied:

(1) Given $n, m \geqq 1$, and $a_i \in (\mathfrak{M})_n$, $\eta_i \in H \otimes \mathbf{C}^n$ for $i = 1, \cdots, m$ and $\epsilon > 0$, there exists $w \in F$ such that

$$|\langle \alpha_n(a_i)\eta_i, \eta_i\rangle - \langle w_n(a_i)\eta_i, \eta_i\rangle| < \epsilon \qquad i = 1, \cdots, m.$$

(2) Given $n, p \geqq 1$, and $a_j^i \in \mathfrak{N}_\alpha$, $\eta_j^i \in H$, $i = 1, \cdots, p$, $j = 1, \cdots, n$ and $\epsilon > 0$, there exists $w \in F$ such that

$$\left| \sum_{j,k=1}^n \langle \alpha[(a_j^i)^* a_k^i]\eta_k^i, \eta_j^i\rangle - \sum_{j,k=1}^n \langle w[(a_j^i)^* a_k^i]\eta_k^i, \eta_j^i\rangle \right| < \epsilon$$

for $i = 1, \cdots, p$.

(3) Given $q \geqq 1$, $a_i \in \mathfrak{N}_\alpha$, $\eta_i \in H$, $i = 1, \cdots, q$, and $\epsilon > 0$, there exists $w \in F$ such that

$$0 \leqq \langle \alpha(a_i^* a_i)\eta_i, \eta_i\rangle - \langle w(a_i^* a_i)\eta_i, \eta_i\rangle < \epsilon$$

for $i = 1, \cdots, q$.

It is clear that an element of $cp(A; H)$ can be in the closure of a family in $CP(A; H)$, which it dominates, without being its upper envelope.

If α is the upper envelope of an ϵ-filtering family F in $CP(A; H)$, then α belongs to the closure of F for the topology of simple convergence. In particular if $\alpha \in cp(A; H)$ with $\mathfrak{M}_\alpha$ dense in A, then α_0 belongs to the closure of $\mathcal{F}^\alpha$ for the topology of simple convergence.

LEMMA 4.2. *Let $\alpha \in cp(A; H)$ and $cp(A; H) \supseteq F$, a family majorised by α. If $w \in F$, let T_w be the corresponding element of $\pi_\alpha(A)'$ as defined in 2.5. Then α belongs to the closure of F for the topology of simple convergence iff 1 is in the weak closure of $\{T_w : w \in F\} \subseteq \pi_\alpha(A)'$.*

Proof. The family $\{T_w\}$ satisfies $\|T_w\| \leqq 1$ for all $w \in F$. Hence 1 lies in their weak closure iff given $\epsilon > 0$, $p, n \geqq 1$, $\eta_j^i \in H$, $a_j^i \in \mathfrak{N}_\alpha$ $i = 1, \cdots, p$, $j = 1, \cdots, n$, there exists $w \in F$ such that

$$\left\langle (1 - T_w)\Lambda_\alpha\left(\sum_{j=1}^n a_j^i \otimes \eta_j^i\right), \Lambda_\alpha\left(\sum_{j=1}^n a_j^i \otimes \eta_j^i\right) \right\rangle < \epsilon, \qquad i = 1, \cdots, p$$

i.e.

$$\sum_{j,k=1}^n \{\langle \alpha[(a_j^i)^* a_k^i]\eta_k^i, \eta_j^i\rangle - \langle w[(a_j^i)^* a_k^i]\eta_k^i, \eta_j^i\rangle\} < \epsilon$$

for $i = 1, \cdots, p$.

PROPOSITION 4.3. *Let $\alpha \in cp(A;H)$ and F be a family in $CP(A;H)$ majorised by α. If $w \in F$, V^w denotes the corresponding element in $B(H, \mathfrak{K}_\alpha)$ such that $w(x) = (V^w)^* \pi_\alpha(x) V^w$, $\forall x \in A$. If α belongs to the closure of F for the topology of simple convergence, then the set $\{V^w H: w \in F\}$ is cyclic for π_α.*

COROLLARY 4.4. *Suppose $\alpha \in cp(A;H)$, A and H are separable, and that there exists a countable family F in $CP(A;H)$ majorised by α such that α belongs to the closure of F for the topology of simple convergence, then $\mathfrak{K}_\alpha$ is separable.*

PROPOSITION 4.5. *$\{\beta_i\}_{i \in I}$ is a family in $cp(A;H)$ majorised by a single element α in $cp(A;H)$ and such that α belongs to the closure of $\{\beta_i\}_{i \in I}$ for the topology of simple convergence. Moreover suppose that for each i in I, there exists a family F_i in $CP(A;H)$ such that β_i majorises each element of F_i and belongs to the closure of F_i for the topology of simple convergence. If $\mathfrak{M}_\alpha$ is dense in A, then the representations π_α and $\bigoplus_i \pi_{\beta_i}$ are quasiequivalent.*

COROLLARY 4.6. *Let $\alpha, \beta \in cp(A;H)$ such that $\beta \leqslant \alpha$ and $\mathfrak{M}_\alpha^- = A$. Suppose that there exists a family F(respectively G) in $CP(A;H)$ majorised by α (respectively β) such that α (respectively β) belongs to the closure of F (respectively G) for the topology of simple convergence, then π_β is a quasiequivalent to a subrepresentation of π_α.*

COROLLARY 4.7. *Suppose $\alpha, \beta \in cp(A;H)$ with $\mathfrak{M}_\alpha \subseteq \mathfrak{M}_\beta$, $\alpha = \beta|_{\mathfrak{M}_\alpha}$,* (i.e. $\alpha \subseteq \beta$), *and $\mathfrak{M}_\alpha^- = A$. If there exists a family F in $CP(A;H)$, majorised by β and such that β belongs to the closure of F for the topology of simple convergence, then π_α and π_β are quasiequivalent.*

DEFINITION 4.8. Let $\{\alpha_i\}_{i \in I}$ be a family in $cp(A;H)$. Then $E = \{x \in \cap \mathfrak{M}_{\alpha_i}^+ : \exists \lambda < \infty \text{ s.t. } \Sigma_J \alpha_i(x) \leqq \lambda 1$ for all finite subsets $J \subseteq I\}$ is a face in A^+ and hence is the positive part of a facial subalgebra $\mathfrak{M}$ in A. If $x \in E$, $\Sigma_{i \in I} \alpha_i(x)$ exists as an ultraweak limit, and we define this limit to be $\alpha'(x)$. Let α be the unique linear extension of α' to $\mathfrak{M}$. Then $\alpha \geqslant \Sigma_J \alpha_i$ for all finite subsets J of I, and in particular $\alpha \in cp(A;H)$. We write $\alpha = \Sigma_{i \in I} \alpha_i$.

(Alternatively, we could consider $(\Sigma\, \hat{\alpha}_i)^{\check{}}$).

PROPOSITION 4.9. *Suppose $\{\alpha_i\}_{i \in I}$ is a family in $cp(A;H)$, and suppose $\alpha \in cp(A;H)$ with $\alpha \subseteq \Sigma\, \alpha_i$. Then the representation π_α of A is equivalent to a subrepresentation of $\bigoplus_i \pi_{\alpha_i}$.*

Proof. For each i in I, let $\mathfrak{M}_i$, $\mathfrak{N}_i$, Λ_i, $\mathfrak{K}_i$, π_i be the canonical

Stinespring objects defined by α_i for each i in I. Since $\alpha_i \leqslant \alpha$, we have by 2.5 a partial isometry $W_i\colon \mathfrak{K}_\alpha \to \mathfrak{K}_i$, and an operator $T_i \in \pi_\alpha(A)'$ such that the subrepresentation induced by the support of T_i is equivalent to a subrepresentation of π_{α_i}. For $\eta \in H$ put $W\eta = (W_i(\eta)) \in \bigoplus_{i\in I}\mathfrak{K}_i$. For $x_j \in \mathfrak{N}_\alpha \subseteq \mathfrak{N}_i$, $\eta_j \in H$, $j = 1, 2, \cdots, n$ we have

$$\begin{aligned}
\left\| W\Lambda_\alpha \sum x_j \otimes \eta_j \right\|^2 &= \sum_{i\in I} \left\| W_i\Lambda_i \sum_{j=1}^n x_j \otimes \eta_j \right\|^2 \\
&= \sum_{i\in I} \left\| \Lambda_i \sum_{j=1}^n x_j \otimes \eta_j \right\|^2 \\
&= \sum_{i\in I} \sum_{j,k=1}^n \langle w_i(x_j^* x_k)\eta_k, \eta_j \rangle \\
&= \sum_{j,k} \langle \alpha(x_j^* x_k)\eta_k, \eta_j \rangle \\
&= \left\| \Lambda_\alpha \sum x_j \otimes \eta_j \right\|^2 .
\end{aligned}$$

Hence W is an isometry from $\mathfrak{K}$ into $\bigoplus \mathfrak{K}_i$. For each i in I, $W_i(\mathfrak{K}_\alpha)$ is stable under π_i, hence $W(K)$ is stable under $\bigoplus_{i\in I}\pi_i$, and defines a subrepresentation ρ of $\bigoplus \pi_i$.

For all z in A, x in $\mathfrak{N}_\alpha$, $\eta \in H$,

$$\begin{aligned}
W\pi_\alpha(z)\Lambda_\alpha x \otimes \eta &= W\Lambda_\alpha zx \otimes \eta = (\Lambda_i zx \otimes \eta)_{i\in I} \\
&= (\pi_i(z)\Lambda_i x \otimes \eta)_{i\in I} = \bigoplus \pi_i(z) W\Lambda_\alpha x \otimes \eta \\
&= \rho(z) W\Lambda_\alpha x \otimes \eta.
\end{aligned}$$

Hence W gives the required equivalence between π_α and ρ.

DEFINITION 4.10. We say $\alpha \in cp(A; H)$ is of type I (respectively II, III, etc.) if the representation π_α is of type I (respectively II, III, etc.), and α is factorial if π_α is factorial.

COROLLARY 4.11. *Suppose $\{\alpha_i\}_{i\in I}$ is a family in $cp(A; H)$, and consider $\alpha = \Sigma\, \alpha_i$. If for each $i \in I$, α_i is of* type I (*respectively* II, II_1, II, III etc.) *then α is of the same type.*

COROLLARY 4.12. *Those elements in $cp(A; H)$ of a fixed type, form a convex subcone of $cp(A; H)$.*

PROPOSITION 4.13. *Suppose $\{\alpha_i\}_{i\in I}$ is a family in $cp(A; H)$ such that for each i in I, there exists a family F_i in $CP(A; H)$ majorised by α_i, with α_i belonging to the closure of F_i for the topology of simple convergence.*

Suppose that $\alpha \in cp(A;H)$ is such that $\mathfrak{M}_\alpha^- = A$ and $\alpha \subseteq \Sigma\,\alpha_i$. Then the representations π_α and $\bigoplus \pi_{\alpha_i}$ are quasiequivalent.

Proof. This follows from 4.5 and 4.9.

PROPOSITION 4.14. *Let $\alpha \in cp(A;H)$ with $\mathfrak{M}_\alpha^- = A$. Suppose that there exists at least one family in $CP(A;H)$ whose sum is α. Then the following conditions are equivalent:*

(i) *α is factorial.*

(ii) *There exists a family of quasi equivalent factorial maps in $CP(A;H)$ whose sum is α.*

If α is also of type I, they are also equivalent to:

(iii) *There exists a family of pure elements of $CP(A;H)$ with equivalent representations whose sum is α.*

Proof. (i) $\Rightarrow$ (ii). Let $\{w_i : i \in I\}$ be a family in $CP(A;H)$ such that $\alpha = \Sigma\, w_i$ and $w_i \neq 0$, $\forall i$. Then by 2.5 for each j in I, π_{w_j} is equivalent to a subrepresentation of π_α. If α is factorial, π_{w_j} is factorial for all j, and quasi equivalent to π_α.

(ii) $\Rightarrow$ (i). If $\{w_i : i \in I\}$ is a family of quasi equivalent factorial maps in $CP(A;H)$, then $\bigoplus \pi_{w_i}$ is factorial. If $\alpha = \Sigma\, w_i$, α is factorial by Proposition 4.9.

Now suppose α is of type I.

(ii) $\Rightarrow$ (iii). Let $\{w_i : i \in I\}$ be a family of quasi equivalent factorial maps in $CP(A;H)$. For each i in I, π_{w_i} is factorial of type I. We can decompose π_{w_i} as a sum of irreducible representations

$$\pi_{w_i} = \bigoplus_{j\in I(i)} \pi_j \quad \text{and such that} \quad \mathfrak{K}_{w_i} = \bigoplus_{j\in I(i)} H_j.$$

If we write $V_{w_i}\eta = (\bigoplus_{j\in I(i)} V^j\eta)$ $\eta \in H$, where $V^j \in B(H, H_j)$ and define $w^j(x) = (V^j)^*\pi_j(x)V^j$ $x \in A$, $j \in I(i)$, then

$$\pi_i(x)V_{w_i}\eta = \bigoplus_{j\in I(i)} \pi_j(x)\left(\bigoplus_{j\in I(i)} V^j\eta\right)$$
$$= \bigoplus_{j\in I(i)} \pi_j(x)V^j\eta \qquad \forall x \in A, \quad \eta \in H.$$

Thus

$$\langle w_i(x^*x)\eta, \eta\rangle = \sum_{j\in I(i)} \langle w^j(x^*x)\eta, \eta\rangle \qquad x \in A, \quad \eta \in H,$$

and

$$\sum_i \langle w_i(x^*x)\eta, \eta\rangle = \sum_{i\in I}\sum_{j\in I(i)} \langle w^j(x^*x)\eta, \eta\rangle.$$

It follows that

$$\alpha = \sum_{i \in I} \sum_{j \in I(i)} w^j.$$

Moreover since each pair w^j, w^k are quasi equivalent and pure, they are equivalent by [**8**, 5, 3, 3],

(iii) $\Rightarrow$ (ii) is trivial.

5. Von Neumann algebras. We now wish to see how our study of the representations of operator valued weights carries over to the enveloping von Neumann algebra. It is natural to include here the following operator valued version of [**3**, Lemma 4.3] concerning scalar valued weights.

PROPOSITION 5.1. *Let M be a von Neumann algebra, and $\alpha \in cp(M;H)$. Then π_α is normal iff $z \to \alpha(x^*zx)$ is normal for each x in $\mathfrak{N}_\alpha$. In which case, if B denotes the norm closure of $\mathfrak{M}_\alpha$, then every element of $CP(B;H)$ majorised by α, is the restriction to B of a normal element in $CP(M;H)$.*

We then define $cp^w(M;H)$ to be the set of α in $cp(M;H)$ such that π_α is normal, and $CP^w(M;H) = CP(M;H) \cap cp^w(M;H)$.

Let A be a C^*-algebra. If $w \in CP(A;H)$, there exists an unique CP linear extension $\tilde{w}$ to $\tilde{A}$ the enveloping von Neumann algebra. The support s_w of $\tilde{w}$ is called the enveloping support of w, and sc_w denotes the central support of s_w in $\tilde{A}$. $1 - sc_w$ is the largest projection in $\ker \tilde{w}$. If w_1, w_2 are two such maps, they give quasi equivalent representations iff $sc_{w_1} = sc_{w_2}$. We wish to extend this for unbounded completely positive maps, and in this way our results 5.2–5.5 generalise [**3**, 4.8–4.11] concerning scalar valued weights. Suppose that $F \subseteq CP(A;H)$ is an ϵ-filtering family, and β given by

$$\beta(x)(\varphi) = \sup_{w \in F} \langle w(x), \varphi \rangle, \qquad x \in A^+, \quad \varphi \in T(H)^+$$

is the associated cp additive, positive homogeneous map from A^+ into $B(H)^\wedge_+$. Put $\alpha = \dot{\beta}$. Then $\tilde{F} = \{\tilde{w} : w \in F\}$ is ϵ-filtering, and define $\tilde{\beta} : (\tilde{A})^+ \to B(H)^\wedge_+$ by

$$\tilde{\beta}(x)(\varphi) = \sup_{\tilde{w} \in \tilde{F}} \langle \tilde{w}(x), \varphi \rangle \qquad x \in \tilde{A}^+, \quad \varphi \in T(H)^+$$

We define $\tilde{\alpha} = (\tilde{\beta})$. Note that for each φ in $T(H)_{+}$, $x \to \tilde{\beta}(x)(\varphi)$ is ultraweakly lower semicontinuous on $\tilde{A}$. Suppose M is a von Neumann algebra, and $\gamma: M^{+} \to B(H)_{+}^{\wedge}$ is additive, positive homogenous and such that $x \to \gamma(x)(\varphi)$ is u.w. ls.c. on M^{+}.

Then $n_{\gamma} = \{x \in M: \gamma(x^*x) = 0\}$ is a σ-weakly closed left ideal in M. If $\delta = \dot{\gamma}$, we let p_{δ} be the largest projection in n_{γ}. Then $1 - p_{\delta} = s_{\delta}$ is called the support of δ, and sc_{δ} is the central support of s_{δ}. If A is a C^*-algebra, and α is the upper envelope of an ϵ-filtering family F in $CP(A; H)$, we let $s_{\alpha} = s_{\tilde{\alpha}}$, and $sc_{\alpha} = sc_{\tilde{\alpha}}$.

LEMMA 5.2. *If M is a von Neumann algebra, and β is the upper envelope of an ϵ-filtering family G in $CP^{w}(M; H)$, then* $\ker \pi_{\beta} = \bigcap_{w \in G} \ker \pi_{w}$, *and* $1 - sc_{\beta}$ *is the largest projection in* $\ker \pi_{\beta}$.

PROPOSITION 5.3. *M is a von Neumann algebra, and $\beta \in cp(M; H)$ is the upper envelope of an ϵ-filtering family G in $CP^{w}(M; H)$. Then the normal representations π_{β} and $\bigoplus_{w \in G} \pi_{w}$ are quasi equivalent.*

COROLLARY 5.4. *A is a C^*-algebra, and $\alpha \in cp(A; H)$ is the upper envelope of an ϵ-filtering family F in $CP(A; H)$ such that $\mathfrak{M}_{\alpha}^{-} = A$. Then the representations $\tilde{\pi}_{\alpha}$ and $\pi_{\tilde{\alpha}}$ of $\tilde{A}$ are quasi equivalent.*

Proof. $\pi_{\tilde{\alpha}}$ and $\bigoplus_{w \in F} \pi_{\tilde{w}}$ are quasi-equivalent by 5.3. Moreover, $\tilde{\pi}_{\alpha}$ and $\bigoplus_{w \in F} \pi_{\tilde{w}}$ are quasi-equivalent by 4.5.

COROLLARY 5.5. *If A is a C^*-algebra, $\alpha, \beta \in cp(A; H)$ such that $\mathfrak{M}_{\alpha}^{-} = \mathfrak{M}_{\beta}^{-} = A$, each the upper envelopes of ϵ-filtering families in $CP(A; H)$, then π_{α} is quasi-equivalent to a subrepresentation of π_{β} (respectively quasi-equivalent) iff $sc_{\alpha} \leqq sc_{\beta}$ (respectively $sc_{\alpha} = sc_{\beta}$).*

6. Invariant completely positive maps. Our investigation of the structure of operator valued weights now proceeds by studying those completely positive maps which are invariant under a group of $*$-automorphisms of the algebra. This work is an extension of van Daele's. If we take the hilbert space H to be one dimensional, then our operator valued results 6.1 and 6.3–6.17 reduce to 2.1–2.4, 2.6, 2.8–2.10 of **[15]** concerning scalar valued weights.

A will always denote a C^*-algebra, unless otherwise stated. We say that $\alpha \in cp(A; H)$ is G-invariant, where G is a group of $*$-automorphisms of A, if $\mathfrak{M}_{\alpha}$ is G-invariant and $\alpha g(a) = \alpha(a)$ for all a in $\mathfrak{M}_{\alpha}$. This is equivalent to

$$(\alpha g)^{\wedge}(a) = \alpha(a) \qquad \forall g \in G, \quad a \in A^{+}.$$

LEMMA 6.1. *Let $\alpha \in cp(A;H)$ be G-invariant. Then $\mathfrak{N}_\alpha$ and N_α are G-invariant and there exists an unitary representation U_g of G in $\mathfrak{K}_\alpha$ such that*

(i) $U_g \Lambda_\alpha x \otimes \eta = \Lambda_\alpha g(x) \otimes \eta \qquad \forall x \in \mathfrak{N}_\alpha,\ \eta \in H,\ g \in G.$

(ii) $U_g \pi_\alpha(y) U_g^{-1} = \pi_\alpha g(y) \qquad \forall y \in A,\ g \in G.$

DEFINITION 6.2. If $\alpha \in cp(A;H)$ is G-invariant, we denote by $\mathcal{F}_0 = \mathcal{F}_0^\alpha$, and $\mathcal{K}_0 = \mathcal{K}_0^\alpha$ the G-invariant elements of $\mathcal{F} = \mathcal{F}^\alpha$ and $\mathcal{K} = \mathcal{K}^\alpha$ respectively, where $\mathcal{F}$, $\mathcal{K}$ are as defined in 2.6.

As expected from 2.8, there is a natural relation between $\mathcal{F}_0$ and $\mathcal{K}_0$ as the next lemma shows.

LEMMA 6.3. *Suppose $\alpha \in cp(A;H)$ is G-invariant. Then*

(i) *$\mathcal{K}$ is a G-invariant left ideal for $\pi_\alpha(A)'$.*

(ii) *$\mathcal{K}_0$ is a left ideal in the fixed point algebra of $\pi_\alpha(A)'$. For any S in $\mathcal{K}_0$, there is a G-invariant map $V = V^S$ in $B(H; [\pi_\alpha(\mathfrak{N}_\alpha^*)\mathfrak{K}]^-)$ (i.e. $U_g V = V\ \forall g \in G$) such that for each fixed η in H, V_η is the unique G-invariant vector in $[\pi_\alpha(\mathfrak{N}_\alpha^*)\mathfrak{K}]^-$ such that*

$$S \Lambda_\alpha x \otimes \eta = \pi_\alpha(x) V_\eta \qquad \forall x \in \mathfrak{N}_\alpha.$$

LEMMA 6.4. *Suppose $\alpha \in cp(A;H)$ is G-invariant. Then for any w in $\mathcal{F}_0$, there is an unique S in $\mathcal{K}_0$ such that $0 \leqq S \leqq 1$, and*

$$\langle w(x^*x)\eta, \eta\rangle = \|S\Lambda_\alpha x \otimes \eta\|^2 \qquad \forall x \in \mathfrak{N}_\alpha,\ \eta \in H.$$

Conversely, for any S in $\mathcal{K}_0$, such that $\|S\| \leqq 1$, there is a w in $\mathcal{F}_0$ such that

$$\langle w(x^*x)\eta, \eta\rangle = \|S\Lambda_\alpha x \otimes \eta\|^2 \qquad \forall x \in \mathfrak{N}_\alpha,\ \eta \in H.$$

COROLLARY 6.5. (i) *Let $\alpha \in cp(A;H)$ be G-invariant with $\mathfrak{M}_\alpha$ norm dense in A, then $\mathcal{F}_0$ is ϵ-filtering.*

(ii) *Let A be a von Neumann algebra, $\alpha \in cp^w(A;H)$ with $\mathfrak{M}_\alpha$ ultraweakly dense in A. Then $\mathcal{F}_0$ is in $CP^w(A;H)$ and is ϵ-filtering.*

Proof. (ii) The only nontrivial point is to show that $\mathcal{F}_0$ in $CP^w(A;H)$ is ϵ-filtering. Suppose $\epsilon > 0$, $w_1, w_2 \in \mathcal{F}_0$ and for all a_i in $\mathfrak{N}_\alpha$, η_i in H $i = 1, \cdots, n$ that

$$\sum (1-\epsilon)\langle w_1(a_i^*a_j)\eta_j, \eta_i\rangle \leqq \sum \langle w_2(a_i^*a_j)\eta_j, \eta_i\rangle.$$

Then $(1-\epsilon)w_1 - w_2 \geqslant 0$ follows from either 2.1 or [**6**, 2.4] (consider $[a_i^* p_\lambda a_j]$, where p_λ are projections in $\mathfrak{N} \cap \mathfrak{N}^*$, $p_\lambda \uparrow 1$).

This leads us naturally to:

THEOREM 6.6. (i) *Let* $\alpha \in cp(A;H)$ *be* G*-invariant with* $\mathfrak{M}_\alpha$ *norm dense in* A. *Then there exists a* G*-invariant element* α_0 *of* $cp(A;H)$ *which is the upper envelope of* $\mathcal{F}_0$. α_0 *majorises any other* G*-invariant element of* $cp(A;H)$ *which is majorised by* α *and is the upper envelope of a* G*-invariant family in* $CP(A;H)$.

(ii) *Let* A *be a von Neumann algebra,* $\alpha \in cp^w(A;H)$ *with* $\mathfrak{M}_\alpha$ σ*-weakly dense in* A. *Then there exists a* G*-invariant map* α_0 *which is the upper envelope of* $\mathcal{F}_0$. α_0 *majorises any other* G*-invariant map* γ *in* $cp(A;H)$ *which is majorised by* α *and such that* γ *is the upper envelope of a* G*-invariant family in* $CP^w(A;H)$.

Following van Daele's one dimensional theory **[15]**, we will construct a unique normal G-invariant projection map ϕ of the ultraweak closure $\bar{\mathcal{K}}$ of $\mathcal{K}$ onto the ultraweak closure $\bar{\mathcal{K}}_0$ of $\mathcal{K}_0$, and which projects $\mathcal{K}$ onto $\mathcal{K}_0$, and $\mathcal{K}^*\mathcal{K}$ onto $\mathcal{K}_0^*\mathcal{K}_0$. It will then be possible to define a unique G-invariant projection map ϕ' of $\mathcal{F}$ onto $\mathcal{F}_0$, which is BW continuous **[1]** on bounded sets.

Let $\alpha \in cp(A;H)$ be G-invariant. We will denote by E_0 the projection onto the fixed points in $\mathfrak{K}_\alpha$. Then $U_g \cdot E_0 = E_0 \cdot U_g = E_0$ $\forall_g \in G$. Moreover there exists a net of convex combinations $\{\Sigma \lambda^i(g) U_g\}_{g\in G}$ converging strongly to E_0. We can then show that if $S \in \mathcal{K}$,

$$\sum \lambda^i(g) U_g S U_g^{-1}$$

converges strongly to an operator ϕS in $\mathcal{K}_0$. The arguments of [**15**, Prop. 3.2] lead us to the following conclusions:

PROPOSITION 6.7. *Let* $\alpha \in cp(A;H)$ *be* G*-invariant. There exists a unique normal positive* G*-invariant projection map* ϕ *of* $\bar{\mathcal{K}}$ *onto* $\bar{\mathcal{K}}_0$ *(the* σ*-weak closures). We have* $\phi(S)E_0 = E_0 S E_0$ *for any* S *in* $\mathcal{K}$. *In particular,*

$$\phi(\mathcal{K}) = \mathcal{K}_0, \qquad \phi(\mathcal{K}^*\mathcal{K}) = \mathcal{K}_0^*\mathcal{K}_0.$$

COROLLARY 6.8. *Let* F, F_0 *be the largest projections in* $\bar{\mathcal{K}}$ *and* $\bar{\mathcal{K}}_0$ *respectively, (the ultraweak closures). Then* $\phi F = F_0$.

REMARK 6.9. It is clear from 4.2 that α belongs to the closure of $\mathcal{F}$ (respectively $\mathcal{F}_0$) for the topology of simple convergence iff $F = 1$ ($F_0 = 1$ respectively). If α belongs to the closure of $\mathcal{F}$ for the topology of simple convergence, it does not follow that α always belongs to the closure of $\mathcal{F}_0$ for the topology of simple convergence. See [**15**, 5.2] where $F = 1$, but $F_0 = 0$.

PROPOSITION 6.10. *Let $\alpha \in cp(A; H)$ be G-invariant. Then there exists a G-invariant projection map ϕ' of $\mathscr{F}$ into $\mathscr{F}_0$, satisfying $\phi'(\lambda f) = \lambda\phi'(f)$, and $\phi'(f+g) = \phi'(f) + \phi'(g)$ whenever $\lambda \geqq 0$, and f, g, λf, $f+g \in \mathscr{F}$. If moreover $\mathfrak{M}_\alpha$ is norm dense in A, then ϕ' is onto $\mathscr{F}_0$, BW continuous on bounded sets and unique.*

REMARK 6.11. Note that when $\mathfrak{M}_\alpha$ is dense in A, the formula

$$\langle w(x^*x)\eta, \eta\rangle = \| S_w \Lambda_\alpha x \otimes \eta \|^2 \qquad x \in N_\alpha, \quad \eta \in H$$

gives a bijection $w \mapsto S_w^* S_w$ between $\mathscr{F}$ and $\{S^*S \colon \|S\| \leqq 1,\ S \in \mathscr{K}\}$, the elements of $(\mathscr{K}^*\mathscr{K})^+$ with norm $\leqq 1$; and similarly for $\mathscr{F}_0$ and $\mathscr{K}_0$. Then on any bounded set in $\mathscr{F}$, $w_\lambda \to w$ in the BW topology iff $S_{w_\lambda}^* S_{w_\lambda} \to S_w^* S_w$ in the weak operator topology.

Our next results concern a study of the upper envelope α_0 constructed in 6.6, and the relation of the existence of fixed points in $\mathfrak{K}_\alpha$ to the existence of fixed points in $\mathscr{F}$.

THEOREM 6.12. *Let $\alpha \in cp(A; H)$ be G-invariant, and $\mathfrak{M}_\alpha$ norm dense in A. Let F_0 be the largest projection in the ultraweak closure $\bar{\mathscr{K}}_0$ of $\mathscr{K}_0$, and α_0 the upper envelope of $\mathscr{F}_0$. Then*

$$\langle \alpha_0(x^*x)\eta, \eta\rangle = \langle \mathscr{F}_0 \Lambda_\alpha x \otimes \eta, \Lambda_\alpha x \otimes \eta\rangle \qquad \text{for all } x \text{ in } \mathfrak{N}_\alpha, \quad \eta \text{ in } H.$$

COROLLARY 6.13. *Let $\alpha \in cp(A; H)$ be G-invariant with $\mathfrak{M}_\alpha$ norm dense in A, and α belonging to the closure of $\mathscr{F}$ for the topology of simple convergence. Then $F_0 = [\pi_\alpha(A)E_0\mathfrak{K}_\alpha]^-$. Moreover α majorises no nonzero G-invariant element of $CP(A; H)$ iff $\mathfrak{K}_\alpha$ has no nonzero fixed points.*

COROLLARY 6.14. *Let $\alpha \in c\dot{p}(A; H)$ be G-invariant with $\mathfrak{M}_\alpha^- = A$. Then there is an increasing net $\{w_i \colon i \in I\}$ in $\mathscr{F}_0$ such that*

$$(\alpha_0)_n(z) = \sup_i (w_i)_n(z) \quad \textit{for all } z \textit{ in } \quad (\mathfrak{M}_{\alpha_0})_n \cap (A_n)^+.$$

COROLLARY 6.15. *Let $\alpha \in cp(A; H)$ be G-invariant with $\mathfrak{M}_\alpha^- = A$, and suppose there is a family $\{w_i \colon i \in I\}$ in $\mathscr{F}$ such that $\alpha \subseteq \Sigma_{i \in I} w_i$. Then there is a family $\{w_i^0 \colon i \in I\}$ in $\mathscr{F}_0$ such that*

$$\alpha_0(x) = \sum_{i \in I} w_i^0(x), \qquad \forall x \in \mathfrak{M}_\alpha^+.$$

Our final observations of this section are concerned with Radon-Nikodym statements for operator valued weights. If $\alpha \in cp(A;H)$, we have seen the relation between the set $\{S \in \mathcal{K}^*\mathcal{K}: 0 \leqq S \leqq 1\}$ and $\mathcal{F} = \{w \in CP(A;H): w \preccurlyeq \alpha\}$, (2.8). One may then wonder if for all T in $\pi_\alpha(A)'$, $0 \leqq T \leqq 1$, there is a β in $cp(A;H)$, $\beta \preccurlyeq \alpha$ and

$$\langle \beta(x^*x)\eta, \eta\rangle = \langle T\Lambda_\alpha x \otimes \eta, \Lambda_\alpha x \otimes \eta\rangle \qquad \forall x \in \mathfrak{N}_\alpha, \ \eta \in H.$$

This is the case if A is a von Neumann algebra. In the C^*-algebra situation, the first part of the following theorem, when G is the identity automorphism, gives a partial converse to 2.5.

THEOREM 6.16. *Let $\alpha \in cp(A;H)$ be G-invariant with $\mathfrak{M}_\alpha^- = A$. For any G-invariant T in $\pi_\alpha(A)'$, $0 \leqq T \leqq F_0$, there is a G-invariant element β of $cp(A;H)$ $\beta \preccurlyeq \alpha$, with*

$$\langle \beta(x^*x)\eta, \eta\rangle = \langle T\Lambda_\alpha x \otimes \eta, \Lambda_\alpha x \otimes \eta\rangle \qquad \forall x \in \mathfrak{N}_\alpha, \ \eta \in H.$$

For any β in $cp(A;H)$ with $\beta \preccurlyeq \alpha$, and $\beta|_{\mathfrak{M}_\alpha}$ belonging to the closure of a family $\mathcal{G}$ of G-invariant elements of $CP(A;H)$ majorised by $\beta|_{\mathfrak{M}_\alpha}$, there exists an operator T in $\pi_\alpha(A)'$, $0 \leqq T \leqq F_0$, and

$$\langle \beta(x^*x)\eta, \eta\rangle = \langle T\Lambda_\alpha x \otimes \eta, \Lambda_\alpha x \otimes \eta\rangle, \qquad \forall x \in \mathfrak{N}_\alpha, \ \eta \in H.$$

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