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RESIDUALLY CENTRAL WREATH PRODUCTS

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This paper is concerned with the problem of determining which standard restricted wreath products of two groups A and G are residually central. Complete characterizations are obtained in the case where G is orderable and in the case where A and G are locally nilpotent.

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A group G is said to be residually central if for all $1 \neq x \in G$, $x \not\in [x, G]$. Other definitions may be found in **[10]** and **[11]**. Residually central groups were first studied by Durbin in **[3]** and **[4]**. Further information may be found in papers by Ayoub **[1]**, Slotterbeck **[12]**, and Stanley **[13]** and **[14]**.

The wreath product of two groups A and G is the semi-direct product $W = \bar{A}\,]\,B$, where $\bar{A}$ is the direct sum $\Pi\{A_g \mid g \in G\}$ of copies of A. If $\alpha \in \bar{A}$, then α can be written as $\alpha = \Pi_{i=1}^m a_i^{g_i}$, meaning that $\alpha(g_i) = a_i$, $1 \leqq i \leqq m$, and $\alpha(g) = 1$ if $g \not\in \{g_1, \cdots, g_m\}$. If $g \in G$, then $\alpha^g = \Pi_{i=1}^m a_i^{g_i g}$. The subgroup $\bar{A}$ is called the base group of W. Note that if $a \in A$, the element a^1 in $\bar{A}$ can be identified with a. Note also that if $B \lhd G$, then $(A/B) wr G$ is a homomorphic image of $A\, wr G$ in the obvious way; the kernel of the homomorphism is $\bar{B} = \Pi\{B_g \mid g \in G\}$. Throughout this paper W will denote the wreath product $A\, wr G$ and $\bar{A}$ its base group.

LEMMA 1. *If $g_1, \cdots, g_n \in G$, then $\Pi_{i=1}^n [g_i, G] = [\langle g_1, \cdots, g_n \rangle, G]$.*

Proof. Since each $[g_i, G] \leqq [\langle g_1, \cdots, g_n \rangle, G]$, $\Pi_{i=1}^n [g_i, G] \leqq [\langle g_1, \cdots, g_n \rangle, G]$. Let $K = \Pi_{i=1}^n [g_i, G]$, a normal subgroup of G. If Z/K is the center of G/K, then each $g_i \in Z$. Hence $\langle g_1, \cdots, g_n \rangle \leqq Z$, and so $[\langle g_1, \cdots, g_n \rangle, G] \leqq K$.

THEOREM 1. *Suppose that $W = A\, wr G$ is residually central. If G is infinite, then A is a Z-group.*

Proof. Let $a_1, \cdots, a_m \in A$, $K = \langle a_1, \cdots, a_m \rangle$. By a theorem of Hickin and Phillips **[7]**, it suffices to show that $K \not\leqq [K, A]$. Let $g_1, \cdots, g_m$ be distinct elements of G, and set $\alpha = \Pi_{i=1}^m a_i^{g_i} \in \bar{A}$. Since W is residually central, $\alpha \not\in [\alpha, W] \geqq [\alpha, \bar{A}] = \Pi_{i=1}^m [a_i, A]^{g_i}$ as a direct sum. Let $b_i \in [a_i, A]$, $1 \leqq i \leqq m$. Then $b_i^{g_i} \in [a_i, A]^{g_i} \leqq [\alpha, W] \lhd W$; thus $b_i^{1_G} = (b_i^{g_i})^{g_i^{-1}} \in [\alpha, W]$. Hence $\Pi_{i=1}^m [a_i, A] = [K, A] \leqq [\alpha, W] \lhd W$, and so $\Pi_{i=1}^m \{[K, A]^g \mid g \in G\} \leqq [\alpha, W]$. If $K \leqq [K, A]$, then $a_i \in [K, A]$, $1 \leqq i \leqq m$, and $\alpha = \Pi_{i=1}^m a_i^{g_i} \in \Pi_{i=1}^m [K, A]^{g_i} \leqq [\alpha, W]$, a contradiction.

LEMMA 2. *Let A and G be residually central groups. Then $W = A\,wr\,G$ is residually central if and only if for all $1 \neq \alpha \in \bar{A}$, $\alpha \not\in [\alpha, G][\alpha, \bar{A}]^G$.*

Proof. The necessity of the condition follows from the definition of residual centrality.

Let $w \in W$. Since W is a semi-direct product $\bar{A}\,]\,G$, w can be expressed uniquely in the form αg, where $\alpha \in \bar{A}$ and $g \in G$. Now $[\alpha g, W] \leqq [\alpha, W][g, \bar{A}G] \leqq \bar{A}[g, G]$. If $g \neq 1$, then $g \not\in [g, G]$, since G is residually central. Thus $\alpha g \not\in [\alpha g, W]$. If $g = 1$, then $[\alpha, W] \leqq [\alpha, G][\alpha, \bar{A}]^G$. Hence if $\alpha \not\in [\alpha, G][\alpha, \bar{A}]^G$, then W is residually central.

A group G is ordered if it possesses a total order $\leqq$ which is preserved under right and left multiplication. Further information may be found in **[8]**. Orderable groups must be torsion-free. Examples of orderable groups are free groups [**8**, p. 17] and torsion-free locally nilpotent groups [**8**, p. 16].

THEOREM 2. *If G is a residually central orderable group, and A is a Z-group, then $W = A\,wr\,G$ is residually central.*

Proof. Let $\alpha = \Pi_{i=1}^m a_i^{g_i} \in \bar{A}$, where $g_i \in G$, $a_i \in A$, and $a_i \neq 1$, $1 \leqq i \leqq m$. By Lemma 2 it is enough to assume that $\alpha \in [\alpha, G][\alpha, \bar{A}]^G$ and reach a contradiction. Let $L = [\langle a_1, \cdots, a_m \rangle, A]$. Since A is a Z-group, some $a_i \not\in L$, by **[7]**. If $\bar{L} = \Pi\{L^g \mid g \in G\}$, then $\alpha \not\in \bar{L}$, but $\alpha\bar{L} \in \zeta_1(\bar{A}/\bar{L})$, where $\zeta_n(H)$ denotes the nth center of a group H. Let $A_1 = A/L$, and $W_1 = A_1\,wr\,G$, a homomorphic image of W. Then $\alpha \in [\alpha, W]$ implies that $\alpha\bar{L} \in [\alpha\bar{L}, W_1]$. Because $\alpha\bar{L} \in \zeta_1(\bar{A}_1)$, a characteristic subgroup of $\bar{A}_1$, $[\alpha\bar{L}, W_1] \leqq \zeta_1(\bar{A}_1)$. Let $A_2 = \zeta_1(\bar{A}_1)$; then $W_2 = A_2\,wr\,G$ is not residually central, and so we may assume that the base group $\bar{A}$ is abelian. We may also assume that $A = \langle a_1, \cdots, a_m \rangle$.

With these assumptions, there is a prime p and subgroup B of index p in A. Since some $a_i \not\in B$, $\alpha \not\in B^G$, so that we may factor out B and assume that A is cyclic of prime order p. Denoting the field of p elements by Z_p, we note that $\bar{A}$ is a free Z_pG-module of rank 1. Let $\Delta = (1 - g \mid g \in G)$ denote the augmentation ideal of Z_pG. If $g \in G$, then $[\alpha, g]$ may be written in (additive) module notation as $-\alpha + \alpha g =$

$-\alpha(1-g)$; thus the assumption that $\alpha \in [\alpha, G]$ means, in module notation, that $\alpha \in \alpha\Delta$. Hence there exists $\delta \in \Delta$ such that $\alpha = \alpha\delta$. Then $\alpha(1-\delta) = 0$, and $\alpha \neq 0$, $1-\delta \neq 0$. But since G is orderable, Z_pG can have no zero divisors [**10**, 26.2 and 26.4], a contradiction.

This shows that if G is a residually central, orderable group, then $A \, wr \, G$ is residually central if and only if A is a Z-group. For example, free groups are orderable and are residually nilpotent; thus the wreath product of two free groups is residually central.

LEMMA 3. *Suppose that $W = A \, wr \, G$ is residually central, and G has an element g of prime order p. Then every element of A and of G of finite order has p-power order.*

Proof. Suppose $a \in A$ has prime order $q \neq p$. As elements of $\bar{A}$, $a \neq a^g$. However, in a residually central group, elements of relatively prime, finite orders commute [**10**, Theorem 6.14], and so $a = a^g$, which is impossible.

Suppose $h \in G$ has prime order $q \neq p$. Then g and h commute, and $\langle g, h\rangle$ is cyclic of order pq. Let $1 \neq a \in A$ and $A_1 = \langle a\rangle$. Then $W_1 = A_1 \, wr \, \langle g, h\rangle$ is residually central with an abelian base group. Let $\alpha = [a, g, h]$. Modulo $[\alpha, g]$ we have

$$1 = [a, g, h^q] \equiv [a, g, h]^q = \alpha^q.$$

Since h and g commute, and $\bar{A}_1$ is abelian,

$$\begin{aligned}\alpha = [a, g, h] &= [a, g]^{-1}[a, h]^{-1}[a, gh] = [a, h]^{-1}[a, g]^{-1}[a, hg] \\ &= [a, h, g].\end{aligned}$$

As before, modulo $[\alpha, G]$,

$$1 = [a, h, g^p] \equiv [a, h, g]^p = \alpha^p.$$

Thus $\alpha^p \in [\alpha, G]$, $\alpha^q \in [\alpha, G]$ for the distinct primes p and q, so that $\alpha \in [\alpha, G]$, implying that W is not residually central, a contradiction.

THEOREM 3. *Suppose A and G are locally nilpotent. Then $W = A \, wr \, G$ is residually central if and only if either*

(1) *G is torsion-free, or*

(2) *For some prime p, all elements of G and of A of finite order have p-power order.*

Proof. The necessity of (1) or (2) follows from Lemma 3.

If (1) holds, then G is orderable [**8**, p. 16], and Theorem 2 applies.

Suppose (2) holds. Since residual centrality is a local property [3], it suffices to show that every finitely generated subgroup $\langle w_1, \cdots, w_m \rangle$ of W is contained in a residually central subgroup. Each $w_i = \alpha_i g_i$, where $g_i \in G$ and $\alpha_i \in \bar{A}$, and each $\alpha_i = \Pi_{j=1}^{n_i} a_{ij}^{g_{ij}}$. Hence

$$\langle w_1, \cdots, w_m \rangle \leqq \langle a_{ij}, g_{ij}, g_i \mid 1 \leqq i \leqq m, i \leqq j \leqq n_i \rangle$$
$$= \langle a_{ij} \rangle \, wr \, \langle g_{ij}, g_i \rangle.$$

Thus we may assume that both A and G are finitely generated and hence nilpotent.

Let $\alpha = \Pi_{k=1}^{l} a_k^{g_k}$. By Lemma 2, it suffices to assume that $1 \neq \alpha \in [\alpha, G][\alpha, \bar{A}]^G$ and reach a contradiction. Since A is nilpotent, there is an integer r such that each $a_i \in \zeta_r(A)$ and some $a_i \notin \zeta_{r-1}(A)$. Then

$$[\alpha, \bar{A}]^G \leqq [\langle a_1, \cdots, a_l \rangle, A]^G \leqq [\zeta_r(A), A]^G \leqq (\zeta_{r-1}(A))^G.$$

$W_1 = (A/\zeta_{r-1}(A)) \, wr G$ is a homomorphic image of W in the obvious way. If $\bar{\alpha}$ denotes the image of α in W_1, then $\bar{\alpha} \in [\bar{\alpha}, G][\bar{\alpha}, \overline{A/\zeta_{r-1}(A)}] = [\bar{\alpha}, G]$ in W_1, since $\alpha \in [\alpha, G][\alpha, A]^G$ in W. Let $A_1 = \zeta_r(A)/\zeta_{r-1}(A)$. Thus $A_1 \, wr G$ is a subgroup of W_1 containing $\bar{\alpha}$. $[\bar{\alpha}, G] \leqq \bar{A}_1$, since A_1 is a characteristic subgroup of $A/\zeta_{r-1}(A)$. By [2, Corollary 2.11], every element of A_1 of finite order has p-power order. By [5, Theorem 2.1], A_1 and G are residually finite p-groups. Because $\bar{\alpha} \in [\bar{\alpha}, G]$, $A_1 \, wr G$ is not residually central and therefore not residually nilpotent. Hartley [6], however, has shown that $A_1 \, wr G$ is residually nilpotent, a contradiction.

Corollary. *If A is abelian and G is locally nilpotent, then $W = A \, wr G$ is residually central if and only if W is locally a residually nilpotent group.*

Proof. The sufficiency of the condition is clear. Theorem 3 and Theorems B1 and B2 of [6] combine to prove the necessity.

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