PACIFIC JOURNAL OF MATHEMATICS
Vol. 67, No. 1, 1976

THE STRUCTURE OF STANDARD C^*-ALGEBRAS AND THEIR REPRESENTATIONS

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We introduce the basic structure space A^b of a C^*-algebra A consisting of all minimal primitive ideals in A. We define a class of C^*-algebras to be called standard. All W^*-algebras and all C^*-algebras with Hausdorff structure spaces are standard. It is proved that a standard C^*-algebra A is isometrically isomorphic to the C^*-algebrade fined by a continuous field of primitive C^*-algebras over its basic structure space A^b. A sufficient condition for a C^*-algebra to be faithfully represented on a separable Hibert space is also presented.

1. Introduction. The Wedderburn structure theorem asserts that every finite dimensional complex semi-simple algebra is a di rect sum of simple algebras, each of which is an $n \times n$ matrix algebra. The structure of a general infinite dimensional complex semi-simple algebra is certainly too complicated to study at present. Instead, we focus our attention on a well-behaved subclass of infinite dimensional algebras: C^*-algebras. A C^*-algebra A is a complex Banach *-algebra such that $\|a^*a\| = \|a\|^2$ for all a in A. By using the Gelfand-Naimark-Segal construction, a C^*-algebra can also be defined as a norm-closed self-adjoint subalgebra of $B(H)$, the algebra of all bounded linear operators on a Hilbert space H. There are structure theorems available for three families of C^*-algebras: (A) the Gelfand-Naimark theorem for commutative C^*-algebras, (B) von Neumann's direct integral decomposition theorem for von Neumann algebras on a separable Hilbert space, (C) Kaplansky's structure theorem for liminal C^*-algebras with Hausdorff spectrum which is actually a noncommutative generalization of (A) or can be viewed as a continuous generalization of the Wedderburn theorem. The main purpose of this paper is to prove a structure theorem for certain C^*-algebras, called standard C^*-algebras, which generalizes (C) as well as providing a continuous version of (B).

Firstly, we introduce the basic structure space of a C^*-algebra: A primitive ideal I of a C^*-algebra A is called *minimal* if it does not contain any other primitive ideal of A. The *basic structure space* A^b of a C^*-algebra A is the set of all minimal primitive ideals in A. A C^*-algebra A is called *bounded* if every primitive ideal of A contains a minimal primitive ideal of A. It follows from the results of Kaplansky [23] and Dixmier [12] that type I C^*-algebras and separable C^*-algebras are bounded. We present another sufficient condition for a C^*-algebra to be bounded in §2. However, we leave the full consideration of the question whether all C^*-algebras are bounded to another occasion. With an additional technical condition, a bounded C^*-algebra A is called **-bounded*, and the space A^b with the hull-kernel topology is a locally compact T_1-space A^*-bounded C^*-algebra is called *normal* if any primitive ideal contains an unique minimal primitive ideal, and a normal C^*-algebra is called *standard* if its basic structure space is Hausdorff. C^*-algebras with Hausdorff structure space and W^*-algebras are standard. Our structure theorem then states that every standard C^*-algebra A is isomorphic to the C^*-algebra defined by a continuous field of primitive C^*-algebras over A^b.

Ever since the publication of Gelfand and Naimark [18], many efforts have been made, notably those by Kaplansky, Fell, Dixmier and Douady, Tomiyama and Takesaki, to generalize (A) to the noncommutative case, at least for some special classes of C^*-algebras (see Dixmier [13]). The best example of such a generalization is (C) which states that a liminal C^*-algebra (called a *CCR-algebra* by Kaplansky) with Hausdorff structure space T is isomorphic to the C^*-algebra defined by a continuous field of elementary C^*-algebras over T. Our method of approach is similar to that of Kaplansky [23], Fell [17], Dixmier and Douady [14]; the main difference being due to the introduction of basic structure space. Recently, Dauns and Hofmann [11], Akemann [1] and Takemoto [31] have all worked on a noncommutative Gelfand-Naimark theorem. Akemann [1] started by considering left ideals, hence his approach is quite different from ours. We will compare our result with that of Dauns and Hofmann [11] in §5. Takemoto [31] gives a continuous decomposition of a von Neumann algebra on a general Hilbert space. Our structure theorem can also be used to give a continuous decomposition of a von Neumann algebra on a general Hilbert space, where each coordinate algebra is primitive. Hence it can be considered as a continuous analogue of the direct integral decomposition theorem of von Neumann [27], where the Hilbert space is restricted to be separable due to measure theoretical difficulties. A continuous decomposition of von Neumann algebras has been studied earlier by Godement [20].

A normal C^*-algebra A is called *pre-standard* if the basic structure space A^b is only T_1 but the relation defined through inseparability by disjoint open neighborhoods between a pair of points in A^b is an equivalence relation. For a pre-standard C^*-algebra A, we can obtain a Hausdorff space X out of A^b by identifying points in each equivalence class, and A is isomorphic to the C^*-algebra defined by a continuous field of "simple" C^*-algebras, where "simple" means that it is a discrete direct sum of primitive C^*-algebras.

The equivalence of the two definitions of C^*-algebras mentioned in the beginning of this introduction is actually a representation theorem for C^*-algebras, which was essentially proved in Gelfand and Naimark [18]. The representation space in [18] is nonseparable in general for an infinite dimensional C^*-algebra. Of course, for separable C^*-algebras the construction can easily be amended to reduce the representation space to a separable one. In the last section, we construct a representation for a densely bounded (see Definition 3 below) C^*-algebra A, where the representation space is separable if every exactly irreducible representation of A (see Definition 1 in §2) can be taken on a separable Hilbert space and A^b is separable. This is a refinement of representation theorem in Gelfand and Naimark [18].

We shall use the terminology of Dixmier [13]. By an isomorphism (homomorphism) of a C^*-algebra into another, we mean a $*$-isomorphism ($*$-homomorphism), and a representation of a C^*-algebra means a $*$-representation of a C^*-algebra on a Hilbert space. In this paper, all algebras and Hilbert spaces are over the field of complex numbers. If A is a C^*-algebra without identity, then A_1 denotes the C^*-algebra obtained from A by adjunction of an identity. We denote by 1 the identity in a C^*-algebra; $B(H)$ (resp. $K(H)$) the algebra of all bounded (resp. compact) operators on a Hilbert space H. We note that $K(H)$ is the unique norm closed ideal in $B(H)$.

2. **The basic structure space of a C^*-algebra.** There are already three spaces associated with a C^*-algebra A which can serve as a base for continuous decomposition of the C^*-algebra A. These are the structure space $A^j = \mathrm{Prim}(A)$ consisting of all primitive ideals in A, the spectrum $\hat{A}$ consisting of all unitary equivalence classes of irreducible representations of A, and the quasi-spectrum $\hat{A}$ consisting of all quasi-equivalence classes of factor representations of $\hat{A}$. Two representations π and ρ of a C^*-algebra A are called *physically equivalent* if there exists an isomorphism φ from $\pi(A)$ onto $\rho(A)$ such that $\varphi(\pi(a)) = \rho(a)$ for all $a \in A$ (p. 107, Emch [15])[1]. Two representations π and ρ of a C^*-algebra are physically equivalent if and only if $\ker \pi = \ker \rho$. Hence A^j can be regarded as the set of all physical equivalence classes of irreducible representations of A. In any algebraic theory, a simple algebra has to be considered as an indecomposable object. Consider an uniformly hyperfinite C^*-algebra which is simple. It has infinitely many unitarily inequivalent irreducible representations, and

[1] This is a special case of weak equivalence of two sets of representations of a C^*-algebra of Fell [16] when both sets are singletons. A more appropriate term is "algebraic equivalence". But the term "algebraic equivalence" has already been used in Dixmier [13] and Powers [26] in two different contexts.

a one-parameter family of nonquasi-equivalent factor representations as shown in Powers [28] (also see Lance [25]). Therefore, in an algebraic structure theorem for C^*-algebras, the underlying base space should be a set of physical equivalence classes of irreducible representations rather than the spectrum $\hat{A}$ or the quasi-spectrum $\hat{A}$ of A.

Our aim is to establish a "continuous" variant of the classical Wedderburn theorem for C^*-algebras, that is, to express a C^*-algebra A as a continuous direct sum of "indecomposable" C^*-algebras. Now, let $A = B(H)$, where H is a separable infinite dimensional Hilbert space. A has only two physical equivalence classes of irreducible representations: one contains the identity representation, and the other contains irreducible representations of A induced by irreducible representation of the quotient (simple) C^*-algebra $B(H)/K(H)$. However, we know that $B(H)$ cannot reasonably be split into a direct sum of two nontrivial subalgebras. Hence, we only consider the identity representation in the decomposition process, and exclude the irreducible representation of the second kind, whose kernel $K(H)$ properly contains $\{0\}$ which is a kernel of another irreducible representation, i.e. the identity representation. The example above illustrates two points: First, we cannot expect a general C^*-algebra to be a subdirect sum of simple C^*-algebras, and hence the basic building blocks in our decomposition theory are primitive C^*-algebras rather than simple C^*-algebras. Second, a subset of the structure space A^j would be a more appropriate base for a continuous direct sum decomposition of a C^*-algebra A. This leads to the following definition:

DEFINITION 1. A primitive ideal I in a C^*-algebra A is called *minimal primitive* if it does not properly contain any other primitive ideal of A. A representation π of A is called *exactly irreducible* if $\ker \pi$ is a minimal primitive ideal.

The set of all minimal primitive ideals in a C^*-algebra A endowed with the hull-kernel topology is called the *basic structure space* of A, and is denoted by A^b. Let A be a C^*-algebra with nonempty basic structure space A^b. A subset S of A^b is closed if and only if

$$J_0 \supset \bigcap_{J \in S} J \quad \text{implies} \quad J_0 \in S .$$

A^b is a T_1-space. In fact, let I be an arbitrary ideal in A^b. The closure $\{\bar{I}\}$ of the singleton $\{I\}$ in A^b consists of all $J \in A^b$ with $J \supset I$. However, if $J \neq I$, then J is not minimal primitive. Hence $\{\bar{I}\} = \{I\}$, and A^b is T_1.

An immediate question arise from the preceding definition is the existence of a minimal primitive ideal in a C^*-algebra. To faciliate the discussion on this question, we introduce the following definitions:

DEFINITION 2. A C^*-algebra A is said to satisfy the *bounded chain condition of primitive ideals* if every maximal descending chain of primitive ideals in A has a smallest element. We call a C^*-algebra *bounded* if it satisfies the bounded chain condition on primitive ideals.

DEFINITION 3. A C^*-algebra A is said to be *densely bounded* if the basic structure space A^b is dense in the structure space A^j of A.

The bounded chain condition on primitive ideals for a C^*-algebra A is equivalent to the condition that any primitive ideal I in A contains a minimal primitive ideal of A. Hence, a bounded C^*-algebra A always has a nonempty basic structure space A^b dense in the struture space A^j. Therefore, a bounded C^*-algebra is densely bounded. And the structure space of a densely bounded C^*-algebra has a dense T_1-subspace. We do not know at present whether the converse to either of these two statements is true.

By Corollary 2 of Dixmier [12], all separable C^*-algebras are bounded. And as Lemma 7.4 of Kaplansky [23] proved that in a postliminal C^*-algebra every prime ideal is primitive, the same argument as that in Corollary 2 of Dixmier [12] shows that a postliminal C^*-algebra is bounded. It is not known whether an arbitrary C^*-algebra is always bounded, and we shall further study this question in another occasion. We would like to point out, however, that there do exist C^*-algebras with infinite descending chains of primitive ideals. This is certainly an infinite dimensional noncommutative phenomena. Although no nonprimitive C^*-algebra with an infinite descending chain of primitive ideals going to $\{0\}$ has been found (such a C^*-algebra is not bounded), examples of primitive C^*-algebras with infinite descending chains of primitive ideals can be found in Dixmier [12], Behncke, Krauss and Leptin [4], where the structure spaces of the C^*-algebras concerned are linearly ordered. Behncke and Bös [3] also constructed examples of primitive C^*-algebras with infinite descending chains of primitive ideals, whose structure spaces are not necessarily linearly ordered. These examples also indicate that the basic building blocks in our structure theorem, i.e., primitive C^*-algebras, unlike that in the finite dimensional case, can have complicated ideal structures themselves. And it is far from easy to classify all primitive C^*-algebras. This situation is likely unavoidable for the study of general infinite dimensional Banach algebras. For in a decompositon theory as deep as von Neumann's direct integral decomposition for von Neumann algebras, there are continuous families of pairwise non-isomorphic factors of type II_1, and of type III (they are all algebraically simple algebras) among the basic building blocks (see McDuff [24], Ching [4], Anastasio and Willig [2], or Sakai [27]).

The following is another sufficient condition for a C^*-algebra to be bounded:

DEFINITION 4. An element I_0 in a descending chain $\{I_\alpha\}$ of primitive ideals in a C^*-algebra A is called a *knot* if (i) the elements in Prim (I_0) are totally ordered by inclusion and (ii) I_0 contains a countable cofinal subchain of $\{I_\alpha\}$. A C^*-algebra A is said to be *essentially ordered* if every maximal descending chain of primitive ideals in A has a knot.

LEMMA 1. *A C^*-algebra A is bounded if it is essentially ordered.*

Proof. Let $\{I_\alpha\}$ be a maximal descending chain of primitive ideals in A with a knot I_0. We can assume that $I_0 \neq \{0\}$, for otherwise $\{I_\alpha\}$ is bounded below by $\{0\}$. By passing to the quotient algebra, we can assume that $\bigcap_\alpha I_\alpha = \{0\}$. For each ideal I_n in the countable cofinal subchain of $\{I_\alpha\}$, let S_n be the set of all primitive ideals in I_0 properly contained in I_n. The complement of S_n in Prim(I_0) is closed since it consists of all primitive ideals in I_0 containing I_n. Hence each S_n is open in Prim(I_0). Furthermore, each S_n is dense in Prim(I_0) because primitive ideals in I_0 are linearly ordered. Now, it follows form Theorem 1 of Dixmier [12] that $\bigcap_{n=1}^\infty S_n \neq \emptyset$. Let $I \in \bigcap_{n=1}^\infty S_n$. Since I is contained in every I_n and $\{I_n\}$ is cofinal in $\{I_\alpha\}$, I is contained in every I_α in the chain. By the preceding assumption, we have $I = \{0\}$. This implies that I_0 has a faithful irreducible representation π_0 on a Hilbert space H. By proposition 2.10.4 of Dixmier [13], we can extend π_0 to an irreducible representation π of A on H. Let $J = \ker \pi$. Let x be a nonzero element of I_0 and y a nonzero element in J. If $xy \neq 0$, then $\pi(xy) = \pi_0(xy) \neq 0$ as π_0 is faithful. But $\pi(xy) = \pi(x)\pi(y) = 0$. This contradiction shows that $xy = 0$. Hence $I_0 \cdot J = 0$. Since each primitive ideal in a C^*-algebra is prime and the intersection of descending family of prime ideals is obviously a prime ideal, we conclude that $I = \{0\}$ is a prime ideal in A. This implies that either $I_0 = \{0\}$ or $J = \{0\}$. Since we assumed $I_0 \neq \{0\}$ in the beginning, we have $J = \{0\}$. Therefore, A is primitive and $\{0\}$ is the smallest element in the chain $\{I_\alpha\}$. This shows that A is bounded.

For some important classes of C^*-algebras, we can actually identify their basic structure spaces with some familiar spaces. A C^*-algebra A is called *liminal* if $\pi(a)$ is a compact operator for every $\pi \in \hat{A}$

and $a \in A$ (Def. 4.2.1 [14]). The structure space A^j of a liminal C^*-algebra A is T_1. In fact, for any primitive ideal J of A, A/J is isomorphic to the (norm) simple algebra $K(H)$ for some Hilbert space H, i.e., J is also a maximal ideal. Hence the singleton $\{J\}$ is closed in A^j. Since each point is closed, A^j is T_1. And $A^j = A^b$, the basic structure space, as every primitive ideal in A is also minimal primitive.

A topological space X is a $T_{1/2}$-space if every pair of distinct points x and y has disjoin closures. A C^*-algebra A with a $T_{1/2}$ structure space is bounded as the following lemma shows:

LEMMA 2. *For a C^*-algebra A, $A^b = A^j$ if and only if A^j is a $T_{1/2}$-space.*

Proof. If $A^b = A^j$, then A^j is a T_1-space. Hence A^j is a $T_{1/2}$-space. Conversely, if A^j is a $T_{1/2}$-space, then every primitive ideal I of A is minimal primitive. For if $I \in A_j$ is not minimal primitive and properly contains another primitive ideal J of A, then $I \in \{\bar{I}\} \cap \{\bar{J}\}$. Hence $\{\bar{I}\} \cap \{\bar{J}\} \neq \emptyset$, and A^j is not $T_{1/2}$. A contradiction. Therefore $A^b = A^j$.

REMARK 1. The relation between the structure space and the basic structure space for postliminal C^*-algebras (Def. 4.3.1 [13]) is quite different from that for liminal C^*-algebras. Let H be a separable infinite dimensional Hilbert space. Let T be a bounded noncompact normal operator of H with spectrum S. Let A be the C^*-algebra generated by all operators of the form $T + K$, where K is a compact operator on H. A is a postliminal C^*-algebra, and it is an extension of the algebra $K(H)$ by a commutative C^*-algebra $C(S)$ of all continuous on the compact set S. A has only one minimal primitive ideal, namely $\{0\}$. Yet A has many primitive ideals, all properly containing $\{0\}$ with the exception of $\{0\}$ itself. Hence $A^b = \{0\}$, and $A^j = X \cup \{0\}$, where X is the maximal ideal space of $C(S)$ which can be identified with S.

The proofs of the following lemmas are adapted from the proofs of the corresponding lemmas in §3.3 [13]. Their inclusion here is partly for the sake of completeness and partly due to certain modifications required by a change of space.

LEMMA 3. *Let A be a C^*-algebra with nonempty structure space A^b. For each $x \in A^b$, let π_x be the quotient map of A onto A/x. Then for each $a \in A$, the function*

$$N_a \colon x \longrightarrow \|\pi_x(a)\|$$

is lower semi-continuous on A^b.

Proof. Since $||\pi_x(a)||^2 = ||\pi_x(a^*a)||$ for all $a \in A$, we only need to prove the lower semi-continuity of N_a when a is a positive element of A. To do so is the same as to show that

$$C_k = \{x \in A^b: ||\pi_x(a)|| \leqq k\} = \{x \in A^b: \mathrm{Sp}\, \pi_x(a) \subset [0, k]\}$$

is closed in A^b for each real $k \geqq 0$, where $\mathrm{Sp}\, \pi_x(a)$ denotes the spectrum of $\pi_x(a)$ (in case A is without identity, we pass to A_1, and $\mathrm{Sp}_{A_1}\pi_x(a) = 0 \cup \mathrm{Sp}_A \pi_x(a)$). Let $y \in A^b$ be in the closure $\bar{C}_k$ of C_k. Suppose $\mathrm{Sp}\, \pi_y(a)$ contains a point t not in $[0, k]$. Let f be a continuous real-valued function vanishing on $[0, k]$ and positive on t. Then,

$$\pi_x(f(a)) = f(\pi_x(a)) = 0 \quad \text{for all} \quad x \in C_k\,, \quad \text{and}$$
$$\pi_y(f(a)) = f(\pi_y(a)) \neq 0\,.$$

Hence,

$$y \not\supset \bigcap_{x \in C_k} x\,, \quad \text{or} \quad y \notin \bar{C}_k\,.$$

This contradiction shows $\mathrm{Sp}\, \pi_y \subset [0, k]$, i.e., $y \in C_k$. Hence C_k is closed and N_a is lower semi-continuous.

LEMMA 4. *Let A be a bounded C^*-algebra. For each $a \in A$, the function N_a defined in Lemma 3 attains its supremum $||a||$ on the basic structure space A^b.*

Proof. The function N_a is also defined on the structure space A^j of A, and it attains its supremum $||a||$ on A^j as shown in Lemma 3.3.6 of [13]. A^b is a subspace of A^j. If $x \in A^j \backslash A^b$, then x is not a minimal primitive ideal. Hence there exists a $y \in A^b$ properly contained in x. Since the quotient algebra A/x is isometrically isomorphic to the quotient algebra $(A/y)/(x/y)$, we have $||\pi_x(a)|| \leqq ||\pi_y(a)||$ by the definition of the quotient norm. Hence the function N_a attains its supremum on A^b.

A bounded C^*-algebra A is said to be *$*$-bounded* if for any primitive ideal J of A with $J \supset \bigcap_{I \in S} I$, where $S \subset A^b$, there exists a minimal primitive ideal $J_0 \subset J$ with $J_0 \supset \bigcap_{I \in S} I$.

LEMMA 5. *The basic structure space A^b of a $*$-bounded C^*-algebra A is locally compact.*

Proof. By exactly the same argument as that in Proposition 3.3.7 of [13], we can prove that for an element a in A and $k > 0$, the subset

$$G_k = \{x \in A^b\colon \|\pi_x(a)\| \geqq k\}$$

is compact in A^b. We only need to change $\pi \in \hat{A}$ to $x \in A^b$, and replace Lemma 3.3.6 of [13] by *-boundedness. Now, let $x \in A^b$, and U be an open neighborhood of x. Since $A^b \backslash U$ is closed, there exists an element a in A such that $a \notin x$ but $a \in y$ for all $y \in A^b \backslash U$. Let V (resp. W) be the set of all z in A^b such that

$$\|\pi_z(a)\| > \frac{1}{2}\|\pi_x(a)\| \quad \left(\text{resp. } \|\pi_z(a)\| \geqq \frac{1}{2}\|\pi_x(a)\|\right).$$

It follows from Lemma 3 that V is an open neighborhood of x. Hence W is a compact neighborhood of x contained in U. This shows that A^b is locally compact.

COROLLARY 1. *Let A be a *-bounded C^*-algebra with Hausdorff basic structure space A_b. Then for each $a \in A$, the function N_a defined in Lemma 3 is continuous on A^b.*

Proof. The set G_k in the preceding lemma is then closed. Hence the function N_a is upper semi-continuous. This together with Lemma 3 implies that N_a is continuous on A^b.

3. **The basic structure space of a W^*-algebra.** A C^*-algebra A is called a *W^*-algebra* if it can be faithfully represented as a von Neumann algebra on some Hilbert space, i.e., a weakly closed self-adjoint subalgebra of $B(H)$. For a W^*-algebra we can easily identify its basic structure space.

LEMMA 6. *Let A be a W^*-algebra with center Z. Let M be the maximal ideal space of Z. For each m in M, let $[m]$ be the smallest closed two-sided ideal of A generated by m. Then $\rho\colon m \mapsto [m]$ is a homeomorphism of M onto the basic structure space A^b of A.*

Proof. By Theorem 4.7 of Halpern [21] for each m in M, $[m]$ is a primitive ideal. Suppose J is another primitive ideal in A contained in $[m]$. Let $J = \ker \pi$ for some irreducible representation π of A. It is not difficult to see that π restricted to Z is a multiplicative functional on the commutative C^*-algebra Z. Hence its kernel $\ker(\pi|Z) = J \cap Z$ is a maximal ideal of Z. However, $J \subset [m]$ implies $J \cap Z \subset [m] \cap Z = m$. Hence $J \cap Z = m$, $J \supset m$. Therefore, $J \supset [m]$ since $[m]$ is the smallest ideal in A containing m. This shows that $[m]$ is a minimal primitive ideal of A. By the same argument we know that if I is any primitive ideal of A, then $I \cap Z$ is a maximal ideal of Z. Hence $[I \cap Z]$ is a minimal primitive ideal contained in I. Consequently, A is bounded. This also shows that ρ is onto. For if I is in A^b, then $I = \rho(I \cap Z)$. The mapping ρ is one-to-one. For if m_1 and m_2 are two maximal ideals of Z such that $[m_1] = [m_2]$, then $m_1 = [m_1] \cap Z = [m_2] \cap Z = m_2$.

Let S be a closed subset of M. Let J be a minimal primitive ideal of A such that $J \supset \bigcap_{I \in \rho(S)} I$. Then we have

$$J \cap Z \supset \bigcap_{I \in \rho(S)} I \cap Z = \bigcap_{I \cap Z \in S} I \cap Z .$$

Hence, $J \cap Z \in S$ and $J \in \rho(S)$. This shows that $\rho(S)$ is closed, and ρ is an open mapping. Conversely, let K be a closed subset of A^b. Let m_0 be a maximal ideal in the closure of $\rho^{-1}(K)$. We claim that

$$[m_0] \supset \bigcap_{m \in \rho^{-1}(K)} [m] .$$

For otherwise there exists an element a in A such that $a \in \bigcap_{[m] \in K} [m]$ but $a \notin [m_0]$. For each $[m]$ in A^b, let a_m be the canonical image of a in $A/[m]$. Hence we have $a_{m_0} \neq 0$ while $a_m = 0$ for all m in $\rho^{-1}(K)$. by Lemma 10 of Glimm [19], the mapping $m \longrightarrow \|a_m\|$ is continuous on M. Consequently $\|a_{m_0}\| = 0$, a contradiction. Hence the claim holds, and

$$m_0 = [m_0] \cap Z \supset \bigcap_{m \in \rho^{-1}(K)} m,\ m_0 \in \rho^{-1}(K) .$$

Hence $\rho^{-1}(K)$ is closed, and ρ is continuous. Therefore ρ is a homeomorphism.

We remark that the fact that A is a W^*-algebra is only used twice: (i) $[m]$ is primitive, (ii) $m \longrightarrow \|a_m\|$ continuous.

Let A be a W^*-algebra and let $S \subset A^b$. Suppose that J is a primitive ideal of A and $J \supset \bigcap_{I \in S} I$. Then $J \cap Z \supset (\bigcap_{I \in S} I) \cap Z = \bigcap_{I \in S} (I \cap Z)$. Hence $\rho(J \cap Z) \supset \bigcap_{I \in S} \rho(I \cap Z) = \bigcap_{I \in S} I$ since ρ is a homeomorphism. Clearly, $\rho(J \cap Z)$ is a minimal primitive ideal of A contained in J. Therefore, a W^*-algebra is *-bounded.

4. The structure of standard C^*-algebras.

We still need another condition on a C^*-algebra to ensure a direct sum decomposition.

DEFINITION 5. A *-bounded C^*-algebra is said to be *normal* if every primitive ideal in A contains a unique minimal primitive ideal of A.

An immediate consequence of the definition is the following. If A is a normal C^*-algebra and I, J two distinct primitive ideals in

A, then $I + J = A$. For if $I + J \neq A$, then $A/I + J$ is a nonzero C^*-algebra. Let π be an irreducible representation of $A/I + J$. π induces an irreducible representation π' of A. $\ker \pi'$ is then a primitive ideal in A containing two distinct minimal primitive ideals I and J. A contradiction. Conversely, if in a C^*-algebra $A, I + J = A$ for all pair of distinct minimal primitive ideals I and J, then A is obviously normal. We point out that there exists a (primitive) C^*-algebra A with two distinct primitive ideals I and J such that $I \not\subset J$ and $J \not\subset I$ but $I + J \neq A$.

Let A be a W^*-algebra with center Z, and I a primitive ideal in A. It is clear from the proof of Lemma 6 that the two-sided ideal J of A generated by $I \cap Z$ is a minimal primitive ideal contained in I. And if J' is another minimal primitive ideal contained in I, then $J \cap Z = I \cap Z = J' \cap Z$. Consequently, $J = J'$. Hence an W^*-algebra is normal.

In general, a $*$-bounded C^*-algebra A is normal if and only if for any pair of distinct points x and y in A^b, the closure of $\{x\}$ and of $\{y\}$ in the structure space A^j of A are disjoint. Therefore, a C^*-algebra A with a $T_{1/2}$ structure space A^j is normal. In particular, a liminal C^*-algebra is normal. We do not know at present whether a postliminal C^*-algebra is necessarily normal. As for the question of boundedness, we shall study the normalcy of a general C^*-algebra in another paper. We introduce the following class of C^*-algebras:

DEFINITION 6. A C^*-algebra A is called *standard* if it is normal and its basic structure space A^b is Hausdorff.

An W^*-algebra is standard since it is normal and its basic structure space being homeomorphic to the maximal ideal space of a commutative C^*-algebra is Hausdorff. Any C^*-algebra with Hausdorff structure space is also standard.

Let A be a standard C^*-algebra with basic structure space $X = A^b$. For each $x \in X$, let π_x be the quotient map from A onto the primitive C^*-algebra A/x which we shall denote by $A(x)$. For each $a \in A$, let $\hat{a}$ be a function on X defined by

$$\hat{a}(x) = \pi_x(a) , \qquad x \in X .$$

By Corollary 1, $x \longrightarrow \|\hat{a}(x)\|$ is continuous for each $a \in A$. Let $\Pi = \prod_{x \in X} A(x)$ be the cartesian product of all primitive C^*-algebras $A(x)$, $x \in X$. Let $\bar{A}$ be the subspace of Π consisting of all functions $\hat{a}$, $a \in A$. Clearly, $\bar{A}$ satisfies conditions (i), (ii), (iii) of Definition 10.1.2 in [13]. Furthermore, $\bar{A}$ is obviously stable under the pointwise involution and multiplication of the product algebra Π. By Proposition 10.2.3 and Proposition 10.3.2 of [13], there exists a subset Γ of Π, containing $\bar{A}$, such that $E = ((A(x))_{x \in X}, \Gamma)$ is a continuous field of primitive C^*-algebras $(A(x))_{x \in X}$ (Def. 10.3.1 [13]). E is called the continuous field of primitive C^*-algebras over X defined by A. Let $\tilde{A}$ be the C^*-algebra defined by the continuous field E of primitive C^*-algebras (10.4.1 [13]). We have the following structure theorem for standard C^*-algebras:

THEOREM 1. *Let A be a standard C^*-algebra with basic structure space X. Let $E = ((A(x))_{x \in X}, \Gamma)$ be the continuous field of primitive C^*-algebras over X defined by A. Let $\tilde{A}$ be the C^*-algebra defined by E. Then the Gelfand transform $a \to \hat{a}, a \in A$, is an isometric isomorphism of A onto $\tilde{A}$.*

Proof. For each $a \in A$, the function $x \to \|\hat{a}(x)\|$ vanishes at infinity on X by Lemma 5. Hence $\hat{a} \in \tilde{A}$ for all $a \in A$. It is easy to see that the Gelfand transform is linear, preserves multiplication and involution. The Gelfand transform is isometric by Lemma 4. We only need to show that the Gelfand transform is onto, i.e., $\bar{A} = \tilde{A}$. Now, for any $x \in X$, $\{\hat{a}(x) \mid a \in A\} = A(x)$. Suppose x and y is a pair of distinct points in X, and $\xi_1 \in A(x)$, $\xi_2 \in A(y)$. There exist a_1 and a_2 in A such that $\pi_x(a_1) = \xi_1$, $\pi_y(a_2) = \xi_2$. Since x and y are distinct minimal primitive ideals of A, we have $x + y = A$. Let $a_i = b_i + c_i$ where $b_i \in x$ and $c_i \in y$, $i = 1, 2$. Let $a = b_2 + c_1$. Then

$$\hat{a}(x) = \pi_x(a) = \xi_1, \ \hat{a}(y) = \pi_y(a) = \xi_2 .$$

Hence $\bar{A} = \tilde{A}$ by Corollary 11.5.3 in [13] which is a consequence of of Glimm's noncommutative generalization of the Stone-Weierstrass theorem for C^*-algebras [19].

We shall apply the structure theorem to a brief study of ideals in a standard C^*-algebra. By an ideal I of a C^*-algebra, we now mean a norm-closed one-sided (left or right), or two-sided ideal. But once the type of I is chosen, all the ideals in the subsequent discussion in this paragraph shall have the same type as that of I. Let I be an ideal of standard C^*-algebra A with basic structure space X. Let $I(x) = \{\hat{a}(x) \mid a \in I\}$, $x \in X$. Then $I(x) = A(x)$ or $I(x)$ is a proper ideal of $A(x)$. Let

$$k(I) = \{x \in X \mid I(x) \neq A(x)\} .$$

$k(I)$ is nonempty for a proper ideal I of A. Let

$$I_x = \{a \in A \mid \hat{a}(x) \in I(x)\} , \qquad x \in k(I) .$$

Then I_x is an ideal in A, and

$$I = \bigcap_{x \in k(I)} I_x$$

by Lemma 10.4.2 of [13]. Hence I is a maximal ideal in A if and only if $k(I)$ is a singleton $\{x_0\}$ and $I(x_0)$ is a maximal ideal in the primitive C^*-algebra $A(x_0)$ (we admit $\{0\}$ as a maximal two-sided ideal of $A(x_0)$ if $A(x_0)$ is topologically simple). In fact, suppose I is a maximal ideal. Since $I \subset I_x$, for $x \in k(I)$,

$$I = I_x \,, \quad \text{for all} \quad x \in k(I) \,.$$

Suppose $x, y \in k(I)$, and $x \neq y$. Then $I(y) \neq A(y)$. Let $\xi \in A(y) \backslash I(y)$. By Theorem 1, there exists an element a of A such that $\hat{a}(x) = 0$ and $\hat{a}(y) = \xi$. Hence $a \in I_x$ but $a \notin I_y$. This shows that $I_x \neq I_y$ in A. Hence we have $k(I) = \{x_0\}$ and $I = I_{x_0}$. If $I(x_0)$ is not maximal, let $J(x_0)$ be an ideal containing $I(x_0)$ in $A(x_0)$. Then

$$J = \{a \,|\, \hat{a}(x_0) \in J(x_0)\}$$

is an ideal properly containing I. A contradiction. Hence $I(x_0)$ is maximal. Conversely, suppose $k(I) = \{x_0\}$ and $I(x_0)$ is maximal. Let $J \supset I = I_{x_0}$. Now for $y \in X$, $y \neq x_0$, $y \notin k(I)$, $J(y) = I(y) = A(y)$, and

$$J(x_0) = \{\hat{a}(x_0) \,|\, a \in J\} \supset I(x_0) \,.$$

Hence $J(x_0) = I(x_0)$. Consequently $J = I$. I is maximal.

It follows that a standard C^*-algebra A is strongly semi-simple if and only if each coordinate algebra $A(x)$, $x \in X$, is a simple C^*-algebra.

5. **Pre-standard C^*-algebras.** Let Y be a T_1 topological space. For a pair of distinct points x and y in Y, we write xRy if x and y do not have disjoint open neighborhoods. A topological space Y is called a *$T_{1\,1/2}$-space* if it is T_1 and the relation R of inseparability between points of Y so defined is an open equivalence relation (5.2 Bourbaki [7]).

DEFINITION 7. A normal C^*-algebra A is called *pre-standard* if its basic structure space A^b is *a* $T_{1\,1/2}$-space.

A C^*-algebra A with a $T_{1\,1/2}$ structure space if pre-standard.

Let A be a pre-standard C^*-algebra with basic structure space A^b. For each $x \in A^b$, let $[x] = \{y \,|\, xRy\}$. Let $X = \{[x] \,|\, x \in A^b\}$ be the quotient space of A^b with respect to the equivalence relation R (equipped with the quotient topology and the quotient map is denoted by q). We call the space X the *base* of the pre-standard C^*-algebra A.

LEMMA 7. *X is a locally compact Hausdorff space.*

Proof. X is locally compact since A^b is locally compact. X is Hausdorff. In fact, let $[x_0] \neq [y_0]$ be two distinct points of X. Since x_0 and y_0 are not in the same equivalence class, we can find two disjoint open sets 0_1 and 0_2 containing x_0 and y_0 respectively. Let $G_i = \{z \mid zRx \text{ for some } x \in 0_i\}$. G_1 and G_2 are open in A^b since R is open. Suppose $z \in G_1 \cap G_2$. Then there exist $x \in 0_1$, $y \in 0_2$, such that xRz, zRy. Hence xRy, i.e., x and y cannot be separated by open sets. This contradiction shows that $G_1 \cap G_2 = \varnothing$. Then $q(G_1)$ and $q(G_2)$ are disjoint open sets in X containing $[x_0]$ and $[y_0]$ respectively.

For each $x \in A^b$, let $\bar{x} = \bigcap_{y \in [x]} y$. We call such an ideal a *pseudo-primitive* ideal. Hence, a pseudo-primitive ideal of A is the intersection of a class of inseparable minimal primitive ideals in A. X can be identified as the set of all pseudo-primitive ideals in A, i.e., $[x] \mapsto \bar{x}$ is a one-to-one correspondence. In fact, if $\bar{x} = \bar{y}$, then $y \supset \bar{y} = \bar{x} = \bigcap \{z \mid z \in [x]\}$. Since $[x] \subset A^b$ is the inverse image of a point in X under the quotient map, $[x]$ is closed in A^b. Hence $y \in [x]$. Consequently, $[x] = [y]$. We shall write $[x]$ and $\bar{x}$ interchangingly for an element of X. Let us denote by $B_A(\bar{x})$ the C^*-algebra of all bounded elements in the discrete direct sum $\bigoplus_{y \in [x]} A(y)$, i.e., all $\hat{a}$ in $\bigoplus_{y \in [x]} A(y)$ with $\sup_{y \in [x]} \|\hat{a}(y)\| < \infty$. And we denote by $C_A(\bar{x})$ the C^*-algebra of all elements vanishing at infinity in the discrete direct sum $\bigoplus_{y \in [x]} A(y)$, i.e., all $\hat{a}$ in $\bigoplus_{y \in [x]} A(y)$ with the property that given any $\varepsilon > 0$, there exists a finite set F_a of $[x]$ such that $\|\hat{a}(y)\| < \varepsilon$ for all $y \notin F_a$. Of course, $B_A(\bar{x}) = C_A(\bar{x})$ if $[x]$ is an equivalence class consisting of finitely many points of A^b. It is clear that the quotient C^*-algebra $A(\bar{x}) = A/\bar{x}$ is a subalgebra of the discrete direct sum $\bigoplus_{y \in [x]} A(y)$ closed under the supremum norm, containing $C_A(\bar{x})$ and contained in $B_A(\bar{x})$. For each $a \in A$, let $\hat{a}(\bar{x}) \in A(\bar{x})$ be the image of a under the quotient map of A onto $A/\bar{x}$, and let

$$(1) \qquad \|\hat{a}(\bar{x})\|_1 = \sup_{y \in [x]} \|\hat{a}(y)\| ,$$

where $\hat{a}(y)$ is as in Theorem 1 the image of a under the quotient map of A onto A/y. For each $a \in A$, the function $y \longrightarrow \|\hat{a}(y)\|$ is lower semi-continuous on A^b by Lemma 3. Since the supremum of lower semi-continuous functions is lower semi-continuous, $x \longrightarrow \|\hat{a}(\bar{x})\|_1$ is lower semi-continuous on A^b and constant on each equivalence class. It follows that for each $a \in A$, $\bar{x} \longrightarrow \|\hat{a}(\bar{x})\|_1$ is a lower semi-continuous function on X since X has the quotient topology. We note that the preceding function on X reaches its supremum $\|a\|$ in X.

Let $[x]^-$ be the closure of $[x]$ (as a subset of A^j) in A^j. We remark that $[x]^-$ is equal to

$S = \{J \in A^j \mid \text{there exists } z \in [x] \text{ such that } z \subset J\}$. In fact, let

$J \in [x]^-$. Then $J \supset \bigcap_{y \in [x]} y$. By *-boundedness of A, there exists $z \in A^b$ such that $J \supset z \supset \bigcap_{y \in [x]} y$. Since $[x]$ is closed in A^b, we have $z \in [x]$. Hence $J \in S$. As $S \subset [x]^-$ is obvious, we have $[x]^- = S$.

For any closed ideal I in A, let $||a + I||$ denote the quotient norm of $a + I$ in A/I. We have the following:

LEMMA 8[2]. *For a pre-standard C^*-algebra A, and $x \in A^b$,*

$$||\hat{a}(\bar{x})||_1 = \sup_{y \in [x]} ||a + y|| = ||a + \bar{x}|| \,, \quad \textit{for all} \quad a \in A \,.$$

Proof. By Theorem 4.9.14 of Rickart [29], for the closed subset $[x]^-$ of A^j, and $a \in A$, there exists a $Q \in [x]^-$ such that

$$||a + Q|| = \sup \{||a + y||: y \in [x]^-\} \,.$$

By the preceding remark, there exists a $z \in [x]$ with $z \subset Q$. Thus $||a + z|| \geqq ||a + Q||$. Consequently, $||a + Q|| = ||a + z|| = ||\hat{a}(\bar{x})||_1$.

Now, $\bigcap \{J \mid J \in [x]^-\} = \bar{x} = \bigcap \{y \mid y \in [x]\}$. Thus $\mathrm{Prim}\,(A/\bar{x}) = \{J/\bar{x} \mid J \in [x]^-\}$. By the proof of Theorem 4.9.14, line 8 in [29], it follows that

$$||a + \bar{x}|| = \sup \{||(a + \bar{x}) + P||: P \in \mathrm{Prim}\,(A/\bar{x})\} \,.$$

But $(A/\bar{x})/(J/\bar{x}) \cong A/J$, so for $P = J/\bar{x}$ with $J \in [x]^-$, $||(a + \bar{x}) + J/\bar{x}|| = ||a + J||$. Thus,

$$||a + \bar{x}|| = \sup \{||a + J||: J \in [x]^-\} \,.$$

By the first paragraph, it now follows that

$$||a + \bar{x}|| = ||a + Q|| = ||a + z|| = ||\hat{a}(\bar{x})||_1 \,.$$

From now on $||\hat{a}(\bar{x})||_1$ will be written as $||\hat{a}(\bar{x})||$.

LEMMA 9. *For each $a \in A$, $N_a \colon x \mapsto ||\hat{a}(\bar{x})||$ is a continuous function on X.*

Proof. We only need to show that N_a is upper semi-continuous. Since X is Hausdorff, it will be sufficient to show that

$$E_k = N_a^{-1}([k, \infty)) = \{\bar{x} \mid ||\hat{a}(\bar{x})|| \geqq k\}$$

is compact for each $k > 0$.

We use the same argument as that in Proposition 3.3.7 [13]. Let $\{Z_i\}$ be a filter base of decreasing nonempty relatively closed subsets of E_k. Let $J_i = \bigcap_{\bar{x} \in Z_i} \bar{x}$, and $J = \overline{\bigcup J_i}$. The norm of the

[2] The author thanks the referee for supplying this lemma.

canonical image of a in the quotient algebra A/J is $\geqq k$. Hence, there exists a minimal primitive ideal x_0 of A such that $x_0 \supset J$. However, for each i

$$J \supset J_i = \bigcap_{x \in q^{-1}(Z_i)} x ,$$

(q is the quotient map of A^b onto X). Since q is continuous, each each $q^{-1}(Z_i)$ is closed. Hence $x_0 \in q^{-1}(Z_i)$ for all i. Therefore, $q(x_0) \in Z_i$ for all i.e., $q(x_0)$ is an adherence point of $\{Z_i\}$. This shows that E_k is compact.

LEMMA 10. *For* $x \in A^b$, $[z] \in X$, $x \notin [z]$, *we have*

$$x + \bigcap_{y \in [z]} y = A .$$

Proof. We first show that for any closed ideals x, y, z in A,

$$(x + y) \cap (x + z) \subset x + (y \cap z) . \tag{2}$$

Let $a \in (x + y) \cap (x + z)$. Then $a = a_1 + b = a_2 + c$, where $a_1, a_2 \in x$, $b \in y, c \in z$. Since each element a is a linear combination of four positive elements, we can assume that a is positive. Then

$$a^2 = a_1a_2 + a_1c + ba_2 + bc \in x + (y \cap z) .$$

Since $x + (y \cap z)$ is also a C^*-algebra, we have $a \in x + (y \cap z)$ (Lemma 7, p. 207 in Bonsall and Duncan [6]). Hence (2) is proved.

Now let $[z]$ be well-ordered, and let y_1 be the first element. We can assume that $[z]$ has a last element y_r. For if it doesn't, we can simply add y_1 to the end, and this will not change $\bigcap_{y \in [z]} y$. We have by normality of A that $x + y_1 = A$. Now suppose that

$$x + \bigcap_{y < y_\alpha} y = A .$$

Then by (2),

$$A = (x + \bigcap_{y < y_\alpha} y) \cap (x + y_\alpha) = x + \bigcap_{y \leqq y_\alpha} y .$$

Hence, by transfinite induction,

$$x + \bigcap_{y \in [z]} y = x + \bigcap_{y \leqq y_r} y = A .$$

Now, $\hat{a}$ can be regarded as a vector field on X. Let $\bar{A}$ be the subspace of the cartesian product $\Pi = \prod_{\bar{x} \in X} A(\bar{x})$ consisting of all $\hat{a}$, $a \in A$. Then $\bar{A}$ clearly satisfies conditions (i), (ii), and (iii) of Definition 10.1.2 of [13]. Also, $\bar{A}$ is obviously stable under the involution and the multiplication of the product algebra Π. By Propositions 10.2.3 and 10.3.2 of [13], there exists a subset Γ of Π containing $\bar{A}$ such that $F = ((A(\bar{x}))_{\bar{x}\in X}, \Gamma)$ is a continuous field of C^*-algebras $(A(\bar{x}))_{\bar{x}\in X}$. Let $\tilde{A}$ be the C^*-algebra defined by the continuous field F of pseudo-primitive C^*-algebras. We have the following:

THEOREM 2. *Let A be a pre-standard C^*-algebra with base X. Let $F = ((A(\bar{x}))_{\bar{x}\in X}, \Gamma)$ be the continuous field of pseudo-primitive C^*-algebras over X defined by A. Let $\tilde{A}$ be the C^*-algebra defined by F. Then the Gelfand transform $a \to \hat{a}$, $a \in A$, is an isometric isomorphism of A onto $\tilde{A}$, where each coordinate algebra $A(\bar{x})$ is a closed *-subalgebra of $B_A(\bar{x})$ containing $C_A(\bar{x})$.*

Proof. The same argument given in the proof of Theorem 1 applies here. We need only check that $\bar{x}_0 + \bar{y}_0 = A$ for any pair $[x_0]$ and $[y_0]$ of distinct equivalence classes. We have for any $x \in [x_0]$

$$x + y = A\,, \quad \text{for all} \quad y \in [y_0]$$

since A is normal. Hence for any $x \in [x_0]$,

$$x + \bar{y}_0 = x + \bigcap_{y\in[y_0]} y = A\,,$$

by Lemma 10. Therefore,

$$\bar{x}_0 + \bar{y}_0 = \bigcap_{x\in[x_0]} x + \bar{y}_0 = \bigcap_{x\in[x_0]} (x + \bar{y}_0) = A\,.$$

REMARK 2. We are not able to give a more precise description of $A(\bar{x})$ other than that it is contained in $B_A(\bar{x})$ and contains $C_A(\bar{x})$. This is perhaps due to the fact that some topological structure is lost when we map the whole equivalence class $[x]$ into one point $\bar{x}$. The algebra $A(\bar{x})$ does not depend on $\{A(y)\}_{y\in[x]}$ alone, it also reflects the structure of A. The following pair of examples will illustrate this situation:

Let B be the set of all sequences $a = \{a_n\}$ of infinite matrices representing bounded linear operators on l^2, each convergent in the operator norm to a scalar matrix. B is a C^*-algebra with the pointwise multiplication and involution (conjugation of matrices), and the norm $\|a\| = \sup_n \|a_n\|$, where $\|a_n\|$ is the operator norm of the matrix a_n, $n = 1, 2, \cdots$. Then B is a pre-standard C^*-algebra with the base X homeomorphic to the subspace $\{0, 1, 1/2, 1/3, \cdots\}$ of the real line. Each pseudo-primitive ideal I of A is actually a minimal primitive ideal and A/I is isomorphic to $B(H)$, where $H = l^2$, except the ideal J corresponding to the point 0, which is the intersection of infinitely many minimal primitive ideals I_i' with each A/I_i' isomorphic to the field of complex numbers, where $I_i' = \{a \mid i\text{th coefficient in the diagonal of } \lim_n a_n \text{ is } 0\}$, $i = 1, 2, \cdots$. A/J is isomorphic to the algebra of all bounded sequences of complex numbers, hence is equal to $B_A(J)$.

Next, let K be the set of all sequences $a = \{a_n\}$ of infinite square matrices representing compact operators on l^2, each of which converges to a scalar matrix in the operator norm. K is a C^*-algebra under pointwise multiplication and involution with norm $\|a\| = \sup_n \|a_n\|$, $n = 1, 2, \cdots$, where $\|a_n\|$ is the operator norm of a_n. Then K is a pre-standard C^*-algebra with the base X homeomorphic to the subspace $\{0, 1, 1/2, 1/3, \cdots\}$ of the real line. Each pseudo-primitive ideal I of A is actually a minimal ideal of A, and A/I is isomorphic to $K(H)$, where $H = l^2$, except the ideal J corresponding to the point 0. J is the intersection of infinitely many minimal primitive ideals I' with each A/I' isomorphic to the field of complex numbers. But A/J is isomorphic to the algebra of all bounded sequences of complex numbers convergent to zero, hence is equal to $C_A(J)$.

Finally, we make some comment about our structure theorem and that of Dauns and Hofmann [11]. First of all, we represent C^*-algebras as C^*-algebras defined by continuous fields of C^*-algebras as in [13]. Dauns and Hofmann [11] developed a more elaborate and somewhat more abstract notion of a field uniform structure. The class of pre-standard C^*-algebras is wide enough to include many interesting examples of C^*-algebras (we actually have difficulty in finding a C^*-algebra which is not pre-standard though such a C^*-algebra may well exist). However, our present result is still limited in the sense that it can not yet be applied to all C^*-algebras. By contrast, the non-commutative Gelfand-Naimark theorem stated in [11] is for general C^*-algebras. On the other hand, we can say something concretely about each coordinate algebra in our decomposition, i.e. a subdirect product (in the sence of [22] of primitive C^*-algebras. The description for each fibre algebra in [11] becomes ambiguous when the C^*-algebra is very general. We get our base X simply by identification of certain points in the basic structure space A^b of a C^*-algebra A while in [11] a more abstract but universal process called Hausdorffization is employed. (Dauns [10] gives a more concrete description for this in case that each primitive ideal in A does not contain the center of A.) It should be pointed out that the memoir of Dauns and Hofmann [11] lays the foundation of a theory that generalizes the studies of sheaves and fibre bundles and the application to C^*-algebras is only one aspect of their work. We conclude this section with two examples to illustrate the difference of of our approaches.

Let $X = [0, 1] \cup \{2\}$ with the induced topology from the real line.

Let A be the algebra of all norm continuous functions a from X to $B(H)$, where H is a separable infinite dimensional Hilbert such that $a(x) \in K(H)$ for all $x \in [0, 1]$ and $a(2)$ is a scalar multiple of the identity operator. For $a, b \in A$, $(ab)(2) = a(2)b(2)$, $(ab)(x) = a(x)b(x) + a(2)b(x) + a(x)b(2)$, $x \in [0, 1]$. We have

$$\operatorname{Prim}(A) = \{I_*, I_x \mid x \in [0, 1]\} ,$$

where $I_* = \{a \mid a(2) = 0\}$, and $I_x = \{a \mid a(x) = a(2) = 0\}$. A is $*$-bounded and normal,

$$A^b = \{I_x \mid x \in [0, 1]\}$$

and can be identified with $[0, 1]$ with the usual topology. Hence A is standard and isometric isomorphic to a C^*-algebra defined by a continuous field of C^*-algebras $A(x)$ over $[0, 1]$, where each $A(x) = A_0$, the primitive C^*-algebra generated by $K(H)$ and $\{cI\}$. (This is not the C^*-algebra of all continuous functions from $[0, 1]$ to A_0. In short, the continuous field of C^*-algebras is not trivial.) The center Z of A consists of scalars, i.e., $\boldsymbol{Z} \cong \boldsymbol{C}$. By Dauns and Hofmann [11] Corollary 8.14 (3), the base space is a singleton, i.e., the algebra A is of a single fibre in their theory.

A variation of the above example is the following: Let A_1 be the subalgebra of A consisting of all $a \in A$ with $a(1)$ a diagonal operator with respect to a fixed orthonormal basis $\{e_i\}_{i=1}^{\infty}$ in H. Again the center of A is $\boldsymbol{C}$, and consequently it is an algebra of a single fibre in the theory of [11]. We have

$$\operatorname{Prim}(A_1) = \{I_* \cap A ,\quad I_x \cap A, I_i \mid x \in [0, 1), i = 1, 2, \cdots\} ,$$

where $I_i = \{a \in A_1 \mid$ the ith coefficient in the diagonal of $a(1)$ is 0, $a(2)$ is $0\}$. A_1 is $*$-bounded and normal,

$$A_1^b = \{I_x \cap A, I_i \mid x \in [0, 1), i = 1, 2, \cdots\}$$

and is not Hausdorff since $\{I_i\}$ can not be separated. However A_1^b is $T_{1\,1/2}$ and the quotient space $X \cong [0, 1]$ with all points corresponding to I_i, $i = 1, 2, \cdots$, identified as $\{1\}$ in $[0, 1]$. A_1 is pre-standard, and isometric isomorphic to a continuous field of C^*-algebras $A(x)$ over $[0, 1]$, where $A(x) = A_0$ for all x $[0, 1)$, but $A(1)$ is the pseudo-primitive algebra of all convergent sequences. We have $C_A(\bar{I}_1) \subsetneqq A(1) \subsetneqq B_A(\bar{I}_1)$ in the notation of Theorem 2.

6. Representation of C^*-algebras.

A state f of a C^*-algebra A is a positive linear functional on A of norm one. For each state f of a C^*-algebra A, we denote by π_f the cyclic representation of A on Hilbert space H_f associated with the standard Gelfand-Naimark-

Segal construction such that

$$(\pi_f(a)\xi \mid \xi) = f(a) \quad \text{for all} \quad a \in A\,;$$

we denote by V_f the associated linear map from A (or A_1 in case A is without identity) onto a dense subspace of H_f, and $\xi = V_f(1)$.

Let A be a densely bounded C^*-algebra with basic structure A^b. Let X be a dense subspace of A^b. X once chosen, is fixed. For each $x \in X$, choose one state f_x of A such that $\ker \pi_{f_x} = x$. We simply write f for f_x if no confusion would arise. Let Y be the set of all states f of A so chosen after each element of X. Y once chosen, is also fixed. Let

$$H = \bigoplus_{f \in Y} H_f\,.$$

For each $v \in H$, let $v(f) \in H_f$ be its f-coordinate. We have

$$\|v\|^2 = \sum_{f \in Y} \|v(f)\|^2 < \infty\,, \quad v \in H\,.$$

An operator T on the Hilbert space H is called *decomposable* if

$$(Tv)(f) = T(f)v(f)\,, \quad v \in H\,,$$

where $T(f) \in B(H_f)$ for each $f \in Y$. We call an operator T on H *A-decomposable* if T is decomposable and $T(f) \in \pi_f(A)$ for each $f \in Y$. For each $a \in A$, define $\hat{a}$ on H as follows:

$$(\hat{a}v)(f) = \pi_f(a)v(f)\,, \quad v \in H\,.$$

$\hat{a}$ is obviously linear on H, and we have

$$\|\hat{a}v\| = \Big(\sum_{f \in Y} \|\pi_f(a)v(f)\|^2\Big)^{1/2} \leqq \|a\|\,\|v\|\,.$$

Hence for each $a \in A$, $\hat{a}$ is a bounded linear operator on H. Clearly, $\varphi\colon a \to \hat{a}$ is a linear map from the C^*-algebra A into the set of all A-decomposable operators on H.

By a straight forward computation, we have

$$\widehat{ab} = \hat{a}\hat{b},\ \hat{a}^* = a^*,\ \text{i.e.,}\ \varphi(ab) = \varphi(a)\varphi(b),\ \varphi(a)^* = \varphi(a^*)$$

for all $a, b \in A$. Since the basic structure space is dense in A^j, X is also dense in A^j. Hence

$$\bigcap_{x \in X} x = \bigcap_{I \in A^j} I = 0\,.$$

If $\hat{a} = 0$, then

$$a \in \bigcap_{f \in Y} \pi_f^{-1}(0) = \bigcap_{x \in X} x = \{0\}\,, \quad \text{where} \quad \pi_f^{-1}(0) = x\,.$$

This shows that φ is injective. As any norm-decreasing isomorphism from a C^*-algebra into a Banach $*$-algebra is isometric, we have the following:

THEOREM 3. *Let A be a densely bounded C^*-algebra, and let X be a dense subset of the basic structure space A^b of A. Then $\varphi: a \rightarrow \hat{a}$ constructed above is an isometric isomorphism of the C^*-algebra A onto a norm-closed self-adjoint subalgebra $B(H)$.*

A representation π of a C^*-algebra A on a Hilbert H is called *separable* if H is separable. Let us call a C^*-algebra A *separably representable* if it has a faithful separable representation. An immediate consequence of Theorem 3 is the following:

COROLLARY 2. *Let A be a densely bounded C^*-algebra with a separable basic structure space A^b. If every exactly irreducible representations of A is physically equivalent to a separable representation, then A is separably representable.*

REMARK 3. Theorem 3 is a refinement of the representation theorem in Gelfand and Naimark [18]. The Hilbert space H in [18] is usually nonseparable for an infinite dimensional C^*-algebra A since the direct sum is taken over all states of A to ensure the faithfulness of the representation. Of course, this is not always necessary. For example, if A is (norm) separable, the direct sum can be taken over a countable separating family F of states f and each H_f is separable; so A is separably representable. But such a judicious choice of a countable separating set F of states is not always possible in general. And many separably representable C^*-algebras are not separable. In fact, any infinite dimensional von Neumann algebra is not norm separable. The separable representability of a C^*-algebra is of considerable interest to theoretical physicists. It seems that the sufficient condition of being separably representable in Corollary 4 is also very close to be a necessary condition. The condition that every exactly irreducibly representation of a C^*-algebra A is physically equivalent to a separable representaion is weaker than the condition that every irreducible representations of A is physically equivalent to a separable representation as the following example shows: Take $A = B(H)$, H a separable infinite dimensional Hilbert space. There is only one physical equivalence class of exactly irreducible reprentations of A, i.e., the one containing the identity representation which is of course a separable representation. However, not every irreducible representations of A is physically equivalent to a separable representation. For all irreducible representations of $B(H)$ induced from a faithful irreducible representation of the Calkin algebra $B(H)/K(H)$ is necessarily taken on a non-separable Hilbert space (see Calkin [8]).

DEFINITION 8. A densely bounded C^*-algebra A with a dense subspace X of the basic structure space A^b is called *X-normal* if any primitive ideal of A does not contain two distinct minimal primitive ideals from X.

It follows from the definition that if $I, J \in X$ and $I \neq J$, then $I + J = A$.

Let A be a densely bounded X-normal C^*-algebra such that X is a locally compact Hausdorff space. Let Y, φ and H be as in the paragraph preceding Theorem 3. Ler Y have the topology transplanted from X. Let D_A denote the subalgebra of $B(H)$ consisting of all A-decomposable operators T on H such that $f \longrightarrow \|T(f)\|$ is continuous and vanishing at infinity on Y. It is not difficult to see that $\varphi(A)$ is a closed *-subalgebra of D_A. $\varphi(A)$ obviously satisfies the conditions (i), (ii), and (iii) of Definition 10.1.2 of [13] with respect to the family $(\pi_f(A))_{f \in Y}$ of primitive C^*-algebras. By Proposition 10.3.2 of [13], there exists a subset Γ of the cartesian product $\prod_{f \in Y} \pi_f(A)$ containing $\varphi(A)$ such that $G = ((\pi_f(A))_{f \in Y}, \Gamma)$ is a continuous field of primitive C^*-algebras $(\pi_f(A))_{f \in Y}$ over Y which can be identified with X. By the same argument as that in the proof of Theorem 1, we conclude that the isomorphism from A to the C^*-algebra defined by the continuous field G of C^*-algebras is onto. In other word, $\Gamma = \varphi(A) = D_A$. We summarize the above discussion as follows:

THEOREM 4. *Let A be a densely bounded C^*-algebra with a dense subspace X of the basic structure space A^b. If A is X-normal and X is a locally compact Hausdorff space, then φ: $a \longrightarrow \hat{a}$ as defined in the paragraph preceding Theorem 3 is an isometric isomorphism of the C^*-algebra A onto the C^*-algebra D_A of all continuous A-decomposable operators on H vanishing at infinity.*

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Received June 10, 1975 and in revised form July 16, 1976. This research was supported by National Science Foundation grant GN-28517.

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