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THE COHOMOLOGICAL DIMENSION OF A n-MANIFOLD IS $n+1$

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It is known that any one-dimensional topological manifold is of cohomological Dimension two. The present paper is devoted to the proof of the conjecture that the cohomological Dimension of any topological n-manifold is $n+1$.

Introduction. Let ϕ be a family of supports on a topological space X. The largest integer n (or ∞) for which there exists a sheaf $\mathscr{A}$ of abelian groups on X such that the Grothendieck cohomology groups $H^n_\phi(X, \mathscr{A}) \neq 0$ is called the *cohomological ϕ-dimension* ($\dim_\phi X$) of X. The supremum of all ϕ-dimensions when ϕ runs over all the families of supports on X is called the *cohomological Dimension* ($\operatorname{Dim} X$) of X. The *extent* $E(\phi)$ of a family of supports ϕ is defined to be the union of all members of ϕ. It is then known that if ϕ and ψ are two paracompactifying families of supports on X such that $E(\phi) \subset E(\psi)$ then $\dim_\phi(X) \leqq \dim_\psi(X)$. It follows, therefore, that if ϕ varies over all those paracompactifying families of supports on X whose extents are equal to X then $\dim_\phi(X)$ is independent of ϕ and is called the *cohomological dimension* ($\dim X$) of X. Thus if a space X admits a paracompactifying family of supports with extent equal to X then $\dim X$ makes sense. Let us call a topological space to be *locally paracompact Hausdorff* if each point of the space has a closed paracompact Hausdorff neighbourhood in it. Then one can easily see that for any space X, $\dim X$ makes sense if and only if X is locally paracompact Hausdorff. An intereting relation between $\dim X$ and $\operatorname{Dim} X$ when X is a nice space is given by the following: *If X is locally completely paracompact Hausdorff then* $\dim X = \operatorname{Dim} X$ or $\dim X = \operatorname{Dim} X - 1$. An open problem as to which one it is, was solved in the case of one-dimensional topological manifolds in [2]. The main objective of this paper is to prove the conjecture of [2] by showing that if X is a topological n-manifold then $\operatorname{Dim} X = n+1$ for any $n \geqq 1$.

1. Preliminaries. By a sheaf in this paper we shall mean a sheaf of abelian groups. If $\mathscr{A}$ is a sheaf on a space X, $\mathscr{C}^0(X, \mathscr{A})$ will denote the sheaf on X generated by the presheaf $U \to C^0(U, \mathscr{A})$ where $C^0(U, \mathscr{A})$ is abelian group of all sections (not necessarilly continuous) of $\mathscr{A}$ on the open set U of X. Similarly, $\mathscr{F}^1(X, \mathscr{A}) = \mathscr{F}^1$ will denote the quotient sheaf $\mathscr{C}^0(X, \mathscr{A})/\mathscr{A}$ on X. For each positive integer n we define inductively

$$\mathscr{C}^n(X, \mathscr{A}) = \mathscr{C}^0(X, \mathscr{F}^n(X, \mathscr{A}))$$

and

$$\mathscr{F}^{n+1} = \mathscr{F}^{n+1}(X, \mathscr{A}) = \mathscr{F}^1(X, \mathscr{F}^n(X, \mathscr{A})) .$$

If $\mathscr{A}$ is a flabby sheaf on a space X then $H^1_\varphi(X, \mathscr{A}) = 0$ for every family of supports ϕ on X. Conversely because an open set U of X is always taut we find ([1], p. 59) that

$$H^1(X, U, \mathscr{A}) \approx H^1_{\mathrm{cld}/X-U}(X, \mathscr{A}) = 0 .$$

Here cld denotes the family consisting of all closed sets of X. But again because (cf. [1] p. 58)

$$0 \longrightarrow H^0(X, U, \mathscr{A}) \longrightarrow H^0(X, \mathscr{A}) \longrightarrow H^0(U, \mathscr{A}) \longrightarrow H^1(X, U, \mathscr{A})$$

with supports in cld is exact we find that $H^0(X, \mathscr{A}) \to H^0(U, \mathscr{A})$ is an epimorphism and hence $\mathscr{A}$ is flabby. Thus we have

PROPOSITION 1.1 ([1] p. 110): *Let $\mathscr{A}$ be a sheaf on X. Then $\mathscr{A}$ is flabby if and only if $H^1_\phi(X, \mathscr{A}) = 0$ for every family of supports φ on X.*

Since the sequence

$$0 \longrightarrow \mathscr{F}^n \longrightarrow \mathscr{C}^n(X, \mathscr{A}) \longrightarrow \mathscr{F}^{n+1} \longrightarrow 0$$

of sheaves on X where $n \geqq 0$ and $\mathscr{F}^0$ stands for $\mathscr{A}$ is exact and $\mathscr{C}^n(X, \mathscr{A})$ is ϕ-acyclic for every ϕ we find that for each $k > 0$

$$H^k_\phi(X, \mathscr{F}^n) \approx H^{k+1}_\varphi(X, \mathscr{F}^{n-1}) \approx \cdots \approx H^{k+n}_\varphi(X, \mathscr{A}) .$$

Hence by above proposition we have

PROPOSITION 1.2 ([1] p. 110): *Let X be a topological space. Then* $\operatorname{Dim} X \leqq n$ *if and only if $\mathscr{F}^n(X, \mathscr{A})$ is flabby for every sheaf $\mathscr{A}$ on X.*

Now we state the following theorems which we shall require in the proof of our main result.

Vietoris-Begle theorem for Sheaf cohomology ([1] p. 55): *Let $f\colon X \to Y$ be a continuous closed map, $\mathscr{B}$ a sheaf on Y and ψ a family of supports on Y. Suppose that each $f^{-1}(y)$ is connected and taut in X and that $H^p(f^{-1}(y), \mathscr{B}_y) = 0$ for $p > 0$ and all y. Then*

$$f^*\colon H^*_\psi(Y, \mathscr{B}) \longrightarrow H^*_{f^{-1}(\psi)}(X, f^*\mathscr{B})$$

is an isomorphism.

Leray spectral theorem for Sheaf cohomology ([1] p. 140): *Let*

$f\colon X \to Y$ be a continuous, ψ any family of supports on X, $\mathscr{A}$ a sheaf on X and ϕ be a paracompactifying family of supports on Y. Then there exists a spectral sequence in which

$$E_2^{pq} = H_\phi^p(Y, \mathscr{H}_\psi^p(f, \mathscr{A}))$$

and which converges to $H_{\phi(\psi)}^{p+q}(X, \mathscr{A})$. Here $\mathscr{H}^*(f, \mathscr{A})$ denotes the Leray Sheaf of the map f and $\phi(\psi)$ denotes the extension of ψ by ϕ.

2. On Dim X. For any space X we have the following form of Subspace Theorem and a result which shows that $\operatorname{Dim} X$, in a sense, is a local property.

THEOREM 2.1. *Let A be any locally closed subspace of X. Then* $\operatorname{Dim} A \leqq \operatorname{Dim} X$. *Furthermore, if each point of X has an open neighbourhood U such that* $\operatorname{Dim} U \leqq n$ *then* $\operatorname{Dim} X \leqq n$.

Proof. For any Sheaf $\mathscr{A}$ on A the sequence

$$0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{C}^0(A, \mathscr{A}) \longrightarrow \mathscr{F}^1(A, \mathscr{A}) \longrightarrow 0$$

is exact which means the sequence

$$0 \longrightarrow \mathscr{A}_A \longrightarrow \mathscr{C}^0(A, \mathscr{A})_A \longrightarrow \mathscr{F}^1(A, \mathscr{A})_A \longrightarrow 0$$

of sheaves on X is exact. But since the sequence

$$0 \longrightarrow \mathscr{A}_A \longrightarrow \mathscr{C}^0(X, \mathscr{A}_A) \longrightarrow \mathscr{F}^1(X, \mathscr{A}_A) \longrightarrow 0$$

is exact and $\mathscr{C}^0(X, \mathscr{A}_A) \approx \mathscr{C}^0(A, \mathscr{A})_A$, we find that

$$\mathscr{F}^1(A, \mathscr{A})_A \approx \mathscr{F}^1(X, \mathscr{A}_A) .$$

By induction, therefore,

$$\mathscr{F}^n(A, \mathscr{A})_A \approx \mathscr{F}^n(X, \mathscr{A}_A) .$$

Now let s be a section of $\mathscr{F}^n(A, \mathscr{A})$ defined on any open set $U \cap A$ of A where U is open in X. This gives canonically a section s' of $\mathscr{F}^n(A, \mathscr{A})_A$ defined on U. But that means we have a section s'' of $\mathscr{F}^n(X, \mathscr{A}_A)$ on U. Suppose $\operatorname{Dim} X = n$. Then $\mathscr{F}^n(X, \mathscr{A}_A)$ is flabby, by Proposition 1.2, and hence we have an extension s''' on X of the section s'' onto U. Then s'''/A on A is an extension of s on $U \cap A$. Whence $\mathscr{F}^n(A, \mathscr{A})$ is flabby and by Proposition 1.2, $\operatorname{Dim} A \leqq n$.

The last statement follows from Proposition 1.2 and the fact that a sheaf $\mathscr{A}$ on a space X is flabby if and only if each point of X has an open neighbourhood U such that $\mathscr{A}/U$ is flabby.

3. Main result. First we have the following

LEMMA 3.1. *Let* $\begin{array}{ccc} \boldsymbol{R}^{n-1} \times \boldsymbol{R} & \xrightarrow{\pi_1} & \boldsymbol{R} \\ \downarrow{\scriptstyle \pi_2} & & \\ \boldsymbol{R}^{n-1} & & \end{array}$ *be the canonical projections and* $\mathscr{B}$ *any sheaf on* $\boldsymbol{R}^{n-1}$. *Let* ϕ *be any family of supports on* $\boldsymbol{R}^{n-1}$ *and* ψ *be the family of supports on* $\boldsymbol{R}^{n-1} \times \boldsymbol{R}$ *which consists of the members of the family* $\pi_2^{-1}(\phi)$ *and their closed subsets. Then the Leray Sheaf* $\mathscr{H}_\psi^*(\pi_1, \pi_2^*\mathscr{B})$ *is a constant sheaf on* $\boldsymbol{R}$ *with stalks* $H_\varphi^*(\boldsymbol{R}^{n-1}, \mathscr{B})$

Proof. Note that the Leray Sheaf on $\boldsymbol{R}$ is generated by the presheaf $U \to {}^*_{\psi \cap \pi_1^{-1}(U)}(\pi_1^{-1}(U), \pi_2^*\mathscr{B} \,|\, \pi_1^{-1}(U))$. Let us regard $H_\varphi^*(\boldsymbol{R}^{n-1}, \mathscr{B})$ to be the constant sheaf on $\boldsymbol{R}$. The projections

$$(\pi_2)_U \colon \pi_1^{-1}(U) \longrightarrow \boldsymbol{R}^{n-1}$$

induce a sheaf map

$$\pi_2^* \colon H_\varphi^*(\boldsymbol{R}^{n-1}, \mathscr{B}) \longrightarrow \mathscr{H}_\psi^*(\pi_1, \pi_2^*\mathscr{B}) .$$

Likewise, the inclusions (chosen once for all for each U) $i_U \colon \boldsymbol{R}^{n-1} \to \pi_1^{-1}(U)$ induce a sheaf map

$$i^* \colon \mathscr{H}_\psi^*(\pi_1, \pi_2^*\mathscr{B}) \longrightarrow H_\phi^*(\boldsymbol{R}^{n-1}, \mathscr{B}) .$$

Now, on the stalks at y in $\boldsymbol{R}$ we find that

$$\begin{aligned} (\pi_2^*)_y \colon & H_\varphi^*(\boldsymbol{R}^{n-1}, \mathscr{B}) \longrightarrow \mathscr{H}_\psi^*(\pi_1, \pi_2^*\mathscr{B})_y \\ &= \xrightarrow{\lim} (\pi_2)_U^* \colon H_\varphi^*(\boldsymbol{R}^{n-1}, \mathscr{B}) \longrightarrow \mathscr{H}_\psi^*(\pi_1, \pi_2^*\mathscr{B})_y \end{aligned}$$

where U runs over all the neighbourhoods of y,

$$= \xrightarrow{\lim} (\pi_2)_W^* \colon H_\varphi^*(\boldsymbol{R}^{n-1}, \mathscr{B}) \longrightarrow \mathscr{H}_\psi^*(\pi_1, \pi_2^*\mathscr{B})_y$$

where W runs over the cofinal family of all compact interval neighbourhoods of y in $\boldsymbol{R}$. But since for each W, $(\pi_2)_W \colon \pi_1^{-1}(W) \to \boldsymbol{R}^{n-1}$ is a closed continuous map and $(\pi_2)_W^{-1}(z)$ is connected and taut (closed subspace of a paracompact space) in $\pi_1^{-1}(W)$ and $H^p((\pi_2)_W^{-1}(y), \mathscr{B}_z) = 0$ for every $p > 0$ and for every z in $\boldsymbol{R}^{n-1}$ we find by Vietoris-Begle theorem that $(\pi_2)_W^*$ is an isomorphism. Hence $(\pi_2^*)_y$ is an isomorphism, which implies that π_2^* is an isomorphism with i^* as its inverse. This proves our lemma.

Now we can prove our

THEOREM 3.2 (Main result). *Let* X *be a topological* n*-manifold. Then* $\operatorname{Dim} X = n + 1$.

Proof. The proof is by induction on n. For $n = 1$ we have already proved in [2] that the Theorem is true. Hence we can assume the theorem to be true for all positive integers less than n, $n > 1$. In view of Proposition 2.1 it suffices to prove that $\mathrm{Dim}\, \boldsymbol{R}^n = n+1$. By our hypothesis, there exists a sheaf $\mathscr{B}$ and a family of supports ϕ on $\boldsymbol{R}^{n-1}$ such that

$$H^n_\phi(\boldsymbol{R}^{n-1}, \mathscr{B}) \neq 0 .$$

Now let π_1 and π_2 and ψ be as in Lemma 3.1. From there it follows that the Leray Sheaf $\mathscr{H}^*_\psi(\pi_1, \pi_2^*\mathscr{B})$ is a constant sheaf on $\boldsymbol{R}$ and that

$$\mathscr{H}^q_\psi(\pi_1, \pi_2^*\mathscr{B})_y = \begin{cases} H^n_\varphi(\boldsymbol{R}^{n-1}, \mathscr{B}) & q = n \\ 0 & q > n . \end{cases}$$

Now let c denote the paracompactifying family of supports of all compact subsets of $\boldsymbol{R}$. Hence by Leray spectral theorem there exists a spectral sequence in which

$$E_2^{pq} = H^p_c(\boldsymbol{R}, \mathscr{H}^q_\psi(\pi_1, \pi_2^*\mathscr{B})) \Longrightarrow H^{p+q}_{c(\psi)}(\boldsymbol{R}^n, \pi_2^*\mathscr{B}) .$$

Since $\mathscr{H}^n_\psi(\pi_1, \pi_2^*\mathscr{B}) \neq 0$ and $E_2^{pq} = 0$ for $p > 1$ or $q > n$ we find that

$$\begin{aligned} H^{n+1}_{c(\psi)}(\boldsymbol{R}^n, \pi_2^*\mathscr{B}) &\approx E_2^{1,n} \\ &\approx H^1_c(\boldsymbol{R}, \mathscr{H}^n_\psi(\pi_1, \pi_2^*\mathscr{B})) \\ &\approx \mathscr{H}^n_\psi(\pi_1, \pi_2^*\mathscr{B}) \neq 0 . \end{aligned}$$

Hence $\mathrm{Dim}\, \boldsymbol{R}^n = n+1$.

REMARK 3.3. If we want to use the Kunneth theorem for Sheaf cohomology ([1] p. 141) then a very short proof of our main result is as follows: By [2] and the definition of Dimension we find that there exists a sheaf $\mathscr{A}$ on $\boldsymbol{R}$ and a family of supports ϕ on $\boldsymbol{R}$ such that $H^2_\phi(\boldsymbol{R}, \mathscr{A}) \neq 0$ and $H^i_\varphi(\boldsymbol{R}, \mathscr{A}) = 0$ for $i > 2$. Since $\boldsymbol{R}$ is a clc_∞ space we have, by Kunneth theorem, the following sequence to be exact.

$$\begin{aligned} 0 \longrightarrow H^2_\phi(\boldsymbol{R}, \mathscr{A}) \bigotimes_{\boldsymbol{Z}} \boldsymbol{Z} \longrightarrow H^{n+1}_{\phi\times c}(\boldsymbol{R} \times \boldsymbol{R}^{n-1}, \mathscr{A} \times \boldsymbol{R}^{n-1}) \\ \longrightarrow H^3_\phi(\boldsymbol{R}, \mathscr{A}) \underset{\boldsymbol{Z}}{*} \boldsymbol{Z} \longrightarrow 0 . \end{aligned}$$

This implies that $H^{n+1}_{\phi\times c}(\boldsymbol{R}^n, \mathscr{A} \times \boldsymbol{R}^{n-1}) \neq 0$, whence the result.

REMARK 3.4. It should be noted that the main result of this paper is not true for zero-dimensional manifolds, which are, as a matter of fact, discrete spaces. Any sheaf on such a space is itself discrete and therefore flabby and hence ϕ-acyclic for every family

ϕ of supports on X. Since the family cld is a paracompactifying family for such a space we find that $\dim X = 0 = \operatorname{Dim} X$. On the other hand the result is clearly true for any manifold with boundary also. Hence $\operatorname{Dim} I^n = n + 1$ where I^n stands for the n-dimensional cube etc.

REMAKR 3.5. As is well known the generalization of the concept of the family of supports to any paracompactifying family of support whose extent equals the space X itself is certainly an interesting one. However, our result shows that the generalization of latter one to any family of supports does not seem to be so interesting because of the fact that such a concept leads to a dimension function defined for all spaces which although distinguishes the Euclidean spaces $\boldsymbol{R}^m$ and $\boldsymbol{R}^n$ $m \neq n$, yet does not agree with other classical dimensions even for Euclidean spaces. But it may be justified on the ground that it, being the supremum of some other dimensions including the classical ones and being consistently greater than the classical ones strictly by one for Euclidean spaces $\boldsymbol{R}^n$, $n \geqq 1$, is not a bad one. Also for manifolds it satisfies the following product formula

$$\operatorname{Dim} X \times Y = \operatorname{Dim} X + \operatorname{Dim} Y - \varepsilon$$

where $\varepsilon = 0$ if $m = 0$ or $n = 0$ and 1 otherwise.

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