PACIFIC JOURNAL OF MATHEMATICS
Vol. 67, No. 2, 1976

ON THE VALUE DISTRIBUTION OF FUNCTIONS MEROMORPHIC IN THE UNIT DISK WITH A SPIRAL ASYMPTOTIC VALUE

JAMES R. CHOIKE

The object in this paper is to examine the value distribution of functions $f(z)$ nonconstant and meromorphic in the unit disk which have an asymptotic value α, finite or infinite, along a spiral boundary path. The main result which we prove is that if $\Delta(r)$ is a component of the set of values z such that $|f(z)-\alpha|<r$, $r>0$, which contains a boundary path on which $f(z)$ tends to α as $|z|\to 1$, then $f(z)$ assumes every value in $|w-\alpha|<r$ infinitely often in $\Delta(r)$ except for at most two values (if $\Delta(r)$ is simply-connected, then there is at most one exceptional value).

1. **Introduction.** A boundary path $S: z = s(t)$, $0 \leqq t < 1$, in $|z| < 1$ shall be called a spiral if arg $s(t) \to +\infty$ as $t \to 1$ or arg $s(t) \to -\infty$ as $t \to 1$. We shall denote by (S) the class of functions, nonconstant and meromorphic in $|z|<1$, which have an asymptotic value α, finite or infinite, along a spiral S. The object in this paper is to examine the value distribution of the functions in class (S). In § 3 we have the main result which is a version of Picard's theorem localized to a neighborhood of a transcendental singularity for functions of class (S). This result is applied in § 4 to show that, in some sense, the number of direct transcendental singularities of functions of class (S) cannot be too large. These results extend some earlier work of K. Noshiro [7] (see also [9, p. 163-167]).

2. **Components of functions of class (S).** Let $f(z) \in (S)$. Then there is a complex value α, finite or infinite, and a spiral $S: z = s(t)$, $0 \leqq t < 1$, in $|z| < 1$ such that $\lim_{t\to 1} f(s(t)) = \alpha$. Let $r > 0$ and ω be a complex number. We form the open set

$$G = \{z \mid |f(z) - \omega| < r\}$$

if ω is finite, and

$$G = \{z \mid |f(z)| > r\}$$

if ω is infinite. Since functions of class (S) are of unbounded characteristic [5, p. 172], the global cluster set $C(f)$ of $f(z)$ in $|z|<1$ is total; hence, $G \neq \varnothing$. We denote by $\Delta(r)$ a nonempty open component of G. If $\mathrm{Fr}\,(\Delta(r)) \cap \{|z| = 1\} = \varnothing$, where $\mathrm{Fr}\,(A)$ denotes the set of frontier points of the set A, we shall call $\Delta(r)$ a finite domain.

If, however, $\mathrm{Fr}(\Delta(r)) \cap \{|z| = 1\} \neq \emptyset$, we shall call $\Delta(r)$ an infinite domain.

By the minimum principle, each finite domain $\Delta(r)$ contains a zero of $f(z) - \omega$. Also, by Rouche's theorem, a finite domain $\Delta(r)$ contains the same number of roots, counting multiplicities, for $f(z) - \beta$ for each value β, $|\beta - \omega| < r$.

If $\Delta(r)$ is an infinite domain, then $\Delta(r)$ is not, in general, simply-connected. However, an easy application of the maximum principle shows $\Delta(r)$ to be simply-connected if either $\omega \neq \infty$ and $f(z)$ is holomorphic in $|z| < 1$, i.e., $f(z)$ omits ∞ in $|z| < 1$, or $\omega = \infty$ and $f(z)$ omits 0 in $|z| < 1$. An infinite domain $\Delta(r)$ with the property that the portion of its boundary $\mathrm{Fr}(\Delta(r))$ which lies within $|z| < 1$ consists entirely of closed analytic curves shall be called an annular domain. An infinite domain $\Delta(r)$ with the property that there exists a spiral S^* in $|z| < 1$ such that $\Delta(r) \subseteqq \{|z| < 1\} - S^*$ shall be called a spiral domain.

THEOREM 1. *If $\Delta(r)$ is an infinite domain for $f(z) \in (S)$, then $\Delta(r)$ is one of the following*: (i) *a spiral domain,* (ii) *an annular domain, or* (iii) *a subset of an annular domain.*

Proof. Suppose $\Delta(r)$ is neither annular domain nor a spiral domain. Let $r_1 > r$. Let $\Delta(r_1)$ be the component of the open set $\{z \mid |f(z) - \omega| < r_1\}$ such that $\Delta(r) \subseteqq \Delta(r_1)$. Suppose $\Delta(r_1)$ is not an annular domain. Then, applying methods found in [3], we can find a boundary path L' in $|z| < 1$ on which $|f(z) - \omega| = r_1$. Since $\Delta(r)$ is not a spiral domain, the spiral S intersects $\Delta(r)$ in $1 - \delta < |z| < 1$ for each $\delta > 0$. Thus, $|\alpha - \omega| \leqq r$, and there exists t_0, $0 < t_0 < 1$, such that the spiral $S^*: z = s(t)$, $t_0 \leqq t < 1$, is disjoint from L'. Thus, L' is also a spiral and $\Delta(r) \subseteqq \Delta(r_1) \subseteqq \{|z| < 1\} - L'$. But this implies that $\Delta(r)$ is a spiral domain, contrary to our assumption. Therefore, $\Delta(r_1)$ is an annular domain.

THEOREM 2. *If $\Delta(r)$ is an infinite domain for $f(z) \in (S)$ and if L is a boundary path in $\Delta(r)$ on which $f(z) \to \omega$ as $|z| \to 1$, then $\Delta(r)$ is either a spiral domain or an annular domain.*

Proof. Recall that α is the asymptotic value of $f(z)$ along the spiral S. Suppose $\omega \neq \alpha$. Then, in this case, the boundary path L must be a spiral. Suppose $\Delta(r)$ is not an annular domain. As in the proof above we apply the methods found in [3] to find a boundary path L' in $|z| < 1$ on which $|f(z) - \omega| = r$. Since L is a spiral, L' is a spiral and $\Delta(r) \subseteqq \{|z| < 1\} - L'$.

Suppose $\omega = \alpha$. Then, there exists t_0, $0 < t_0 < 1$, such that

$|f(s(t)) - \alpha| < r$ for all t, $t_0 \leqq t < 1$. Let $S^*: z = s(t)$, $t_0 < t < 1$. Clearly, either $S^* \subseteqq \Delta(r)$ or $\Delta(r) \subseteqq \{|z| < 1\} - S^*$. If $S^* \subseteqq \Delta(r)$, then we can argue as above to show that $\Delta(r)$ is a spiral domain.

Let $z = \phi(w)$ denote the inverse function of $w = f(z) \in (S)$. The damain of $z = \phi(w)$ is a Riemann surface Φ. We shall write $Q(w; w_0)$ to denote a functional element with center $w = w_0$ for $z = \phi(w)$. Let

$$\Lambda: q(t) = Q(w; w(t)), \quad 0 \leqq t < 1,$$

with $\lim_{t \to 1} w(t) = \omega$, be a curve on the Riemann surface Φ. This curve Λ is said to define a transcendental singularity Ω for $z = \phi(w)$ on Φ, with projection $w = \omega$, if (i) for every positive number δ, $\delta < 1$, the system of functional elements $Q(w; w(t))$, $0 \leqq t \leqq \delta$, defines an analytic continuation (possibly, of algebraic character), but (ii) for any functional element $Q(w; \omega)$, rational or algebraic, with center at $w = \omega$, the system $Q(w; w(t))$, $0 \leqq t \leqq 1$, where $w(1) = \omega$, never defines an analytic continuation. A theorem due to the work of Iversen [6, p. 13] and Noshiro [7, p. 53] states that there is a one-to-one correspondence between the asymptotic paths of $w = f(z)$ and the transcendental singularities of $z = \phi(w)$, the inverse function of $w = f(z)$. In view of this result, we shall say that two asymptotic boundary paths L_1 and L_2 for $f(z) \in (S)$ are equivalent if L_1 and L_2 both correspond (in the sense of the Iversen-Noshiro theorem) to the same transcendental singularity Ω on the Riemann surface Φ of $z = \phi(w)$, and, we shall indicate this equivalence by the notation $[L_1] = [L_2]$. We refer the reader to [2] and [3] where the notions of equivalent and nonequivalent asymptotic paths are analyzed in greater detail.

THEOREM 3. *If $f(z) \in (S)$ and $\Delta(r)$ is an annular domain for all $r > 0$, then the inverse function $z = \phi(w)$ of $w = f(z)$ has exactly one transcendental singularity and it lies above $w = \omega$.*

Proof. Suppose $z = \phi(w)$ has at least two transcendental singularities Ω_1 and Ω_2. Let S_1 and S_2 be the asymptotic boundary paths in $|z| < 1$ for $f(z)$ which correspond to Ω_1 and Ω_2, respectively. Since $f(z) \in (S)$, S_1 and S_2 are spirals. Let $r > 0$. Since $\Delta(r)$ is an annular domain, S_1 and S_2 intersect $\Delta(r) \cap \{\delta < |z| < 1\}$ for each δ, $0 < \delta < 1$. Since r is an arbitrary positive number, we have that ω is the asymptotic value on S_1 and S_2. By [3, Theorem 1], $[S_1] \neq [S_2]$ implies that there exists $b > 0$ and a boundary path L', necessarily a spiral, on which $|f(z) - \omega| = b$. But, then $\Delta(r_0) \subseteqq \{|z| < 1\} - L'$ for r_0, $0 < r_0 < b$, which contradicts our hypothesis that $\Delta(r_0)$ is an annular domain. Hence, $z = \phi(w)$ has exactly one transcendental singularity.

In [10] Valiron offers a construction of a function which shows that the converse of Theorem 3 is false. Valiron's construction is very difficult to follow and prompts the need for another approach to the construction of such a function.

3. Value distribution of functions of class (S) in $\Delta(r)$. We denote by $n(\phi, a)$ the number of functional elements $Q(w; a)$, with center $w = a$, for $z = \phi(w)$ the inverse of $f(z) \in (S)$, where an algebraic functional element is counted k times if its order of ramification is $k - 1$. Noshiro [7, p. 60] proved the following: Let $z = \phi_D(w)$ denote the branch of $z = \phi(w)$ obtained by continuing $Q(w; a)$, $a \in D$, inside D with algebraic elements, where D is an arbitrary domain of the w-plane. If $z = \phi_D(w)$ has no transcendental singularity with projection inside D, then $n(\phi_D, w)$ is a finite or infinite constant in D.

THEOREM 4. *If $\Delta(r)$ is an annular domain for $f(z) \in (S)$ and if the transcendental singularities of $z = \phi(w)$ lying above $|w - \omega| < r$ have the property that they lie above at most a finite set of points $w_1, w_2, \cdots, w_k$ in $|w - \omega| < r$, then every value of $|w - \omega| < r$ is assumed infinitely often by $f(z)$ in $\Delta(r)$, except possibly the values $w_1, w_2, \cdots, w_k$.*

Proof. Let $D' = \{|w - \omega| < r\} - \cup_{j=1}^{k} \{w_j\}$. Then, the branch $z = \phi_{D'}(w)$ of $z = \phi(w)$ has no transcendental singularity with projection inside D'. By Noshiro's theorem above, the function $n(\phi_{D'}, w)$ is constant throughout D'. If $\Delta(r)$ had at most finitely many holes, then, since $\Delta(r)$ is an annular domain, the global cluster set $C(f)$ of $f(z)$ would be contained in the closed disk $|w - \omega| \leqq r$. But this contradicts functions of class (S) having total global cluster sets. Thus, $\Delta(r)$ has infinitely many holes. Each hole is bounded by a closed analytic curve whose image under $f(z)$ covers the circumference $|w - \omega| = r$ completely. Thus, $n(\phi_{D'}, w) = +\infty$ throughout D' and we are done.

THEOREM 5. *If $\Delta(r)$ is a spiral domain for $f(z) \in (S)$, then each value β, $|\beta - \omega| < r$, omitted by $f(z)$ in $\Delta(r)$ is an asymptotic value along a spiral contained in $\Delta(r)$.*

Proof. Since $\Delta(r)$ is a spiral domain, there exists a spiral S' in $|z| < 1$ such that $\Delta(r) \subseteqq \{|z| < 1\} - S'$. Let $\zeta = \zeta(z)$ be a one-to-one conformal map of the simply-connected region $\{|z| < 1\} - S'$ onto

$|\zeta| < 1$ such that the prime end P of $\{|z| < 1\} - S'$ whose impression $I(P)$ is $|z| = 1$ corresponds to $\zeta = 1$. We use $z = z(\zeta)$ to denote the inverse map of $\zeta = \zeta(z)$. Let $\Delta'(r)$ be the image of $\Delta(r)$ in $|\zeta| < 1$ under $\zeta = \zeta(z)$. Since $\Delta(r)$ is a spiral domain, we have that $1 \in \mathrm{Fr}\,(\Delta'(r))$.

The function $F(\zeta) = f(z(\zeta))$ is holomorphic in $\Delta'(r)$ and continuous in $\overline{\Delta'(r)}$, with the exception of $\zeta = 1$. In fact, $|F(\zeta) - \omega| < r$ for $\zeta \in \Delta'(r)$ and $|F(\zeta) - \omega| = r$ for $\zeta \in \mathrm{Fr}\,(\Delta'(r))$, $\zeta \neq 1$.

Suppose $f(z)$ omits β in $\Delta(r)$, $|\beta - \omega| < r$. Then, $F(\zeta)$ omits β in $\Delta'(r)$. Let

$$F_2(\zeta) = \frac{F_1(\zeta) - \beta^*}{1 - \overline{\beta^*} F_1(\zeta)}$$

where $F_1(\zeta) = 1/r(F(\zeta) - \omega)$ and $\beta^* = 1/r(\beta - \omega)$. Then, $F_2(\zeta)$ is holomorphic in $\Delta'(r)$ with $|F_2(\zeta)| < 1$ in $\Delta'(r)$, $|F_2(\zeta)| = 1$ for $\zeta \in \mathrm{Fr}\,(\Delta'(r))$, $\zeta \neq 1$, and $F_2(\zeta) \neq 0$ in $\Delta'(r)$. If $\Delta(r)$ were simply-connected, then $\Delta'(r)$ would be simply-connected and we could, then, apply results of functions belonging to Seidel's class (U) [8, p. 32] to prove our theorem. Unfortunately, $\Delta(r)$ may not be simply-connected, in general. An argument of Doob [4] helps to surmount this difficulty.

Since $F_2(\zeta)$ omits 0 in $\Delta'(r)$, $1/F_2(\zeta)$ is holomorphic in $\Delta'(r)$. Suppose $1/F_2(\zeta)$ is bounded in $\Delta'(r)$. Then there exists a number $K > 0$ such that $1/|F_2(\zeta)| < K$ in $\Delta'(r)$. Let σ be a number such that $0 < \sigma < 1$. Choose a definite branch of the function $(1/2(\zeta - 1))^\sigma$ in $\Delta'(r)$. This branch is holomorphic in $\Delta'(r)$ with $|1/2(\zeta - 1)| \leqq 1$ in $\Delta'(r)$. The function

$$\phi_\sigma(\zeta) = \frac{\left(\dfrac{1}{2}(\zeta - 1)\right)^\sigma}{F_2(\zeta)}$$

is holomorphic and bounded in $\Delta'(r)$. Furthermore,

$$|\phi_\sigma(\zeta)| \leqq \frac{1}{|F_2(\zeta)|} = 1$$

for $\zeta \in \mathrm{Fr}\,(\Delta'(r))$, $\zeta \neq 1$. Also,

$$\lim_{\substack{\zeta \to 1 \\ \zeta \in \Delta'(r)}} \phi_\sigma(\zeta) = 0\ .$$

By the maximum principle, $|\phi_\sigma(\zeta)| \leqq 1$ for $\zeta \in \Delta'(r)$, for every $\sigma > 0$. If we let $\sigma \to 0$, we have that $|F_2(\zeta)| \geqq 1$ in $\Delta'(r)$, and this is a contradiction. Thus, we have that 0 is a cluster value for F_2 at $\zeta = 1$ in $\Delta'(r)$. But, clearly, 0 does not belong to the set of boundary cluster values for F_2 at $\zeta = 1$ in $\Delta'(r)$. Since 0 is omitted by F_2 in

$\Delta'(r)$, by the Gross-Iversen theorem [8, p. 23-24], there exists a path L in $\Delta'(r)$ terminating at $\zeta = 1$ on which $F_2(\zeta) \to 0$ as $|\zeta| \to 1$. Thus, on L $f(z(\zeta)) \to \beta$ as $|\zeta| \to 1$. We simply notice that the image of L is a spiral lying in $\Delta(r)$ on which $f(z) \to \beta$ as $|z| \to 1$.

THEOREM 6. *Let $\Delta(r)$ be an infinite domain for $f(z) \in (S)$ such that $\Delta(r)$ contains a boundary path L on which $f(z) \to \omega$ as $|z| \to 1$. Let $\Delta_\tau = \Delta(r) \cap \{|z| < \tau\}$, $0 < \tau < 1$, and let $A(\tau)$ denote the area of the Riemannian image of Δ_τ under $w = f(z)$. Then*

$$\lim_{\tau \to 1} A(\tau) = +\infty .$$

Proof. Suppose for all r_1, $0 < r_1 < r$, $\Delta(r_1)$ is an annular domain (here it is understood, of course, that $\Delta(r_1)$ is that component which contains the end part of L). Then, by Theorem 3, the inverse $z = \phi(w)$ has exactly one transcendental singularity which lies above $w = \omega$. By Theorem 4, every value in the disk $|w - \omega| < r$, except possibly $w = \omega$, is assumed infinitely often by $f(z)$ in $\Delta(r)$. Thus, in this case,

$$\lim_{\tau \to 1} A(\tau) = +\infty .$$

Let us, next, consider the case that for some r_0, $0 < r_0 < r$, $\Delta(r_0)$ is a spiral domain. Let $\Delta_\tau^0 = \Delta(r_0) \cap \{|z| < \tau\}$, $0 < \tau < 1$, and let $A_0(\tau)$ denote the area of the Riemannian image of Δ_τ^0 under $f(z)$. Since $\Delta(r_0) \subseteq \Delta(r)$, it follows that $A_0(\tau) \leqq A(\tau)$. In view of this, it suffices to assume that $\Delta(r)$ is a spiral domain. Since $\Delta(r) \subseteq \{|z|<1\} - S'$ for some spiral S' in $|z| < 1$, we again map $\Delta(r)$ onto $\Delta'(r)$ using the one-to-one conformal map $\zeta = \zeta(z)$ of $\{|z| < 1\} - S'$ onto $|\zeta| < 1$ that we used in the proof of Theorem 5. Then, the image L' of the spiral L is a path in $\Delta'(r)$ which terminates at $\zeta = 1$.

Let $\Delta''(r)$ denote the image of $\Delta'(r)$ in the t-plane under the map $t = (\zeta - \zeta_0)/(\zeta - 1)$, where ζ_0 is the initial point of the path L'. The image L'' of L' under this map is a path which begins at the interior point $t = 0$ of $\Delta''(r)$ and terminates at the boundary point $t = \infty$ of $\Delta''(r)$. We define

$$G(t) = f\left(z\left(\frac{t - \zeta_0}{t - 1}\right)\right)$$

in $\Delta''(r)$. Let $\Delta_\tau'' = \Delta''(r) \cap \{|t| < \tau\}$, $0 < \tau < +\infty$. Let $A''(\tau)$ denote the area of the Riemannian image of the open set Δ_τ'' under $G(t)$. Since the range of $G(t)$ in $\Delta''(r)$ is identical to that of $f(z)$ in $\Delta(r)$ and since $\Delta''(r)$ is linked to $\Delta(r)$ by means of a one-to-one conformal map, it suffices to show that

$$\lim_{\tau\to+\infty} A''(\tau) = +\infty .$$

Let $\tau_0 > 0$ be fixed so that $|t| < \tau_0$ contains at least one boundary point of $\Delta''(r)$. Denote by L''_τ the part of the path L'' which runs from the last point of intersection t_τ of L'' with $|t| = \tau$, counting from $t = 0$. Since $G(t) \to \omega$ on L'' as $|t| \to +\infty$, there exists a number τ_1, $\tau_1 > \tau_0 > 0$, such that (i) $|G(t) - \omega| < (1/2)r$ for all $t \in L''_{\tau_1}$, and (ii) for any $\tau > \tau_1$, if γ_τ denotes the collection of component arcs of $|t| = \tau$ which fall into $\Delta''(r)$ (there can be at most finitely many such arcs since the boundary of $\Delta''(r)$ is an analytic curve), then γ_τ contains a crosscut of $\Delta''(r)$, call it λ_τ, such that the point $t_\tau \in \lambda_\tau$ and the endpoints of λ_τ lie on $\mathrm{Fr}\,(\Delta''(r))$. Since the image of the arc λ_τ under $G(t)$ is a curve which starts from a point on $|w - \omega| = r$, passes through a point lying in $|w - \omega| < (1/2)r$, and, finally, terminates at a point on $|w - \omega| = r$, we have that the length of the image of λ_τ under $G(t)$ is greater than or equal to r.

Let $L(\tau)$ be the total length of the image of γ_τ under $G(t)$; let $l(\tau)$ be the total length of γ_τ. Then for $t = \tau e^{i\theta}$,

$$L(\tau) = \int_{\gamma_\tau} |G'(t)|\tau d\theta .$$

By the Schwarz inequality,

$$(L(\tau))^2 \leqq \left(\int_{\gamma_\tau} |G'(t)|^2 \tau d\theta\right)\left(\int_{\gamma_\tau} \tau d\theta\right) = l(\tau)\int_{\gamma_\tau} |G'(t)|^2 \tau d\theta .$$

Hence,

$$(1) \qquad \frac{(L(\tau))^2}{l(\tau)} \leqq \int_{\gamma_\tau} |G'(t)|^2 \tau d\theta ,$$

and, for $\tau > \tau_1$,

$$(2) \qquad \int_{\tau_1}^{\tau} \frac{(L(\tau))^2}{l(\tau)} d\tau \leqq \int_{\tau_1}^{\tau}\int_{\gamma_\tau} |G'(t)|^2 \tau d\tau d\theta \leqq A''(\tau) - A''(\tau_1) .$$

Since $l(\tau) \leqq 2\pi\tau$ and $L(\tau) \geqq r$ for all τ, $\tau_1 \leqq \tau \leqq +\infty$,

$$\int_{\tau_1}^{\tau} \frac{(L(\tau))^2}{l(\tau)} d\tau \geqq \frac{r^2}{2\pi}\int_{\tau_1}^{\tau} \frac{d\tau}{\tau} \longrightarrow +\infty$$

as $\tau \to +\infty$. Therefore,

$$\lim_{\tau\to+\infty} A''(\tau) = +\infty .$$

THEOREM 7. *Under the hypothesis of Theorem* 6,

$$\liminf_{t\to 1} \frac{L(\tau)}{A(\tau)} = 0 ,$$

where $L(\tau)$ denotes the length of the images of the collection of arcs of $|z| = \tau$ which fall into $\Delta(r)$ under $f(z)$.

Proof. By (2)

$$\frac{(L(\tau))^2}{l(\tau)} \leqq \frac{dA''(\tau)}{d\tau} .$$

Hence,

$$\frac{d\tau}{l(\tau)} \leqq \frac{dA''(\tau)}{(L(\tau))^2} .$$

Let $E = \{\tau \mid \tau > \tau_0, L(\tau) \geqq A''(\tau)^{1/2+\varepsilon}\}$, $0 < \varepsilon < 1/2$. Since $l(\tau) \leqq 2\pi\tau$,

$$\frac{1}{2\pi}\int_E \frac{d\tau}{\tau} \leqq \int_E \frac{d\tau}{l(\tau)} .$$

Thus,

$$\frac{1}{2\pi}\int_E \frac{d\tau}{\tau} \leqq \int_E \frac{dA''(\tau)}{(L(\tau))^2} \leqq \int_E \frac{dA''(\tau)}{(A''(\tau)^{1/2+\varepsilon})^2}$$
$$\leqq \int_{A''(\tau_0)}^{\infty} \frac{dt}{t^{1+2\varepsilon}} < + \infty .$$

Thus, there exists a sequence of positive numbers $\{\tau_n\}$ such that $\tau_n \to +\infty$ as $n \to +\infty$ with $\tau_n \notin E$ for all n. Therefore,

$$\frac{L(\tau_n)}{A''(\tau_n)} \leqq \frac{A''(\tau_n)^{1/2+\varepsilon}}{A''(\tau_n)} = \frac{1}{A''(\tau_n)^{1/2-\varepsilon}}$$

and, by Theorem 6, we have

$$(3) \qquad \liminf_{\tau\to+\infty} \frac{L(\tau)}{A''(\tau)} = 0 .$$

For $f(z) \in (S)$, the Riemannian image Φ_r of the disk $|z| < r$, $0 < r < 1$, under $f(z)$ is a finite covering of the Riemann sphere of diameter 1 and tangent to the w-plane endowed with the spherical distance as metric. The Riemann images Φ_r exhaust the surface Φ of the inverse $z = \phi(w)$ of $w = f(z)$ [8, p. 90]. Let $A(r)$ denote the spherical area of Φ_r and let $L(r)$ denote the spherical length of the boundary of Φ_r.

COROLLARY. *For $f(z) \in (S)$,*

$$(\text{i}) \qquad \lim_{r\to 1} A(r) = +\infty ,$$

and

(ii) $$\liminf_{r\to 1} \frac{L(r)}{A(r)} = 0 .$$

Proof. Follows immediately from Theorems 6 and 7.

A Riemann surface Φ which satisfies condition (ii) in the above corollary is called regularly exhaustible. For a thorough discussion of regularly exhaustible surfaces and the value distribution theory connected with them we refer the reader to [9, p. 152–170]. In view of this, it is appropriate to state the following corollary.

COROLLARY. *The Riemannian image of a function of class* (S) *is regularly exhaustible.*

The next theorem is a Picard theorem localized to a transcendental singularity of the inverse function $z = \phi(w)$ of $f(z) \in (S)$. The idea behind the proof comes from the work of K. Noshiro [7].

THEOREM 8. *Let* $\Delta(r)$ *be an infinite domain for* $f(z) \in (S)$ *such that* $\Delta(r)$ *contains a boundary path* L *on which* $f(z) \to \omega$ *as* $|z| \to 1$. *Then, the function takes every value of* $|w - \omega| < r$ *infinitely often in* $\Delta(r)$, *except for at most two values. In particular, if* $\Delta(r)$ *is simply-connected, then* $f(z)$ *assumes every value of* $|w - \omega| < r$ *infinitely often in* $\Delta(r)$ *with one possible exception.*

Proof. We assume, first, that $\Delta(r)$ is simply-connected. Let $\Delta''(\tau)$, $A''(\tau)$, γ_τ, $L(\tau)$, and $G(t)$ be as found in the proof of Theorem 6. Suppose that w_1 and w_2 are distinct values in $|w - \omega| < r$ such that $G(t)$ omits w_1 and w_2 in $\Delta''(r) \cap \{|t| > \tau_0\}$ for some $\tau_0 > 0$. Consider the open set $\Delta''_\tau = \Delta''(r) \cap \{\tau_0 < |t| < \tau\}$. Since each component of Δ''_τ contains at least one arc of either γ_{τ_0} or γ_τ, we must have that Δ''_τ consists of finitely many simply-connected components

$$\Delta''_\tau(1), \Delta''_\tau(2), \cdots, \Delta''_\tau(m), m = m(\tau) .$$

The Riemannian image of Δ''_τ under $G(t)$ consists of m simply-connected covering surfaces $\Phi''_\tau(j)$ corresponding to $\Delta''_\tau(j)$, $j = 1, 2, \cdots, m$, of the base surface

$$B = \{|w - \omega| < r\} - \{w_1, w_2\} .$$

The Euler characteristic of B is 1 [5, p. 136].

Applying Ahlfors' theorem of covering surfaces [5, p. 137] to $\Delta''_\tau(j)$ and $\Phi''_\tau(j)$, we have $S^j \leqq hL^j$, $j = 1, 2, \cdots, m$, where S^j denotes the ratio between the area of $\Phi''_\tau(j)$ and the area of B, and L^j

denotes the length of the boundary of $\Phi''_\tau(j)$ relative to B, h being a constant dependent on B. Thus,

$$\sum_{j=1}^{m} S^j \leqq h \sum_{j=1}^{m} L^j ,$$

and

$$S(\tau) \leqq h(L(\tau) + L(\tau_0))$$

where $S(\tau) = A''(\tau)/\pi r^2$. Thus, $(L(\tau) + L(\tau_0))/S(\tau) \geqq 1/h$ for all τ, $\tau > \tau_0$. Therefore,

$$\lim_{\tau \to +\infty} \frac{L(\tau)}{A''(\tau)} \geqq \frac{\pi r^2}{h} > 0 .$$

But this contradicts (3). Thus, $G(t)$ in $\Delta''(r)$ and, clearly, $f(z)$ in $\Delta(r)$ assumes all values of $|w - \omega| < r$, except possibly one, infinitely many times.

Suppose $\Delta(r)$ is an arbitrary infinite domain which contains L; $\Delta(r)$ may not be simply-connected. Assume that there are three distinct values w_1, w_2, w_3 of $|w - \omega| < r$ which are assumed only finitely many times by $f(z)$ in $\Delta(r)$. We draw a simple closed analytic curve L in $|w - \omega| < r$ which encloses ω, w_1, w_2, and passes through w_3. Since $f(z) \to \omega$ on L as $|z| \to 1$, we may assume that the image of L under $f(z)$ lies entirely in the interior H of L.

Let Δ_H be the component of $\{z \mid f(z) \in H\}$ which contains the path L. Then, clearly, $\Delta_H \subseteqq \Delta(r)$. Choose r_0, $0 < r_0 < 1$, such that $f(z)$ omits w_1, w_2, and w_3 in $\Delta(r) \cap \{r_0 < |z| < 1\}$. Let Δ_H^* be the component of $\Delta_H \cap \{r_0 < |z| < 1\}$ which contains the end part of L. Then, $\Delta_H^* \subseteqq \Delta_H$, and Δ_H^* is simply-connected. Indeed, if it were not, then on the boundary of each hole $f(z)$ would assume the value w_3. But by the construction of Δ_H^* this is impossible. Also, Δ_H^* is clearly a spiral domain.

We can, now, apply the above argument to the simply-connected spiral domain Δ_H^* to show that it cannot omit two values. Thus, our theorem is proved.

THEOREM 9. *Each exceptional value of Theorem* 8 *is an asymptotic value for $f(z)$ along a spiral contained in $\Delta(r)$.*

Proof. Suppose $\beta \neq \omega$ is assumed by $f(z)$ in $\Delta(r)$ only finitely many times. Then, there exists δ, $0 < \delta < 1$, such that $f(z)$ omits β in $\Delta(r) \cap \{\delta < |z| < 1\}$. Choose δ_1, $\delta < \delta_1 < 1$, so that $f(z) \neq \beta$ on $|z| = \delta_1$. Let

$$\varepsilon = \min |f(z) - \beta| \quad \text{for} \quad |z| = \delta_1 ,$$

and, let $\rho = 1/4 \min(\varepsilon, r - |\beta - \omega|, |\beta - \omega|)$. Clearly, $\rho > 0$. By Theorem 8, there exists a value z_0 such that $z_0 \in \Delta(r) \cap \{\delta_1 < |z| < 1\}$ and $|f(z_0) - \beta| < \rho$. Let $\Delta_\beta(\rho)$ be that component of $\{z \,||\, f(z) - \beta| < \rho\}$ which contains z_0. By the choice of ρ, we have $\Delta_\beta(\rho) \subseteqq \{\delta_1 < |z| < 1\}$ and $\Delta_\beta(\rho) \subseteqq \Delta(r)$. Since $f(z)$ omits β in $\Delta_\beta(\rho)$, we have that $\Delta_\beta(\rho)$ is an infinite domain. We, also, point out that the end part of the spiral on which $f(z) \to \omega$ as $|z| \to 1$ is disjoint from $\Delta_\beta(\rho)$. Thus, $\Delta_\beta(\rho)$ is a spiral domain in which β is an omitted value of $f(z)$. By Theorem 5, β is an asymptotic value along a spiral contained in $\Delta_\beta(\rho)$.

4. **Direct transcendental singularities.** Let $f(z) \in (S)$ and $\Delta(r)$ be an infinite domain in $|z| < 1$ such that $\Delta(r)$ contains a boundary path L on which $f(z) \to \omega$. If $f(z)$ omits ω in $\Delta(r)$ for $r > 0$ sufficiently small, then the transcendental singularity Ω of $z = \phi(w)$ which corresponds to L is said to be a direct transcendental singularity.

THEOREM 10. *Let $f(z) \in (S)$ and let $z = \phi(w)$ be its inverse function. Then, the set of values ω in the w-plane which are projections of direct transcendental singularities of $z = \phi(w)$ is at most countable.*

Proof. Let $\{\omega_n\}$ be the rational points in the w-plane and let $\{r_n\}$ be the rationals of the interval (0, 1). Let $G_n = \{z \,||\, f(z) - \omega_n| < r_n\}$. We set $H = \bigcup_{n=1}^{\infty} H_n$, where H_n is the set of points of $|w - \omega_n| < r_n$ which are not covered by the image of at least one component of G_n under $f(z)$. By Theorem 9, H_n is at most countable, and, hence, so is H.

Suppose $f(z) \to \omega$ on a spiral S and S corresponds to a direct transcendental singularity for $z = \phi(w)$. Then, there exists $r > 0$ such that $S \subset \Delta(r)$ and $f(z)$ omits ω in $\Delta(r)$. But there exists an integer n such that a component Δ_n of G_n is contained in $\Delta(r)$ with $|\omega - \omega_n| < r_n$. Therefore, $\omega \in H_n$ and the theorem is proven.

We, next, present an example of a holomorphic function $f(z) \in (S)$ such that its inverse function has uncountably many transcendental singularities above $w = \infty$, and we note that since $f(z)$ is holomorphic these are direct transcendental singularities. This example places the last result in clearer perspective.

Let A be the extended complex plane and let M be the Cantor set on the interval $0 \leqq \theta \leqq 2\pi$ with $\{I_n\}$ denoting the sequence of the open middle-third intervals of $0 \leqq \theta \leqq 2\pi$ which are deleted to construct M. The order of the sequence $\{I_n\}$ is as follows:

$$I_1 = \left(\frac{2\pi}{3}, \frac{4\pi}{3}\right), \quad I_2 = \left(\frac{2\pi}{9}, \frac{4\pi}{9}\right),$$
$$I_3 = \left(\frac{14\pi}{9}, \frac{16\pi}{9}\right), \quad I_4 = \left(\frac{2\pi}{27}, \frac{4\pi}{27}\right), \cdots .$$

Let $\{J_n\}$ be the sequence of open intervals of the sequence of open sets $[0, 2\pi] - \bar{I}_1$, $[0, 2\pi] - \overline{I_1 \cup I_2}$, $\cdots$, with the ordering

$$J_1 = \left(0, \frac{2\pi}{3}\right), \quad J_2 = \left(\frac{4\pi}{3}, 2\pi\right), \quad J_3 = \left(0, \frac{2\pi}{9}\right),$$
$$J_4 = \left(\frac{4\pi}{9}, \frac{2\pi}{3}\right), \quad J_5 = \left(\frac{4\pi}{3}, \frac{14\pi}{9}\right), \cdots .$$

Since A is an analytic set and M is a closed nowhere dense set we can apply a theorem of Bagemihl and Seidel [1, p. 198-199] to claim the existence of a function $f(z)$, holomorphic in $|z| < 1$, with the following properties:

(i) for every $\theta \in M$, $\lim_{r\to 1} f(re^{i(\theta+1/(1-r))}) = w_\theta$ exists (possibly infinite);

(ii) if I is any subinterval of $0 \leqq \theta \leqq 2\pi$ such that $I \cap M \neq \emptyset$, then $A = \{w_\theta | \theta \in I \cap M\}$, and for every $a \in A$, there are uncountably many values of $\theta \in I \cap M$ for which $w_\theta = a$.

By (ii) every value of the extended complex plane is an asymptotic value on uncountably many spiral paths S_θ: $z = s_\theta(r) = re^{i(\theta+1/(1-r))}$, $0 \leqq r < 1$, for $\theta \in M$. Let $\theta_1, \theta_2 \in M$, $\theta_1 < \theta_2$ such that $f(z) \to \omega$ on S_{θ_1} and S_{θ_2} as $|z| \to 1$. We can find two intervals I_{n_1} and I_{n_2} of $\{I_n\}$ such that $I_{n_1} \subseteqq (\theta_1, \theta_2)$ and $I_{n_2} \subseteqq [0, 2\pi] - [\theta_1, \theta_2]$. Thus, for n_1', n_2' sufficiently large, there exist intervals $J_{n_1'}$ abutting I_{n_1} and $J_{n_2'}$ abutting I_{n_2} such that $J_{n_1'} \subseteqq (\theta_1, \theta_2)$ and $J_{n_2'} \subseteqq [0, 2\pi] - [\theta_1, \theta_2]$. But since $J_{n_1'} \cap M \neq \emptyset$ and $J_{n_2'} \cap M \neq \emptyset$, by (ii), there exist spirals separating S_{θ_1} and S_{θ_2} on which $f(z)$ has asymptotic values different from $w = \omega$. Thus $[S_{\theta_1}] \neq [S_{\theta_2}]$. Since ω is an arbitrary value of the extended complex plane, we see that the inverse $z = \phi(w)$ has uncountably many transcendental singularities above every value of the extended w-plane. In particular, since $f(z)$ omits $w = \infty$, $z = \phi(w)$ has uncountably many direct transcendental singularities above $w = \infty$.

REFERENCES

1. F. Bagemihl and W. Seidel, *Some boundary properties of analytic functions*, Math. Z., **61** (1954), 186-199.

2. J. R. Choike, *On the asymptotic boundary paths of analytic functions*, J. reine angew. Math., **264** (1974), 29-39.

3. ———, *On the asymptotic boundary behavior of functions analytic in the unit disk*, (submitted for publication).

4. J. L. Doob, *On a theorem of Gross and Iversen*, Ann. of Math., (2) **33** (1932), 753-757.
5. W. K. Hayman, *Meromorphic Functions*, Oxford Mathematical Monographs, Oxford, 1964.
6. F. Iversen, *Recherches sur les fonctions inverses des fonctions méromorphes*, Thèse, Helsingfors, 1914.
7. K. Noshiro, *On the singularities of analytic functions*, Japan J. Math., **17** (1940), 37-96.
8. ———, *Cluster Sets*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1960.
9. L. Sario and K. Noshiro, *Value Distribution Theory*, D. Van Nostrand Co., Inc., New York, 1966.
10. G. Valiron, *Sur les singularités de certaines fonctions holomorphes et de leur inverses*, J. Math. Pures et Appl., **15** (1936), 423-435.

Received October 22, 1974.

OKLAHOMA STATE UNIVERSITY