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WEYL'S INEQUALITY AND QUADRATIC FORMS ON THE GRASSMANNIAN

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This paper is concerned with the largest absolute value taken on by an m-square principal subdeterminant in any unitary transform of an n-square complex matrix A. For $m=1$ this maximum coincides with the numerical radius of A. The results obtained constitute generalizations of the Gohberg-Kreĭn analysis of the case of equality in Weyl's inequalities relating eigenvalues and singular values.

Introduction. Let A be an n-square complex matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$, $|\lambda_1| \geqq \cdots \geqq |\lambda_n|$, and singular values $\alpha_1(A) \geqq \cdots \geqq \alpha_n(A)$. The *numerical radius* of A, $r(A)$, is the maximum absolute value assumed by a diagonal element in any unitary transform of of A, i.e., in any matrix unitarily similar to A. Of course,

$$(1) \qquad |\lambda_1| \leqq r(A) .$$

Matrices for which equality holds in (1) are called *spectral*. In this paper we consider $r_{d,m}(A)$, the largest absolute value taken on by an m-square principal subdeterminant in any unitary transform of A. As we shall see in the sequel

$$(2) \qquad |\lambda_1 \cdots \lambda_m| \leqq r_{d,m}(A) .$$

For $m=1$, (2) collapses to (1). Matrices for which equality holds in (2) will be called *m-decomposably spectral*. One of the purposes of this paper is to examine the structure of matrices A which are m-decomposably spectral for each $m=1, \cdots, n$. Such results are related to the inequalities of Weyl [5],

$$(3) \qquad |\lambda_1 \cdots \lambda_k| \leqq \alpha_1(A) \cdots \alpha_k(A),\ k=1, \cdots, n ,$$

and to the case of equality in (3) for $k=1, \cdots, n$ discussed by Gohberg and Kreĭn [1]. We also examine the case where A is m-decomposably spectral for a particular m and, in fact, show that if A has s eigenvalues of maximum modulus $|\lambda_1|$, $s>m$, then spectral and m-decomposably spectral are equivalent. To examine the concept of m-decomposably spectral we require the machinery of induced maps on the mth Grassmann space.

2. Preliminary notions and theorems. Let V be an n-dimensional unitary space with an inner product (x, y). Let $T: V \rightarrow V$ be

a linear transformation with eigenvalues $\lambda_1, \cdots, \lambda_n$, $|\lambda_1| \geqq \cdots \geqq |\lambda_n|$, and singular values $\alpha_1(T) \geqq \cdots \geqq \alpha_n(T)$. Let $E = \{e_1, \cdots, e_n\}$ be an o.n. basis of V and let $A = [T]_E^E$, the matrix representation of T with respect to E. We will consider A as a linear transformation on C^n, the space of complex n-tuples. For each m, $1 \leqq m \leqq n$, let $\bigwedge^m V$ be the mth Grassmann space over V where the inner product induced on $\bigwedge^m V$ by (x, y) is defined by

$$(x_1 \wedge \cdots \wedge x_m, y_1 \wedge \cdots \wedge y_m) = \det [(x_i, y_j)]$$

for any decomposable tensors $x^\wedge$ and $y^\wedge$ in $\bigwedge^m V$, i.e., $x^\wedge = x_1 \wedge \cdots \wedge x_m$, $y^\wedge = y_1 \wedge \cdots \wedge y_m$ where x_i and y_i are in V, $i = 1, \cdots, m$. The space $\bigwedge^m V$ has an ordered o.n. basis $E^\wedge = \{e_{\omega(1)} \wedge \cdots \wedge e_{\omega(m)} = e_\omega^\wedge : \omega \in Q_{m,n}\}$ where $Q_{m,n}$ is the totality of strictly increasing sequences ω of length m, $1 \leqq \omega(1) < \cdots < \omega(m) \leqq n$, and where the ω's are assumed to be ordered lexicographically. The compound $C_m(T)$: $\bigwedge^m V \to \bigwedge^m V$ is defined by

$$C_m(T) x_1 \wedge \cdots \wedge x_m = Tx_1 \wedge \cdots \wedge Tx_m$$

for any decomposable $x^\wedge \in \bigwedge^m V$. Let $C_m(A) = [C_m(T)]_{E^\wedge}^{E^\wedge}$. Then $C_m(A)$ has eigenvalues $\lambda_\beta = \lambda_{\beta(1)} \cdots \lambda_{\beta(m)}$, $\beta \in Q_{m,n}$ and singular values $\alpha_\gamma = \alpha_{\gamma(1)}(A) \cdots \alpha_{\gamma(m)}(A)$, $\gamma \in Q_{m,n}$.

The *numerical radius* of A is defined by

$$r(A) = \max_{||x||=1} |(Ax, x)| .$$

and the *spectral norm* of A by

$$\alpha_1(A) = \max_{||x||=1} ||Ax|| .$$

The Grassmannian in $\bigwedge^m V$ is the set

$$G_m = \left\{x^\wedge \in \bigwedge^m V : ||x^\wedge|| = 1 \text{ and } x^\wedge \text{ is decomposable}\right\} ,$$

and the *decomposable numerical radius* of $C_m(A)$ is defined by

$$r_d(C_m(A)) = \max_{x^\wedge \in G_m} |(C_m(A)x^\wedge, x^\wedge)| . \tag{4}$$

In (4) we may assume without loss of generality that for each $x^\wedge = x_1 \wedge \cdots \wedge x_m$ the vectors $x_1, \cdots, x_m$ are o.n. Since the α, β entry of $C_m(A)$ is $\det A[\alpha|\beta]$, where $A[\alpha|\beta]$ indicates the submatrix of A lying in rows α and columns β, $\alpha, \beta \in Q_{m,n}$, we see that by taking $Ue_i = x_i$, $i = 1, \cdots, m$, U unitary, we have

$$\begin{aligned} r_d(C_m(A)) &= \max_{x^\wedge \in G_m} |(C_m(A)x^\wedge, x^\wedge)| \\ &= \max_{U \text{ unitary}} |(C_m(A)C_m(U)e_1 \wedge \cdots e_m, C_m(U)e_1 \wedge \cdots \wedge e_m)| \end{aligned}$$

$$= \max_{U \text{ unitary}} |\det U^*AU[1, \cdots, m|1, \cdots, m]|$$
$$= r_{d,m}(A) .$$

Of course if $m = 1$, $r_d(C_m(A)) = r(A)$. In general,

$$r_d(C_m(A)) \leqq r(C_m(A)) . \tag{5}$$

It is possible to have strict inequality in (5) as the following example shows. Let

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so that

$$C_2(A)_{\alpha,\beta} = \begin{cases} 1, & \text{if } \alpha = (12), \beta = (34) \\ 0, & \text{otherwise} . \end{cases}$$

If $x^\wedge \in G_m$ then $x^\wedge = \sum_{\alpha \in Q_{2,4}} p(\alpha) e_\alpha^\wedge$ where

$$\sum_{\alpha \in Q_{2,4}} |p(\alpha)|^2 = 1 \tag{6}$$

and the $p(\alpha)$ satisfy the quadratic Plücker relations [4]:

$$p(\alpha)p(\beta) = \sum_{t=1}^{m} p(\alpha[s, t: \beta])p(\beta[t, s: \alpha], \quad s = 1, \cdots, m \tag{7}$$

where $\alpha[s, t: \beta]$ is the sequence $(\alpha(1), \cdots, \alpha(s-1), \beta(t), \alpha(s+1), \cdots, \alpha(m))$ and $p(\alpha)$ is defined for any sequence α of length m by skew-symmetry. We have for $x^\wedge \in G_m$

$$\begin{aligned} |(C_2(A)x^\wedge, x^\wedge)| &= |p(12)p(34)| \\ &= |p(32)p(14) + p(42)p(31)| , \quad \text{(from (7) with } s = 1) \\ &\leqq |p(23)||p(14)| + |p(24)||p(13)| \\ &\leqq \frac{1}{2}(|p(23)|^2 + |p(14)|^2 + |p(24)|^2 + |p(13)|^2) \\ &= \frac{1 - |p(12)|^2 - |p(34)|^2}{2} , \quad \text{(from (6))} . \end{aligned} \tag{8}$$

Thus

$$(|p(12)| + |p(34)|)^2 \leqq 1 ,$$
$$(|p(12)| + |p(34)|) = c \leqq 1 ,$$
$$|p(12)||p(34)| = |p(12)|(c - |p(12)|) ,$$

and

$$|(p(12)p(34)| \leqq \frac{c^2}{4} \leqq \frac{1}{4}.$$

From (8) we see that

$$r_d(C_2(A)) \leqq \frac{1}{4}.$$

If we consider the quadratic form evaluated on the indecomposable unit tensor $1/\sqrt{2}(e_1 \wedge e_2 + e_3 \wedge e_4)$ we have

$$\left(C_2(A)\frac{1}{\sqrt{2}}(e_1 \wedge e_2 + e_3 \wedge e_4), \frac{1}{\sqrt{2}}(e_1 \wedge e_2 + e_3 \wedge e_4)\right) = \frac{1}{2},$$

so that

$$r(C_2(A)) \geqq \frac{1}{2}.$$

The explanation of this phenomenon is that not every tensor on the unit sphere in $\bigwedge^2 V$ is decomposable.

The following results are well known [3]:

(i) For M any principal sub-matrix of A,

$$r(M) \leqq r(A). \tag{9}$$

(ii) (The Elliptical Range Theorem.) For a 2×2 matrix the numerical range is an ellipse with foci the eigenvalues of the matrix; if $A = \begin{bmatrix} \lambda_1 & a \\ 0 & \lambda_2 \end{bmatrix}$ then the semi-minor axis of the ellipse has length $|a|/2$.

(iii)

$$|\lambda_1| \leqq r(A) \leqq \alpha_1(A). \tag{10}$$

We may generalize (10) for $1 \leqq m \leqq n$ to

$$|\lambda_1 \cdots \lambda_m| \leqq r_d(C_m(A)) \leqq r(C_mA)) \leqq \alpha_1(A) \cdots \alpha_m(A). \tag{11}$$

The first inequality may be seen as follows. Let

$$U^*AU = \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ \bigcirc & & \lambda_n \end{bmatrix}.$$

Then $C_m(U^*AU)$ is also upper triangular and

$$\lambda_1 \cdots \lambda_m = C_m(U^*AU)_{(1,\cdots,m),(1,\cdots,m)} = (C_m(A)u^\wedge, u^\wedge)$$

for an appropriate $u^\wedge \in G_m$. If A is normal then equality holds throughout (10) and (11). A proof of the Weyl inequalities (3) is now immediate. The first follows from (10) and the subsequent ones from (11). Since $r_{d,m}(A) = r_d(C_m(A))$ we will say that $C_m(A)$, $1 \leqq m \leqq n$, is *decomposably spectral* if

$$|\lambda_1 \cdots \lambda_m| = r_d(C_m(A)) .$$

M. Goldberg, E. Tadmor and G. Zwas [2] have shown that if $|\lambda_1| = \cdots = |\lambda_s| > |\lambda_{s+1}| \geqq \cdots \geqq |\lambda_n|$ then A is spectral iff A is unitarily similar to a matrix of the form $T \dotplus B$ where

$$T = \begin{bmatrix} \lambda_1 & & O \\ & \ddots & \\ O & & \lambda_s \end{bmatrix}, \qquad B = \begin{bmatrix} \lambda_{s+1} & & * \\ & \ddots & \\ O & & \lambda_n \end{bmatrix} \tag{12a}$$

and

$$r(B) \leqq |\lambda_1| . \tag{12b}$$

THEOREM 1 (Gohberg and Kreĭn). *Equality holds in* (3) *for* $k = 1, \cdots, n$ *iff* A *is normal.*

We include a proof of this theorem based on properties of the Grassmann algebra which suggests a proof of the following stronger result:

THEOREM 2. *For each* $m = 1, \cdots, n$

$$|\lambda_1 \cdots \lambda_m| \leqq r_d(C_m(A)) , \quad m = 1, \cdots, n . \tag{13}$$

Equality holds in (13) *for* $m = 1, \cdots, n$ *iff* A *is normal. Equivalently, the largest absolute value taken on by an m-square principal subdeterminant in any unitary transform of* A *is at least* $|\lambda_1 \cdots \lambda_m|$, $m = 1, \cdots, n$. *This largest absolute value is equal to* $|\lambda_1 \cdots \lambda_m|$ *for* $m = 1, \cdots, n$ *iff* A *is normal.*

We will also investigate the case of equality in a single one of the inequalities in (13).

THEOREM 3. *Assume that* A *has s eigenvalues of maximum modulus,* $s > m$:

$$|\lambda_1| = \cdots = |\lambda_s| > |\lambda_{s+1}| \geqq \cdots \geqq |\lambda_n| .$$

Then $C_m(A)$ *is decomposably spectral iff* A *is spectral.*

3. Proofs and examples.

Proof of Theorem 1. Clearly if A is normal then $|\lambda_1 \cdots \lambda_k| =$

$\alpha_1(A) \cdots \alpha_k(A)$, $k = 1, \cdots, n$. Suppose now that $|\lambda_1 \cdots \lambda_k| = \alpha_1(A) \cdots \alpha_k(A)$, $k = 1, \cdots, n$. By Schur's theorem we may assume

$$A = \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ \bigcirc & & \lambda_n \end{bmatrix}.$$

Let

$$|\lambda_1| \geqq \cdots \geqq |\lambda_t| > 0 = |\lambda_{t+1}| = \cdots = |\lambda_n| ,$$

for some t, $1 \leqq t \leqq n$. We have

$$\begin{aligned} (AA^*)_{11} &= |\lambda_1|^2 + |a_{12}|^2 + \cdots + |a_{1n}|^2 \\ &\leqq \alpha_1^2(A) . \end{aligned}$$

Since $|\lambda_1| = \alpha_1(A)$ we must have $a_{1i} = 0$, $i \neq 1$ and

$$A_{(1)} = \lambda_1 e_1 .$$

($A_{(1)}$ is the first row of A, i.e., the n-tuple $(a_{11}, \cdots, a_{1n})$.) Applying this argument to $C_m(A)$, $1 \leqq m \leqq n$, we have

$$\begin{aligned} C_m(A)_{(1)} &= A_{(1)} \wedge \cdots \wedge A_{(m)} \\ &= \lambda_1 \cdots \lambda_m e_1 \wedge \cdots \wedge e_m . \end{aligned} \tag{14}$$

Assume now that we have shown

$$A_{(i)} = \lambda_i e_i,\ i = 1, \cdots, k-1,\ k \leqq t . \tag{15}$$

Then

$$\begin{aligned} A_{(1)} \wedge \cdots \wedge A_{(k)} &= \lambda_1 \cdots \lambda_{k-1} e_1 \wedge \cdots \wedge e_{k-1} \wedge \left(\lambda_k e_k + \sum_{i=k+1}^{n} a_{ki} e_i\right) \\ &= \lambda_1 \cdots \lambda_k e_1 \wedge \cdots \wedge e_k + \lambda_1 \cdots \lambda_{k-1}\left(\sum_{i=k+1}^{n} a_{ki} e_1 \wedge \cdots \wedge e_{k-1} \wedge e_i\right) . \end{aligned} \tag{16}$$

Since the representation of $A_{(1)} \wedge \cdots \wedge A_{(k)}$ with respect to the basis $E^\wedge$ is unique and since $\lambda_1 \cdots \lambda_k \neq 0$, (14) and (16) imply $a_{ki} = 0$, $i = k+1, \cdots, n$. We have

$$A = \operatorname{diag}(\lambda_1, \cdots, \lambda_t) \dotplus B$$

where

$$B = \begin{bmatrix} 0 & & * \\ & \ddots & \\ \bigcirc & & 0 \end{bmatrix}.$$

However, $|\lambda_1 \cdots \lambda_{t+1}| = \alpha_1(A) \cdots \alpha_{t+1}(A)$ implies that

$$\alpha_{t+1}(A) = \cdots = \alpha_n(A) = 0 .$$

Thus AA^*, and hence A, has rank t so that $B = 0_{n-t}$. Thus A is normal.

Proof of Theorem 2. If A is normal then obviously equality holds in (13) for $m = 1, \cdots, n$. Conversely, assume that (13) is equality, $m = 1, \cdots, n$. Without loss of generality we can assume

$$A = \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ \bigcirc & & \lambda_n \end{bmatrix} .$$

Suppose there exists an a_{1i}, $i \neq 1$, such that a_{1i} is nonzero. Then from (9),

$$|\lambda_1| = r(A) \geqq r\begin{bmatrix} \lambda_1 & a_{1i} \\ 0 & \lambda_i \end{bmatrix} \geqq |\lambda_1| ,$$

so that by the Elliptical Range Theorem $a_{1i} = 0$ and

$$A_{(1)} = \lambda_1 e_1 .$$

Let

$$|\lambda_1| \geqq \cdots \geqq |\lambda_t| > 0 = |\lambda_{t+1}| = \cdots = |\lambda_n|$$

for some t, $1 \leqq t \leqq n$, and suppose we have shown that

$$A_{(i)} = \lambda_i e_i,\ i = 1, \cdots, k-1,\ k \leqq t .$$

Let $1 \leqq r \leqq n - k$ and consider the function

$$\begin{aligned} e(u, v) = (C_k(A)e_1 \wedge \cdots \wedge e_{k-1} \wedge (ue_k + ve_{k+r}) ,\\ e_1 \wedge \cdots \wedge e_{k-1} \wedge (ue_k + ve_{k+r})) \end{aligned}$$

where $|u|^2 + |v|^2 = 1$. Then

$$\begin{aligned} &e(u, v) \\ &\quad = \Big(\lambda_1 \cdots \lambda_{k-1}\Big(u\lambda_k e_1 \wedge \cdots \wedge e_{k-1} \wedge e_k + v\sum_{i=k}^{k+r} a_{i,k+r} e_1 \wedge \cdots \wedge e_{k-1} \wedge e_i\Big) , \\ (17) &\qquad ue_1 \wedge \cdots \wedge e_{k-1} \wedge e_k + ve_1 \wedge \cdots \wedge e_{k-1} \wedge e_{k+r}\Big) \\ &\quad = \lambda_1 \cdots \lambda_{k-1}\{|u|^2\lambda_k + v\bar{u}a_{k,k+r} + |v|^2\lambda_{k+r}\} . \end{aligned}$$

Let

$$C = \begin{bmatrix} \lambda_k & a_{k,k+r} \\ 0 & \lambda_{k+r} \end{bmatrix} .$$

If $a_{k,k+r} \neq 0$ then from the Elliptical Range Theorem $r(C) > |\lambda_k|$, i.e., there exist u and v, $|u|^2 + |v|^2 = 1$, such that the expression in curly brackets on the right side of (17) has absolute value greater than $|\lambda_k|$. Since $\lambda_1 \cdots \lambda_{k-1}$ is nonzero we conclude that $|e(u, v)| > |\lambda_1 \cdots \lambda_k|$. But $e(u, v)$ is a value of the quadratic form associated with $C_k(A)$ on a decomposable tensor of unit length, and thus it follows that $r_d(C_k(A)) > |\lambda_1 \cdots \lambda_k|$. Therefore $a_{k,k+r} = 0$, $r = 1, \cdots, n-k$ and thus

$$A = \operatorname{diag}(\lambda_1 \cdots \lambda_t) \dotplus B$$

where

$$B = \begin{bmatrix} 0 & & * \\ & \ddots & \\ \bigcirc & & 0 \end{bmatrix}.$$

Next assume $a_{t+1,i} \neq 0$ for some $i > t+1$. Then the $(1, \cdots, t, t+1)$, $(1, \cdots, t, i)$ element of $C_{t+1}(A)$ is $\lambda_1 \cdots \lambda_t a_{t+1,i} \neq 0$. Letting $x^\wedge$ be the decomposable unit tensor $1\sqrt{2}(e_1 \wedge \cdots \wedge e_t \wedge e_{t+1} + e_1 \wedge \cdots \wedge e_t \wedge e_i)$ we have

$$\begin{aligned}(C_{t+1}(A)x^\wedge, x^\wedge) &= \frac{1}{2}\Big(\lambda_1 \cdots \lambda_t e_1 \wedge \cdots \wedge e_t \wedge \Big(a_{t+1,i}e_{t+1} + \sum_{j=t+2}^{n} a_{ji}e_j\Big), \\ &\qquad e_1 \wedge \cdots \wedge e_t \wedge e_{t+1} + e_1 \wedge \cdots \wedge e_t \wedge e_i\Big) \\ &= \frac{1}{2}\lambda_1 \cdots \lambda_t a_{t+1,i} \\ &\neq 0 .\end{aligned}$$

But then $r_d(C_{t+1}(A)) \geqq 1/2|\lambda_1 \cdots \lambda_t a_{t+1,i}| > |\lambda_1 \cdots \lambda_t \lambda_{t+1}| = 0$, contradicting the assumption that (13) is equality for $m = t+1$. Thus

$$A_{(t+1)} = 0 .$$

Suppose that we have shown

$$A_{(t+r)} = 0, r = 1, \cdots, k-1 .$$

If there exists an element $a_{t+k,i}$, $i > t+k$, which is nonzero we see that the $(1, \cdots, t, t+k)$, $(1, \cdots, t, i)$ element of $C_{t+1}(A)$ is $\lambda_1 \cdots \lambda_t a_{t+k,i} \neq 0$. Let $x^\wedge = 1/\sqrt{2}(e_1 \wedge \cdots \wedge e_t \wedge e_{t+k} + e_1 \wedge \cdots \wedge e_t \wedge e_i) \in G_{t+1}$ and note that

$$\begin{aligned}(C_{t+1}(A)x^\wedge, x^\wedge) &= \frac{1}{2}\lambda_1 \cdots \lambda_t a_{t+k,i} \\ &\neq 0 ,\end{aligned}$$

contradicting the fact that $r_d(C_{t+1}(A)) = 0$. We conclude that $B = 0_{n-t}$

and hence that A is normal.

Proof of Theorem 3. Once again we may assume that

$$A = \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ \bigcirc & & \lambda_n \end{bmatrix},$$

so that $C_m(A)$ is also upper triangular. Let $\alpha \in Q_{m,s}$, $\gamma \in Q_{m,n}$, and assume $\gamma > \alpha$, i.e., γ follows α in the lexicographic ordering. Moreover suppose that $|\alpha \cap \gamma| = m - 1$, i.e., $\operatorname{Im}\alpha$ and $\operatorname{Im}\gamma$ overlap in $m - 1$ places. Then if $|s|^2 + |t|^2 = 1$, $se_\alpha^\wedge + te_\gamma^\wedge \in G_m$ and

$$|(C_m(A)(se_\alpha^\wedge + te_\gamma^\wedge), se_\alpha^\wedge + te_\gamma^\wedge)| \leqq |\lambda_1 \cdots \lambda_m| ; \tag{18}$$

$$\begin{aligned} |(C_m(A)(se_\alpha^\wedge + te_\gamma^\wedge), se_\alpha^\wedge + te_\gamma^\wedge)| &= |s|^2 C_m(A)_{\alpha,\alpha} + s\bar{t} C_m(A)_{\gamma,\alpha} \\ &\quad + t\bar{s} C_m(A)_{\alpha,\gamma} + |t|^2 C_m(A)_{\gamma,\gamma} \\ &= |s|^2 \lambda_\alpha + t\bar{s} p(\gamma) + |t|^2 \lambda_\gamma , \end{aligned}$$

where $p(\gamma) = C_m(A)_{\alpha,\gamma}$;

$$|(C_m(A)(se_\alpha^\wedge + te_\gamma^\wedge), se_\alpha^\wedge + te_\gamma^\wedge)| = |\lambda_1|^{m-1} \left| |s|^2 \lambda_i + \frac{t\bar{s}}{c} p(\gamma) + |t|^2 \lambda_j \right|$$

where $|\lambda_i| = |\lambda_1|$ and $c \neq 0$. From (18) we have

$$\left| |s|^2 \lambda_i + \frac{t\bar{s}}{c} p(\gamma) + |t|^2 \lambda_j \right| \leqq |\lambda_1| .$$

Applying the Elliptical Range Theorem to the matrix

$$\begin{bmatrix} \lambda_i & \dfrac{p(\gamma)}{c} \\ 0 & \lambda_j \end{bmatrix}$$

tells us that unless $p(\gamma) = 0$ there exists an s and t, $|s|^2 + |t|^2 = 1$, for which $||s|^2\lambda_i + t\bar{s}/c p(\gamma) + |t|^2\lambda_j| > |\lambda_1|$. Thus

$$C_m(A)_{\alpha,\gamma} = 0 \quad \text{if} \quad \alpha \in Q_{m,s}, \gamma > \alpha, \quad \text{and} \quad |\alpha \cap \gamma| = m - 1 .$$

The elements of row α of $C_m(A)$ are the Plücker coordinates of the decomposable tensor $A_{\alpha(1)} \wedge \cdots \wedge A_{\alpha(m)}$ and therefore satisfy the quadratic Plücker relations:

$$p(\alpha)p(\gamma) = \sum_{t=1}^{m} p(\alpha[s, t: \gamma])p(\gamma[t, s: \alpha]) , \quad s = 1, \cdots, m . \tag{19}$$

For $\gamma > \alpha$, $|\alpha \cap \gamma| = m - 1$, we have seen that $p(\gamma) = 0$. Let $\gamma > \alpha$, $|\alpha \cap \gamma| \neq m - 1$. Pick s in (19) so that $\alpha(s) \notin \operatorname{Im} \gamma$. Then

$|\alpha[s, t: \gamma] \cap \alpha| = m - 1$ so that the first factor in each summand of (19) is zero. Since $p(\alpha) = \lambda_{\alpha(1)} \cdots \lambda_{\alpha(m)} \neq 0$ we have $p(\gamma) = 0$, i.e.,

$$(C_m(A))_{\alpha,\gamma} = 0,\ \alpha \in Q_{m,s},\ \alpha \neq \gamma\ . \tag{20}$$

From (20),

$$A_{\alpha(1)} \wedge \cdots \wedge A_{\alpha(m)} = \lambda_{\alpha(1)} \cdots \lambda_{\alpha(m)} e_{\alpha(1)} \wedge \cdots \wedge e_{\alpha(m)}\ ,$$

which in turn implies the equality of the subspaces spanned by the two sets of vectors, i.e.,

$$\langle A_{\alpha(1)}, \cdots, A_{\alpha(m)} \rangle = \langle e_{\alpha(1)}, \cdots, e_{\alpha(m)} \rangle,\ \alpha \in Q_{m,s} \tag{21}$$

($\langle x_1, \cdots, x_m \rangle$ means the linear span of $x_1, \cdots, x_m$). Since $s > m$, for each $i \in \{1, \cdots, s\}$ there exist sequences $\alpha_1, \cdots, \alpha_m \in Q_{m,s}$ such that $\{i\} = \bigcap_{j=1}^m \operatorname{Im} \alpha_j$. If $\alpha \in Q_{m,s}$ then each $\alpha(i) \in \{1, \cdots, s\}$, $i = 1, \cdots, m$, so that there exist sequences $\alpha_1, \cdots, \alpha_m$ such that $\{\alpha(i)\} = \bigcap_{j=1}^m \operatorname{Im} \alpha_j$. Therefore,

$$\begin{aligned} A_{\alpha(i)} \in \bigcap_{j=1}^{m} \langle A_{\alpha_j(1)}, \cdots, A_{\alpha_j(m)} \rangle &= \bigcap_{j=1}^{m} \langle e_{\alpha_j(1)}, \cdots, e_{\alpha_j(m)} \rangle, \text{ (from (21))} \\ &= \langle e_{\alpha(i)} \rangle\ . \end{aligned}$$

Hence $A = T \dotplus B$ where

$$T = \operatorname{diag}(\lambda_1, \cdots, \lambda_s),\ B = \begin{bmatrix} \lambda_{s+1} & & * \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$

Finally, suppose there exists $u \in C^{n-s}$, $\|u\| = 1$, such that $|(Bu, u)| > |\lambda_1|$. Let

$$\begin{aligned} x_i &= e_i,\ i = 1, \cdots, m-1\ , \\ x_m &= 0 \dotplus u = (0, \cdots, 0, u_1, \cdots, u_{n-s})\ . \end{aligned}$$

Then

$$\begin{aligned} |(C_m(A)x^\wedge, x^\wedge)| &= \left| \det \begin{bmatrix} \lambda_1 & & & \bigcirc \\ & \ddots & & \\ & & \lambda_{m-1} & \\ * & \cdots & * & (Bu, u) \end{bmatrix} \right| \\ &= |\lambda_1 \cdots \lambda_{m-1}(Bu, u)| \\ &> |\lambda_1 \cdots \lambda_m|\ , \end{aligned}$$

contradicting the hypothesis that $C_m(A)$ is decomposably spectral. Therefore $r(B) \leqq |\lambda_1|$ and by (12), A is spectral.

To prove the converse, observe that $r_d(C_m(A)) \geqq |\lambda_1|^m$. Suppose

$r_d(C_m(A)) > |\lambda_1|^m$. Then there exists $x^\wedge \in G_m$ such that

$$|C_m(A)x^\wedge, x^\wedge)| > |\lambda_1|^m .$$

Without loss of generality we can assume $x_1, \cdots, x_m$ are o.n. Let $Ue_i = x_i$, $i = 1, \cdots, U$ unitary, and compute that

$$\begin{aligned}|(C_m(A)x^\wedge, x^\wedge)| &= |(C_m((U^*AU)e_1 \wedge \cdots \wedge e_m, e_1 \wedge \cdots \wedge e_m)| \\ &= |\det U^*AU[1, \cdots, m|1, \cdots, m]| .\end{aligned}$$

Letting $B = U^*AU[1, \cdots, m|1, \cdots, m]$, we have

$$|\det B| > |\lambda_1|^m ,$$

so that B has an eigenvalue $\tilde{\lambda}$ satisfying $|\tilde{\lambda}| > |\lambda_1|$. There exists a unitary m-square V for which

$$V^*BV = \begin{bmatrix} \tilde{\lambda} & & \\ & \ddots & * \\ \bigcirc & & * \end{bmatrix} .$$

Let $W = V \dot{+} I_{n-m}$ and note that

$$W^*U^*AUW = \left[\begin{array}{ccc|c} \tilde{\lambda} & & & \\ & \ddots & * & * \\ \bigcirc & & * & \\ \hline & * & & * \end{array}\right] .$$

Let $X = UW$; $X^{(1)}$, the first column of X, is a unit vector and

$$\begin{aligned}|(AX^{(1)}, X^{(1)})| &= |(X^*AX)_{11}| \\ &= |\tilde{\lambda}| > |\lambda_1| .\end{aligned}$$

But this contradicts the fact that $r(A) = |\lambda_1|$. Therefore, $r_d(C_m(A)) = |\lambda_1|^m$.

In the second part of Theorem 3 the hypothesis $s \geqq m$ is necessary. For, let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} ,$$

and note that

$$\frac{1}{2}(C_2(A)\{e_1 \wedge e_3 + e_1 \wedge e_2\}, \{e_1 \wedge e_3 + e_1 \wedge e_2\}) = 1 > \lambda_1\lambda_2 = 0 .$$

Also the hypothesis $s > m$ in the first part of Theorem 3 is necessary as the following examples illustrate:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$C_2(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

then $r_d(C_2(A)) = 1 = \lambda_1\lambda_2$, but $r(A) \geqq r\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) > 1$;

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C_2(A) = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

then $r_d(C_2(A)) = 1/2 = \lambda_1\lambda_2$, but $r(A) \geqq r\left(\begin{bmatrix} 1 & 1 \\ 0 & 1/2 \end{bmatrix}\right) > 1$. Also observe that although Theorem 3 implies that if $C_m(A)$ is spectral, $m < s$, then A is spectral, the converse is false. For example, let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \dotplus \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \dotplus \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}.$$

Then $r(A) = 1$ but $r(C_2(A)) \geqq r\left(\begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix}\right) = 2$ so that $C_2(A)$ is not spectral.

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