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A PERTURBATION THEOREM FOR SPECTRAL OPERATORS

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This note is concerned with analytic perturbations of spectral operators. It is shown that under small perturbations simple isolated eigenvalues remain simple and isolated and depend holomorphically upon the perturbation parameter. As one would expect the bounds are rather complicated in the case of a spectral operator with general quasinilpotent part. For scalar operators, however, these bounds become simple and reproduce in the selfadjoint case those given by F. W. Schäfke.

For the sake of simplicity we deal only with bounded operators. The method used is an appropriate modification of the elegant Hilbert space method introduced by Schäfke in **[3]** to settle the analogous problem for selfadjoint operators. The generalization to the unbounded case is then straightforward.

Let X be a Banach space over $\mathbf{C}$ and $B(X)$ the algebra of bounded linear operators on X with the norm topology. Let further $\sigma(T)$, $\rho(T)$, and $R_\lambda(T) := (T - \lambda I)^{-1}$ for $\lambda \in \rho(T)$ denote the spectrum, the resolvent set, and the resolvent operator for $T \in B(X)$. Spectral measures, spectral operators and scalar spectral operators are defined as in Dunford–Schwartz **[1]**. Especially, if $S \in B(X)$ is a scalar spectral operator, we have

$$S = \int_{\mathbf{C}} \lambda E(d\lambda)$$

with an uniquely determined spectral measure E. A spectral operator can be uniquely decomposed as $T = S + N$, where S is a scalar spectral operator, and N is a bounded quasinilpotent operator commuting with the spectral measure E of S. If E is a spectral measure we denote by $\omega(E)$ the minimum of all reals c which obey

$$\left\| \int_{\mathbf{C}} f(\lambda) E(d\lambda) \right\| \leqq c \cdot \sup_{\lambda \in \mathbf{C}} |f(\lambda)|$$

for all bounded, Borel measurable, $\mathbf{C}$-valued functions on $\mathbf{C}$. Then $\omega(E) \geqq 1$.

If $T = S + N$ as above is a spectral operator we have

$$\sigma(T)=\sigma(S)$$

and

$$\ker(T-\lambda I)^n = \ker N^n \cap \ker(S-\lambda I)$$

for each $n \in \mathbf{N}$ and $\lambda \in \mathbf{C}$ (**[1]**).

Especially, the point spectrum of T is contained in the point spectrum of S (the converse is not true, in general). From the second equation we get also that $\lambda \in \mathbf{C}$ is a simple eigenvalue of T (that means, $\dim\ker(T-\lambda I)=1$) if λ is a simple eigenvalue of S and an eigenvalue of T. If for some $n \in \mathbf{N}$ $N^n = 0$, the point spectra of S and T coincide.

Now let $T = S + N$ be a spectral operator with spectral measure E. Further, let $\lambda_0 \in \mathbf{C}$ be an isolated point of $\sigma(T)$ and a simple eigenvalue of T, and $y_0 \in X$ with $\|y_0\| = 1$ an eigenvector to λ_0. We put

$$d := \operatorname{dist}\{\lambda_0, \sigma(T)-\{\lambda_0\}\},$$

$$\alpha := \omega(E)\cdot\frac{2}{d}\cdot\sum_{n=0}^{\infty}\frac{2^n\|N^n\|}{d^n} \quad (\geqq 1),$$

$$\beta := \omega(E)\cdot\frac{1}{d}\cdot\sum_{n=0}^{\infty}\frac{\|N^n\|}{d^n}.$$

Now, let $G_n \in B(X)$ be given for $n \in \mathbf{N}$ such that

$$\rho := \max\left\{\mu>0, \sum_{n=1}^{\infty}\mu^n\|G_n\| \leqq \frac{1}{\alpha}\right\}$$

obeys $0<\rho<\infty$, that means that not all the G_n are zero. There exists a maximal $\rho_0 \leqq \infty$ such that

$$\sum_{n=1}^{\infty}|\mu|^n\|G_n\|$$

exists for all $\mu \in \mathbf{C}$ with $|\mu|<\rho_0$, and

$$G(\mu) := \sum_{n=1}^{\infty}\mu^n G_n$$

defines on $\{\mu \in \mathbf{C}, |\mu|<\rho_0\}$ a holomorphic $B(X)$-valued function G.

Now we can state our perturbation theorem.

THEOREM 1. *There exist holomorphic functions y and λ on $\{\mu \in \mathbf{C}, |\mu|<\rho_0\}$ with $y(0)=y_0$ and $\lambda(0)=\lambda_0$ which obey*

(i) $E(\lambda_0)y(\mu) = y_0$;
(ii) $[T - G(\mu)]y(\mu) = \lambda(\mu)y(\mu)$;
(iii) $\omega(E)\cdot \Sigma_{n=0}^{\infty}\|N^n\|/(|\lambda(\mu) - \lambda_0|^{n+1}) \geqq (\Sigma_{n=1}^{\infty}|\mu|^n\|G_n\|)^{-1}$
(iv) $\|y(\mu) - y_0\| \leqq \beta\cdot\Sigma|\mu|^n\|G_n\|/(1 - \beta\cdot\{|\lambda(\mu) - \lambda_0| + \Sigma|\mu|^n\|G_n\|\})$
(v) If $[T - G(\mu)]y = \lambda y$ with $y \neq 0$, $|\mu| < \rho$, $|\lambda - \lambda_0| < \beta^{-1} - \alpha^{-1}$, then we have $\lambda = \lambda(\mu)$ and $y = c\cdot y(\mu)$ with some $c \in \mathbf{C}$.

Proof. (i), (ii) We put

$$R := \sum_{n=0}^{\infty} N^n \int_D \frac{1}{(\lambda - \lambda_0)^{n+1}} E(d\lambda),$$

where $D := \{\lambda \in \mathbf{C}, |\lambda - \lambda_0| \geqq d/2\}$.
Then for every pair (λ, μ) with

$$|\mu| < \rho_0, \qquad \beta\cdot\left(|\lambda - \lambda_0| + \sum_{n=1}^{\infty}|\mu|^n\|G_n\|\right) < 1$$

there is exactly one solution $z = z(\lambda, \mu)$ of the equation

$$z = R((\lambda - \lambda_0)z + G(\mu)z) + RG(\mu)y_0. \tag{1}$$

z depends holomorphically upon λ and μ.
In fact, we have

$$\|R\| \leqq \sum_{n=0}^{\infty}\|N^n\|\frac{\omega(E)}{d^{n+1}} = \beta,$$

so for $|\mu| < \rho_0$

$$\|RG(\mu)\| \leqq \beta\cdot\sum_{n=1}^{\infty}|\mu|^n\|G_n\|.$$

Then, if $a_1, b_1 \in X$ and

$$a_2 := R((\lambda - \lambda_0)a_1 + G(\mu)a_1) + RG(\mu)y_0$$

$$b_2 := R((\lambda - \lambda_0)b_1 + G(\mu)b_1) + RG(\mu)y_0,$$

we get

$$\|a_2 - b_2\| \leqq \|a_1 - b_1\|\cdot\beta\cdot\left(|\lambda - \lambda_0| + \sum_{n=1}^{\infty}|\mu|^n\|G_n\|\right).$$

Since the second factor on the right side is smaller than 1, uniqueness and existence of the solution z follow immediately as well as the holomorphy of z. Starting for example with $z_0 = 0$, z is the limit of the iteration given by

$$z_{n+1} = R((\lambda - \lambda_0)z_n + G(\mu)z_n) + RG(\mu)y_0.$$

Especially, we get $z(\lambda, 0) = 0$.
We now consider the X-valued holomorphic function

$$\Delta(\lambda, \mu) := E(\lambda_0)[(\lambda - \lambda_0)(z(\lambda, \mu) + y_0) + G(\mu)(z(\lambda, \mu) + y_0)]$$

for $|\mu| < \rho_0$, $\beta \cdot (|\lambda - \lambda_0| + \Sigma |\mu|^n \|G_n\|) < 1$.

It is $\Delta(\lambda, 0) = E(\lambda_0)[(\lambda - \lambda_0)(z(\lambda, 0) + y_0)] = (\lambda - \lambda_0)y_0$. The implicit function argument used by Schäfke can be employed in the vector valued case, too (e.g. Lang **[2]**). This gives us the existence of a holomorphic function λ on $|\mu| < \rho$ with

(2) $E(\lambda_0)[(\lambda(\mu) - \lambda_0)(z(\lambda(\mu), \mu) + y_0) + G(\mu)(z(\lambda(\mu), \mu) + y_0) = 0.$

We put $y(\mu) = y_0 + z(\lambda(\mu), \mu)$.
Then $y(\mu)$ obeys

(ii) $[T - G(\mu)]y(\mu) = \lambda(\mu)y(\mu)$.

In fact, we have $E(\lambda_0)R = RE(\lambda_0) = 0$, so (1) gives

(i) $E(\lambda_0)y(\mu) = E(\lambda_0)y_0 + E(\lambda_0)z(\lambda(\mu), \mu) = E(\lambda_0)y_0 = y_0$.

Further, let be $x \in X$. Then, $x = x_1 + x_2$ with $x_1 \in E(\lambda_0)X$, $x_2 \in E(\mathbf{C} - \{\lambda_0\})X$, so

$$(T - \lambda_0 I)Rx = (T - \lambda_0 I)Rx_1 + (T - \lambda_0 I)Rx_2 = x_2,$$

since $R \,|\, E(\mathbf{C} - \{\lambda_0\})X = [(T - \lambda_0 I) \,|\, E(\mathbf{C} - \{\lambda_0\})X]^{-1}$. On the other hand, we have $x_2 = x - x_1 = x - E(\lambda_0)x$, so

$$(T - \lambda_0 I)Rx = [I - E(\lambda_0)]x.$$

If (1) is fulfilled, we get with $z = z(\lambda(\mu), \mu)$ and $\lambda = \lambda(\mu)$

$$\begin{aligned} Tz - \lambda_0 z &= [(\lambda - \lambda_0)z + G(\mu)z + G(\mu)y_0] - E(\lambda_0)[\cdots] \\ &= [(\lambda - \lambda_0)(z + y_0) + G(\mu)(z + y_0)] - E(\lambda_0)[\cdots], \end{aligned}$$

since $(\lambda - \lambda_0)y_0 = E(\lambda_0)((\lambda - \lambda_0)y_0)$.

Then, (2) gives that

$$Tz - \lambda_0 z = (\lambda - \lambda_0)(z + y_0) + G(\mu)(z + y_0).$$

But this is equivalent to

(ii) $[T - G(\mu)]y(\mu) = \lambda(\mu)y(\mu)$.

(iii) We assume that for some $\lambda = \lambda(\mu)$

$$\omega(E) \cdot \sum_{n=0}^{\infty} \frac{\|N^n\|}{|\lambda - \lambda_0|^{n+1}} < \left(\sum_{n=1}^{\infty} |\mu|^n \|G_n\|\right)^{-1}.$$

Let $y \neq 0$ be an eigenvector to λ. Then $\lambda \in \rho(T)$, so from $Ty - \lambda y = G(\mu)y$ we get $y = R_\lambda(T)G(\mu)y$ and

$$0 \neq Ty - \lambda y = G(\mu)y = G(\mu)R_\lambda(T)G(\mu)y,$$

which gives

$$0 < \|G(\mu)y\| \leqq \|G(\mu)R_\lambda(T)\| \cdot \|G(\mu)y\| \quad \text{or} \quad 1 \leqq \|G(\mu)R_\lambda(T)\|.$$

Now we have according to **[1]**

$$R_\lambda(T) = -\sum_{n=0}^{\infty} N^n \int_C (\lambda - \xi)^{-n-1} E(d\xi),$$

from which we get

$$\begin{aligned} \|R_\lambda(T)\| &\leqq \sum_{n=0}^{\infty} \|N^n\| \cdot \omega(E) \cdot \operatorname{dist}(\lambda, \sigma(T))^{-n-1} \\ &= \sum_{n=0}^{\infty} \|N^n\| \cdot \omega(E) \cdot |\lambda - \lambda_0|^{-n-1}. \end{aligned}$$

So

$$1 \leqq \|G(\mu)R_\lambda(T)\| \leqq \sum_{n=1}^{\infty} |\mu|^n \|G_n\| \cdot \omega(E) \cdot \sum_{n=0}^{\infty} \|N^n\| \cdot |\lambda - \lambda_0|^{-n-1} < 1$$

according to what we have assumed. This contradiction proves (iii).

(iv) We put $z(\mu)$ instead of $z(\lambda(\mu), \mu)$. Then,

$$\|y(\mu) - y_0\| = \|z(\mu)\| \leqq \beta \cdot \Big\{ |\lambda(\mu) - \lambda_0| \|z(\mu)\| + \|G(\mu)z(\mu)\| + \sum_{n=1}^{\infty} |\mu|^n \|G_n\| \Big\} \tag{3}$$

so $\|G(\mu)z(\mu)\| \leqq \beta \cdot \Sigma_{n=1}^{\infty} |\mu|^n \|G_n\| \cdot \{\cdots\}$ and

$$|\lambda(\mu)-\lambda_0|\,\|z(\mu)\|+\|G(\mu)z(\mu)\| \leqq \beta\cdot\left(|\lambda(\mu)-\lambda_0|+\sum_{n=1}^{\infty}|\mu|^n\|G_n\|\right)\cdot\{\cdots\}.$$

The last inequality gives

$$|\lambda(\mu)-\lambda_0|\,\|z(\mu)\|+\|G(\mu)z(\mu)\| \leqq \sum_{n=1}^{\infty}|\mu|^n\|G_n\| \cdot\frac{\beta\cdot\{|\lambda(\mu)-\lambda_0|+\Sigma|\mu|^n\|G_n\|\}}{1-\beta\cdot\{|\lambda(\mu)-\lambda_0|+\Sigma|\mu|^n\|G_n\|\}}.$$

Substituting this into (3), we get

$$\|y(\mu)-y_0\| \leqq \beta\cdot\sum_{n=1}^{\infty}|\mu|^n\|G_n\|\cdot\left[\frac{\beta\cdot\{|\lambda(\mu)-\lambda_0|+\Sigma|\mu|^n\|G_n\|\}}{1-\beta\cdot\{|\lambda(\mu)-\lambda_0|+\Sigma|\mu|^n\|G_n\|\}}+1\right]$$

$$=\frac{\beta\cdot\sum_{n=1}^{\infty}|\mu|^n\|G_n\|}{1-\beta\cdot\left\{|\lambda(\mu)-\lambda_0|+\sum_{n=1}^{\infty}|\mu|^n\|G_n\|\right\}}.$$

(v) We assume that y obeys $E(\lambda_0)y=0$. In this case we get $y\in E(\mathbf{C}-\{\lambda_0\})X$, so

$$y=(T-\lambda_0 I)Ry=R(T-\lambda_0 I)y=R((\lambda-\lambda_0)y+G(\mu)y).$$

But this leads to

$$|\lambda-\lambda_0|\,\|y\|+\|G(\mu)y\| \leqq |\lambda-\lambda_0|\cdot\beta\cdot(|\lambda-\lambda_0|\,\|y\|+\|G(\mu)y\|) +\|G(\mu)y\|$$

and

$$\|G(\mu)y\| \leqq \sum_{n=1}^{\infty}|\mu|^n\|G_n\|\cdot\beta\cdot(|\lambda-\lambda_0|\,\|y\|+\|G(\mu)y\|).$$

Altogether we get

$$|\lambda-\lambda_0|\,\|y\|+\|G(\mu)y\| \leqq \beta\cdot(|\lambda-\lambda_0|+\Sigma|\mu|^n\|G_n\|) \cdot\{|\lambda-\lambda_0|\,\|y\|+\|G(\mu)y\|\}.$$

If $|\lambda-\lambda_0|<\beta^{-1}-\alpha^{-1}$, further if $|\mu|<\rho$, the first factor on the right side is smaller than

$$\beta \cdot \{\beta^{-1} - \alpha^{-1} + \alpha^{-1}\} = 1,$$

and we have got a contradiction, unless $y = 0$.

Since the range of $E(\lambda_0)$ is one-dimensional, we must have

$$E(\lambda)y = c \cdot y_0 \quad \text{with some} \quad c \in \mathbf{C},\ c \neq 0.$$

As in [3] one then can see, that also $\lambda = \lambda(\mu)$.

Some remarks are in order:

1. Since $\alpha \geqq 2\beta$ we have $\beta^{-1} - \alpha^{-1} \geqq \alpha^{-1}$. So, in (v) the assertion concerning λ is true especially if $|\lambda - \lambda_0| < \alpha^{-1}$.

2. Our conditions and bounds become simpler, if N is small in norm, that is if $\|N\| < d/2$. In this case we get

$$\alpha \leqq \frac{2\omega(E)}{d - 2\|N\|}$$

$$\beta \leqq \frac{\omega(E)}{d - \|N\|}.$$

3. Of course a considerable simplification occurs in the scalar case, that is if $N = 0$. In this case we have

$$\alpha = \frac{2\omega(E)}{d}, \quad \beta = \frac{\omega(E)}{d} = \frac{\alpha}{2}, \quad \beta^{-1} - \alpha^{-1} = \frac{d}{2\omega(E)}.$$

The theorem then takes the following form:

THEOREM 2. *There exist holomorphic functions λ and y on $\{\mu \in \mathbf{C}, |\mu| < \rho\}$ with $y(0) = y_0$ and $\lambda(0) = \lambda_0$ which obey*

(i) $E(\lambda_0)y(\mu) = y_0$;

(ii) $[T - G(\mu)]y(\mu) = \lambda(\mu)y(\mu)$;

(iii) $|\lambda(\mu) - \lambda_0| \leqq \omega(E) \cdot \sum_{n=1}^{\infty} |\mu|^n \|G_n\|$;

(iv) $$\|y(\mu) - y_0\| \leqq \frac{\omega(E) \cdot \sum_{n=1}^{\infty} |\mu|^n \|G_n\|}{d - \omega(E)(\omega(E) + 1) \sum_{n=1}^{\infty} |\mu|^n \|G_n\|}.$$

(v) *If $[T - G(\mu)]y = \lambda y$ with $|\mu| < \rho$, $|\lambda - \lambda_0| < d/2\omega(E)$, $y \neq 0$, we have $\lambda = \lambda(\mu)$ and $y = c \cdot y(\mu)$ with some $c \in \mathbf{C}$.*

If X is a Hilbert space and T selfadjoint, we have $\omega(E) = 1$. In this case Theorem 2 reproduces exactly the bounds given by Schäfke.

Finally, we remark that in Theorem 1(iii) one can get more explicit estimates by familiar methods for localizing zeros of polynomials if N is assumed nilpotent.

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