PACIFIC JOURNAL OF MATHEMATICS
Vol. 68, No. 2, 1977

COMPLETELY SEMISIMPLE INVERSE Δ-SEMIGROUPS ADMITTING PRINCIPAL SERIES

P. G. TROTTER AND TAKAYUKI TAMURA

A Δ-semigroup is a semigroup whose lattice of congruences is a chain with respect to inclusion. A completely semisimple inverse Δ-semigroup that admits a principal series is characterized here as a semigroup that results from a particular series of ideal extensions of Brandt semigroups by Brandt semigroups. A characterization is given of finite inverse Δ-semigroups in terms of groups, Brandt semigroups, and one to one partial transformations of sets.

1. Introduction.

A *Δ-semigroup* is a semigroup whose lattice of congruences is a chain with respect to inclusion. Schein **[8]** and Tamura **[11]** showed that a commutative Δ-semigroup is either a quasicyclic group A, or a commutative nil semigroup B with the divisibility chain condition, or A^0, or B^1. We study here the structure of completely semisimple inverse Δ-semigroups with principal series. Such semigroups will be characterized in terms of Δ-groups, idempotent properties, and ideal extensions of Brandt semigroups by Brandt semigroups.

In **[11]** it was shown that the least semilattice congruence on a Δ-semigroup has at most two classes. We begin by characterizing completely semisimple inverse semigroups admitting principal series and having this property.

In the final section we show that each finite inverse Δ-semigroup determines a set of structure data that involves groups, Brandt semigroups and one to one partial transformations of sets. Conversely the semigroup can be reconstructed from the structure data.

2. Preliminaries.

We call a semigroup S an $\mathcal{S}_1$-, or $\mathcal{S}_2$-*semigroup* if the smallest semilattice congruence on S has one, or two congruence classes respectively. S is a Δ-semigroup only if it is an $\mathcal{S}_1$- or $\mathcal{S}_2$-semigroup. In this section we characterize completely semisimple inverse $\mathcal{S}_1$-, or $\mathcal{S}_2$-semigroups that admit principal series.

A subsemigroup H of a semigroup S is $\mathcal{S}$-*unitary* if and only if whenever $HxyH \subseteq H$ for $x, y \in S^1$ then $Hx, yH \subseteq H$. Notice that if E is a semilattice and $efg = e$ in E then $ef = e = ge$. Hence, any class of a semilattice congruence on S is $\mathcal{S}$-unitary. Let $\mathcal{J}^*$ denote the least congruence on S containing Green's relation $\mathcal{J}$. For $a \in S$ let J_a be the $\mathcal{J}$-class of a and $J(a) = S^1aS^1$.

THEOREM 2.1. *Let S be a regular semigroup. The following are equivalent*:

(i) *S is an $\mathcal{S}_1$-semigroup.*

(ii) $\mathcal{J}^* = S \times S$.

(iii) *Each $\mathcal{S}$-unitary subsemigroup of S that is a union of $\mathcal{J}$-classes is an ideal.*

Proof. Howie and Lallement **[2]** have shown that $\mathcal{J}^*$ is the least semilattice congruence on S. Hence (i) and (ii) are equivalent.

(ii) *implies* (iii). Let H be an $\mathcal{S}$-unitary subsemigroup that is a union of $\mathcal{J}$-classes but is not an ideal. Suppose $xay \in H$ for some $x, y \in S^1$, $a \in S$. Then $HxayH \subseteq H$ so $Hxa \subseteq H$. Hence $HxaH \subseteq H$ so $aH \subseteq H$ and $HaH \subseteq H$. If $J_a = J_b$, $b \in S$, then there exists $r, s \in S^1$ so that $HxrbsyH \subseteq H$ and similarly $HbH \subseteq H$. So $HJ_aH \subseteq H$. Since H is not an ideal and is $\mathcal{S}$-unitary there is a $d \in S$ so that $HdH \not\subseteq H$. Define

$$(1) \qquad \begin{aligned} C_h &= \{a \in S;\ xJ_ay \cap H \neq \square \text{ for some } x, y \in S^1\} \text{ and} \\ \bar{C}_H &= \{d \in S;\ d \notin C_H\}. \end{aligned}$$

Let ρ_H denote the equivalence relation on S with classes C_H and $\bar{C}_H$. If $a \in C_H$ then we have $HJ_aH \subseteq H \subseteq C_H$. Furthermore, since H is $\mathcal{S}$-unitary, $HabH \subseteq H$ if and only if $HaH, HbH \subseteq H$, for $a, b \in S^1$. Hence C_H is a unitary semigroup, $\bar{C}_H$ is an ideal, and ρ_H is a nonuniversal semilattice congruence.

(iii) *implies* (ii). Since a $\mathcal{J}^*$-class is $\mathcal{S}$-unitary, it is an ideal. But ideals of S intersect nontrivially.

The next theorem is an immediate consequence of results in **[5]**, **[6]** or **[9]**.

THEOREM 2.2. *For any semigroup S the following are equivalent*:

(i) *S is an $\mathcal{S}_1$-semigroup.*

(ii) *Each ideal of S is an $\mathcal{S}_1$-semigroup.*

(iii) *S is an ideal extension of an $\mathcal{S}_1$-semigroup I by an $\mathcal{S}_1$-semigroup T.*

Note that T has zero divisors.

COROLLARY 2.3. *Let S be a regular semigroup with a principal series. S is a $\mathcal{S}_1$-semigroup if and only if each 0-simple principal factor of S has a zero divisor.*

Proof. By Theorem 2.2, the condition is clearly necessary.

Conversely let $S_0 \subset S_1 \subset \cdots \subset S_n = S$ be a principal series. Since S_0 is simple it is an $\mathcal{S}_1$-semigroup. Continuing by induction, assume S_{i-1} is an $\mathcal{S}_1$-semigroup and S_i/S_{i-1} has zero divisors for some i, $1 \leqq i \leqq n$. S_i/S_{i-1} is 0-simple so is an $\mathcal{S}_1$-semigroup. Hence, by Theorem 2.2 (iii), S_i is an $\mathcal{S}_1$-semigroup.

Let $B(G, I)$ denote the Brandt semigroup that is a Rees matrix semigroup over the group with zero G^0 and with the identity $I \times I$ sandwich matrix. We call G the *basic group* of $B(G, I)$. $B(G, I)$ has zero divisors if $|I| > 1$ and is isomorphic to G^0 if $|I| = 1$. Since an inverse semigroup is completely [0]-simple if and only if it is a group [Brandt semigroup], we have from Corollary 2.3:

COROLLARY 2.4. *Let S be a completely semisimple inverse semigroup with principal series $S_0 \subset S_1 \subset \cdots \subset S_n = S$. S is an $\mathcal{S}_1$-semigroup if and only if* (i) *S_0 is a group, and* (ii) *$S_i/S_{i-1} \cong B(G_i, I_i)$ with $|I_i| > 1$ for $1 \leqq i \leqq n$.*

We conclude this section with a similar result for inverse $\mathcal{S}_2$-semigroups.

THEOREM 2.5. *Let S be a completely semisimple inverse semigroup with principal series $S_0 \subset S_1 \subset \cdots \subset S_n = S$. S is an $\mathcal{S}_2$-semigroup if and only if* (i) *S_0 is a group and* (ii) *$S_i/S_{i-1} \cong B(G_i, I_i)$ for $1 \leqq i \leqq n$ where $|I_r| = 1$ for exactly one r, $1 \leqq r \leqq n$.*

Proof. We first observe that if $|I_r| = 1$ and $J_r = S_r \backslash S_{r-1}$, $1 \leqq r \leqq n$, then J_r is an $\mathcal{S}$-unitary subgroup of S that is a $\mathcal{J}$-class but not an ideal. As in the proof of Theorem 2.1 there is a semilattice congruence ρ_{J_r} with classes C_{J_r}, $\bar{C}_{J_r}$ defined as in (1).

Assume that S is an $\mathcal{S}_2$-semigroup then S_0 is a group and by Corollary 2.4 there exists an r, $1 \leqq r \leqq n$, so that $|I_r| = 1$. Suppose also that $|I_t| = 1$, $1 \leqq t \leqq n$. Then $\rho_{J_r} = \mathcal{J}^* = \rho_{J_t}$. Hence $C_{J_r} = C_{J_t}$ and since J_r, J_t are $\mathcal{J}$-classes, $r = t$.

Conversely assume (i) and (ii). As in the proof of Theorem 2.1, C_{J_r} is a unitary subsemigroup and $\bar{C}_{J_r}$ is an ideal of S. Then the $\mathcal{J}$-classes of C_{J_r} and the $\mathcal{J}$-classes of $\bar{C}_{J_r}$ are $\mathcal{J}$-classes of S. Since $S_0 \subseteq \bar{C}_{J_r}$ and $J_r \subseteq C_{J_r}$ are the only $\mathcal{J}$-classes that are groups then $\bar{C}_{J_r}$, C_{J_r} are $\mathcal{S}_1$-semigroups. Hence $\rho_{J_r} = \mathcal{J}^*$.

3. Characterization.

In this section completely semisimple inverse Δ-semigroups with principal series are characterized.

The following Lemma is an immediate consequence of results of Preston **[7]**. Parts (i) and (ii) are also corollaries of Tamura **[10]**.

LEMMA 3.1. *Let $S = B(G, I)$ be a Brandt semigroup.*
(i) *S is Δ-semigroup if and only if G is a Δ-group.*
(ii) *Each congruence of S is idempotent separating or universal.*
(iii) *S is primitive.*

We need some further results.

LEMMA 3.2. *Let S be an inverse semigroup with ideal I. Any congruence ρ' on I extends to a congruence ρ on S so that*

$$a\rho = \begin{cases} a\rho' & \text{if } a \in I \\ \{a\} & \text{if } a \in S \setminus I. \end{cases}$$

In particular, any ideal of an inverse Δ-semigroup is an inverse Δ-semigroup.

Proof. Let A and B be congruence classes of ρ'. Suppose $xay \in B$ for some $x, y \in S^1$, $a \in A$. Since $xaa^{-1}aa^{-1}ay \in B$ and $xaa^{-1}, a^{-1}ay \in I$ then $xaa^{-1}Aa^{-1}ay \subseteq B$. If $c \in A$ then $aa^{-1}ca^{-1}a \in A$ and $xcc^{-1}aa^{-1}ca^{-1}ac^{-1}cy = xaa^{-1}ca^{-1}ay \in B$ so $xcc^{-1}Ac^{-1}cy \subseteq B$. In particular $xcy \in B$. Hence $xAy \subseteq B$. Since I is an ideal the result follows.

If S is an inverse semigroup with semilattice E, let $C(E)$ denote the centralizer of E in S.

LEMMA 3.3. *Let S be a completely semisimple inverse semigroup with principal series $\{0\} \subset S_1 \subset S$ and with semilattice E. Then on S;*

(i) *Each non idempotent separating congruence has S_1 or S as a congruence class if and only if for any $e, f \in E$ so that $e \in S \backslash S_1$, $f \in S_1 \backslash 0$ there exists $a \in S$ so that $a^{-1}ea = f$ and so that $fa = 0$ if $e > f$.*

(ii) *Each idempotent separating congruence is the identity equivalence on $S \setminus S_1$ if and only if $C(E) \cap (S \setminus S_1) \subseteq E$.*

Proof. (i) Suppose the non idempotent separating congruences have S_1 or S as congruence classes. If $a \in S_1 \backslash 0$ then $S_1 = J(a)$. If $b \notin J(a)$ then considering the Rees congruence modulo $J(b)$ we see that $J(a) \subset J(b) = S$. Hence the principal ideals of S are chain ordered. Let τ be the least congruence so that for some $e \neq f$ in $E \backslash 0$, $(e, f) \in \tau$. Assume that $e \in S \backslash S_1$ and $f \in S_1$. Then τ is universal. Since $0 \in f\tau$ then by Teissier **[12]** there exists $x_1, y_1, \cdots, x_n, y_n \in S^1$ so that

$$f = x_1 i_1 y_1, \quad x_1 j_1 y_1 = x_2 i_2 y_2, \cdots, \quad x_n j_n y_n = 0$$

where $i_p, j_p \in \{e, f\}$, $p = 1, \cdots, n$. But $(x_p i_p y_p)^{-1}(x_p i_p y_p) = z_p^{-1} i_p z_p$ where $z_p = x_p^{-1} x_p y_p$. So

$$f = z_1^{-1} i_1 z_1, \quad z_1^{-1} j_1 z_1 = z_2^{-1} i_2 z_2, \cdots, \quad z_n^{-1} j_n z_n = 0 .$$

Deleting repetitious terms we may assume that $z_p^{-1} i_p z_p \neq z_p^{-1} j_p z_p$, and that $z_p^{-1} i_p z_p > z_p^{-1} j_p z_p$ (otherwise replace z_q by $z_q z_p^{-1} i_p z_p$ for $p \leqq q \leqq n$). If $e > f$, then $z_p^{-1} e z_p \geqq z_p^{-1} f z_p$, so $i_p = e$, $j_p = f$. Furthermore, by Lemma 3.1 (iii) we have $f = z_1^{-1} e z_1 > z_1^{-1} f z_1 = 0$. Hence $(fz_1)^{-1}(fz_1) = 0$ so $fz_1 = 0$.

Conversely, suppose $e \neq f$ in E. If $e \in S \backslash S_1$, and $f \in S_1 \backslash 0$, then $a^{-1} e a = f$ for some $a \in S$, so $J(f) \subset J(e) = S$. Let τ be the least congruence with $(e, f) \in \tau$. By Lemma 3.1 (ii), if $e, f \in S_1$, then $e\tau \supseteq S_1$. If $e, f \in S \backslash S_1$, then, by Lemma 3.1 (iii), $ef \in S_1$ and $ef \in e\tau$. If $e \in S \backslash S_1$ and $f \in S_1$, then $0 \in e\tau$ since either $e > f$ and $0 = a^{-1} f a \leqq a^{-1} e a = f$ for some $a \in S$, or $ef = 0$ by Lemma 3.1 (iii). Then $e\tau = J(e) = S$.

(ii) By [1] the greatest idempotent separating congruence on S has group kernel normal system $\{H_e \cap C(E);\ e \in E\}$ where H_e is the $\mathcal{H}$-class of e.

LEMMA 3.4. *Let S be a completely semisimple inverse $\mathcal{S}_1$-semigroup with principal series $\{0\} \subset S_1 \subset S$. S is a Δ-semigroup if and only if*

(i) *the Brandt semigroups S/S_1 and S_1 have Δ-basic groups,*

(ii) *each non idempotent separating congruence of S has S_1 or S as a congruence class, and*

(iii) *each idempotent separating congruence of S is the identity equivalence on $S \backslash S_1$.*

Proof. Let S be a Δ-semigroup. By Lemmas 3.1 (i) and 3.2, (i) is satisfied. Comparing congruences with the Rees congruence modulo S_1 we see that (iii) holds and that any non universal congruence has its classes in S_1 or $S \backslash S_1$. Hence, applying Lemma 3.1 (ii) to S_1, we see that (ii) holds.

Conversely, by (i), (iii) and Lemma 3.1 (i) applied to S_1, the idempotent separating congruences are chain ordered. By (ii) the other non universal congruences have S_1 as a class and are then chain ordered since, by (i), S/S_1 is a Δ-semigroup. Hence, by (iii), S is a Δ-semigroup.

LEMMA 3.5. *Let S be a completely semisimple inverse $\mathcal{S}_2$-semigroup with principal series $\{0\} \subset S_1 \subset S$. S is a Δ-semigroup if and only if S_1 is an $\mathcal{S}_1$-Δ-semigroup, $S \backslash S_1$ is a Δ-group and S satisfies conditions* (ii) *and* (iii) *of Lemma* 3.4.

Proof. By Theorem 2.5 just one of $S_1 \backslash 0$ or $S \backslash S_1$ is a group. Assume S is a Δ-semigroup. The $\mathcal{J}$-classes of S are chain ordered **[11]**. If $S_1\backslash 0$ is a group then, as in the proof of Theorem 2.5, $\mathcal{J}^*$ has classes $\{0\}$, $S\backslash 0$. But then $\mathcal{J}^*$ is not comparable with the Rees congruence modulo S_1. Hence $S\backslash S_1$ is a group while $S_1\backslash 0$ is not. The remainder of the proof is as for Lemma 3.4.

The following theorem is the main result. Together with the results 2.4, 2.5, 3.1(i), 3.3, 3.4 and 3.5, it provides a characterization of completely semisimple inverse Δ-semigroups with principal series in terms of Δ-groups and idempotent properties.

THEOREM 3.6. *Let S be a completely semisimple inverse semigroup with principal series $S_0 \subset S_1 \subset \cdots \subset S_n = S$. S is a Δ-semigroup if and only if*

(i) *S_0 is a Δ-group; $S_0 = \{0\}$ if $n > 0$,*
(ii) *S_1 is a Brandt semigroup with Δ-basic group if $n > 0$,*
(iii) *S_i/S_{i-2} is an $\mathcal{S}_1$-Δ-semigroup for $i = 2, \cdots, n-1$,*
(iv) *S_n/S_{n-2} is an $\mathcal{S}_1$-, or $\mathcal{S}_2$-Δ-semigroup.*

Proof. Say S is a Δ-semigroup. S and S_0 have the same maximal group homomorphic image and if $n > 0$ the only such group is trivial **[11]**. Hence, by Lemmas 3.2, 3.1(i), (i), 3.4 and 3.5, we see that (i), $\cdots$(iv) are satisfied.

Conversely we prove that for any congruence ρ on S and some i, $0 \leqq i \leqq n$, then $a\rho = S_i$ for $a \in S_i$, $e\rho \cap E = \{e\}$ for $e \in (S_{i+1}\backslash S_i) \cap E$ where E is the semilattice of S, and $a\rho = \{a\}$ for $a \in S\backslash S_{i+1}$. Then S will be a Δ-semigroup. The result holds for $n = 0$ or 1, by Lemma 3.1 (ii). Continue by induction, assuming the result for $n = t$. Since S_{t+1}/S_{t-1} is a Δ-semigroup, then the congruences of S_{t+1} that have their classes in S_t or $S_{t+1}\backslash S_t$ are of the required form by Lemmas 3.4, 3.5. Suppose ρ is a congruence on S_{t+1} with $(a, b) \in \rho$, $a \in S_{t+1}\backslash S_t$, $b \in S_t$. Then the congruence on the Δ-semigroup S_{t+1}/S_{t-1} induced by ρ is universal by Lemmas 3.4, 3.5. Hence there exists $h \in S_t\backslash S_{t-1}$, $k \in S_{t-1}$ so that $(h, k) \in \rho$. But then, by the induction assumption, $S_t \subseteq h\rho$. Since S_{t+1}/S_{t-1} is a Δ-semigroup, then, by Lemmas 3.4, 3.5, $a\rho = S_{t+1}$.

4. Further study of finite case. We now investigate circumstances under which the extensions of Theorem 3.6 are possible for finite inverse semigroups. Some further information is required.

EXAMPLE 1. Let H_X be the subgroup of the symmetric group P_X whose elements displace a finite number of elements of the set X. The alternating group A_X is a simple normal subgroup of H_X with index 2 (see **[3]**) for $|X| \neq 4$. Hence H_X is a Δ-group. In particular if $|X|$ is finite then P_X is a Δ-group.

EXAMPLE 2. The symmetric inverse semigroup $\mathcal{I}_X$, $|X|$ finite, is a Δ-semigroup. To see this, let $D(\alpha)$, $R(\alpha)$ denote the domain and range of $\alpha \in \mathcal{I}_X$ respectively. Any ideal of $\mathcal{I}_X$ is of the form $I_n = \{\alpha \in \mathcal{I}_X; |D(\alpha)| \leqq n\}$. Since $\mathcal{I}_X$ is finite it has a principal series and is completely semisimple. If α is an idempotent its $\mathcal{H}$-class is $\{\beta \in \mathcal{I}_X;\ D(\beta) = R(\beta) = D(\alpha)\}$ (see **[4]**), which is the symmetric group on $D(\alpha)$. So for some α a non group principal factor has Δ-basic group isomorphic to $P_{D(\alpha)}$. If α, γ are idempotents so that $|D(\alpha)| > |D(\gamma)| \geqq 1$ then there is a $\beta \in \mathcal{I}_X$ so that $\beta^{-1}\alpha\beta = \gamma$ and $|D(\beta^{-1}\gamma\beta)| < |D(\gamma)|$. If α is not an idempotent there is an idempotent β so that $|D(\alpha)| - |D(\beta)| \leqq 1$ and $\beta\alpha \neq \alpha\beta$. $\mathcal{I}_X$ can now be seen to satisfy the requirements of Lemmas 3.3, 3.4 and Theorem 3.6.

Let Z_n denote the set $\{1, 2, \cdots, n\}$. If $X = Z_n$ write $P_n = P_X$ and $\mathcal{I}_n = \mathcal{I}_X$.

Suppose S is a finite Δ-semigroup with $\{0\} \subset S_1 \subset S$, $S_1 \cong B(G, Z_n)$ and $S/S_1 \cong B(H, Z_r)$. Let $(G \times Z_n \times Z_n) \cup \{0\}$ denote the set of elements of S_1, with the binary operation $(x, i, j)(y, h, k) = (xy, i, k)$ if $j = h$, and 0 if $j \neq h$.

Denote the semigroup of right translations of S_1 by $P(S_1)$ and for $a \in S$ define $\rho^a \in P(S_1)$ by $b\rho^a = ba$ for all $b \in S_1$. Since inverse semigroups are left reductive there is a unique homomorphism $\theta: S \to P(S_1)$ so that the restriction of θ to S_1 is the regular representation of S_1 (by **[6**; III.1.12]). θ is given by $a\theta = \rho^a$, $a \in S$, and $(S_1)\theta \cong S_1$. Since S is a Δ-semigroup then, by Lemma 3.4 or 3.5, θ is injective. Call θ the *extension homomorphism of S*.

Let 1 denote the identity of G. For $u \in S$, $i \in Z_n$ define $D(u) = \{j \in Z_n;\ (1, i, j)u\theta \neq 0\}$. By **[6**; V.3.6 and V.5.4] there exists $\phi_u \in \mathcal{I}_n$ with domain $D(\phi_u) = D(u)$ and a map $a_u: D(u) \to G$ so that

$$(x, i, j)u\theta = \begin{cases} (x(ja_u), i, j\phi_u) & \text{if } j \in D(u), \\ 0 & \text{if } j \notin D(u). \end{cases}$$

Furthermore the map given by $u\theta \to (a_u, \phi_u)$ defines an isomorphism between $(S)\theta$ and the semigroup $\{(a_u, \phi_u);\ u \in S\}$ with the binary operation $(a_u, \phi_u)(a_v, \phi_v) = (a_u \cdot a_v, \phi_u\phi_v)$ where $j(a_u \cdot a_v) = (ja_u)(j\phi_u a_v)$. Since θ is an isomorphism then $(a_{uv}, \phi_{uv}) = (a_u \cdot a_v, \phi_u\phi_v)$. Note that with the operations $\cdot$ and composition of maps, the sets $\{a_u;\ u \in S\}$ and $\{\phi_u;\ u \in S\}$ respectively are homomorphic images of S. For convenience we will identify $u\theta$ and (a_u, ϕ_u) for each $u \in S$.

Clearly $v \in S_1$ if and only if $|D(\phi_v)| \leqq 1$. Since ϕ_u is a bijection for $u \in S$ then $\{v \in S;\ |D(\phi_v)| \leqq |D(\phi_u)|\}$ is an ideal of S. Hence for $u, v \in S \backslash S_1$, $|D(\phi_u)| = |D(\phi_v)|$; call this number the *rank* of S/S_1. Clearly e is an idempotent of $S \backslash 0$ if and only if $(D(\phi_e))a_e = \{1\}$

and ϕ_e is an identity map. A product of distinct idempotents $e, f \in S$ is in S_1 so $|D(\phi_e) \cap D(\phi_f)| \leqq 1$. Hence if $S \backslash S_1$ is not a group then the rank of S/S_1 is bounded above by $[(n+1)/2]$.

DEFINITION 4.1. For integers m and n, $1 < m \leqq n$, let ${}^n\Gamma_m$ denote the largest number so that $\{Y_i;\ |Y_i| = m,\ i = 1, \cdots, {}^n\Gamma_m\}$ is a family of subsets of Z_n with $|Y_i \cap Y_j| \leqq 1$ for $i \neq j$. For an integer r, $1 < r \leqq {}^n\Gamma_m$, let $\mathscr{A} = \{X_i;\ |X_i| = m,\ i \in Z_r\}$ be a family of subsets of Z_n with $|X_i \cap X_j| \leqq 1$ for $i \neq j$. Let

$$\mathscr{A}^* = \{\alpha \in \mathscr{I}_n;\ \alpha = 0 \quad \text{or} \quad D(\alpha), R(\alpha) \in \mathscr{A}\}$$

with a binary operation $*$ so that

$$\alpha * \beta = \begin{cases} \alpha\beta & \text{if } R(\alpha) = D(\beta) \\ 0 & \text{if } R(\alpha) \neq D(\beta). \end{cases}$$

LEMMA 4.2. *Let S be a Δ-semigroup with principal series $\{0\} \subset S_1 \subset S$ so that $S_1 \cong B(G, Z_n)$ and $S/S_1 \cong B(H, Z_r)$ has rank m. Then*

(i) *either $1 < m \leqq [(n+1)/2]$ and $1 < r \leqq {}^n\Gamma_m$ or $1 < m \leqq n$ and $r = 1$,*

(ii) *H is embeddable in the symmetric group P_m.*

Proof. Part (i) follows from the preceding commets and definition. Let $Q = (S \backslash S_1) \cup \{0\}$ and define a binary operation $*$ so that $u * v = uv$ if $uv \in S \backslash S_1$, and 0 otherwise. Then $Q \cong B(H, Z_r)$. Let $\mathscr{A} = \{D(\phi_u);\ u \in S \backslash S_1\}$. The map $\delta\colon Q \to \mathscr{A}^*$ given by $u\delta = \phi_u$ is a homomorphism. If $u \neq v$ in $S \backslash S_1$ and $\phi_u = \phi_v$ then $|D(\phi_{uv^{-1}})| > 1$ so uv^{-1} is a non idempotent element of $S \backslash S_1$. But then for any idempotent $e \in S$, it can be readily shown that $(euv^{-1})\theta = (uv^{-1}e)\theta$ so $euv^{-1} = uv^{-1}e$. This contradicts Lemmas 3.3 and 3.4 or 3.5. Hence δ is injective. If $e \neq 0$ is an idempotent of Q then it can be easily shown that the $\mathscr{H}$-class of e in Q is $H_e = \{u \in Q;\ D(\phi_u) = R(\phi_u) = D(\phi_e)\}$. Then $(H_e)\delta \cong H_e \cong H$. Part (ii) follows since the elements of $(H_e)\delta$ are permutations of $D(\phi_e)$.

Let $\mathscr{B}^* = \{a_u;\ u \in S \backslash S_1\} \cup \{0\}$ with a binary operation $*$ so that $a_u * a_v = a_u \cdot a_v$ if $uv \in S \backslash S_1$, and 0 otherwise. Since δ is injective, if $\phi_u = \phi_v$ for $u, v \in S \backslash S_1$ then $u = v$ so $a_u = a_v$. Hence there is a homomorphism $\lambda\colon (Q)\delta \to \mathscr{B}^*$ given by $\phi_u\lambda = a_u$ if $u \in S \backslash S_1$ and $0\lambda = 0$. The set $\bar{H} = \{(u\delta\lambda, u\delta);\ u \in Q\}$ with the binary operation so that $(u\delta\lambda, u\delta)(v\delta\lambda, v\lambda) = ((u * v)\delta\lambda, (u * v)\delta)$ is then a semigroup isomorphic to Q.

DEFINITION 4.3. A *structure data set* is a set $\{n, r, m, G, \bar{H}\}$ defined as follows:

(i) n, r and m are integers so that either $1 < m \leqq [(n+1)/2]$ and $1 < r \leqq {}^{n}\Gamma_{m}$, or $1 < m \leqq n$ and $r = 1$.

(ii) G is a Δ-group.

(iii) Let $\mathscr{A} = \{X_i; |X_i| = m, i \in Z_r\}$ be a family of subsets of Z_n so that $|X_i \cap X_j| \leqq 1$ if $i \neq j$. Let H be a Δ-subgroup of the symmetric group P_m. Let K be a subsemigroup of $\mathscr{A}^*$ so that $K \cong B(H, Z_r)$. Let $\lambda: K \to \mathscr{B}^*$ be a surjective homomorphism so that for $\phi \in K$, $\phi\lambda: D(\phi) \to G$ is a map and so that for $j \in D(\phi * \psi)$ then $j(\phi * \psi)\lambda = (j(\phi\lambda))(j\phi(\psi\lambda))$. Define $\bar{H} = \{(\phi\lambda, \phi); \phi \in K\}$ with a binary operation so that $(\phi\lambda, \phi)(\psi\lambda, \psi) = ((\phi * \psi)\lambda, \phi * \psi)$. Write $\phi\lambda * \psi\lambda = (\phi * \psi)\lambda$.

Notice that in the terminology of **[6]**, $\bar{H}$ satisfies this definition if and only if $\bar{H}$ is a subsemigroup of the right wreath product of G and K so that the map $\bar{H} \to K$ given by $(\phi\lambda, \phi) \to \phi$ is an isomorphism.

We have seen that any finite inverse Δ-semigroup S with principal series $\{0\} \subset S_1 \subset S$ determines a structure data set $\{n, r, m, G, \bar{H}\}$. Call this a *structure data set of S*. We say that structure data sets $\{n, r, m, G, \bar{H}\}$ and $\{n', r', m', G', \bar{H}'\}$ are *equivalent* if and only if $n = n'$, $r = r'$, $m = m'$ and there exists an isomorphism $\alpha: G^\circ \to (G')^\circ$ and a bijection $\beta: Z_n \to Z_n$ so that the map $\gamma: \bar{H} \to \bar{H}'$ given by $(a, \phi)\gamma = (\beta^{-1}a\alpha, \beta^{-1}\phi\beta)$ is a bijection.

LEMMA 4.4. *Let S and T be finite inverse Δ-semigroups with principal series $\{0\} \subset S_1 \subset S$ and $\{0\} \subset T_1 \subset T$ respectively. Then $S \cong T$ if and only if the structure data sets of S and T are all equivalent.*

Proof. Lable the elements of S_1 and T_1 so that $S_1 = (G \times Z_n \times Z_n) \cup \{0\}$ and $T_1 = (G' \times Z_{n'} \times Z_{n'}) \cup \{0\}$, with binary operations as defined after Example 2. Then structure data sets $\{n, r, m, G, \bar{H}\}$ and $\{n', r', m', G', \bar{H}'\}$ of S and T respectively can be uniquely determined by the method described above. Depending on the labelling of the elements of S_1, each structure data set of S can be so determined. Let θ_S and θ_T be the extension homomorphisms of S and T respectively and let $\eta: S \to T$ be an isomorphism. Then $n = n'$, $r = r'$ and the restriction of η to S_1 determines an isomorphism $\alpha: G^\circ \to (G')^\circ$ and a bijection $\beta: Z_n \to Z_n$ so that $(x, i, j)\eta = (x\alpha, i\beta, j\beta) \in T_1$. The map $u\theta_S$, $u \in S$, is given by $v(u\theta_S) = vu$ for all $v \in S_1$. Let $u\eta\theta_T = (b_{u\eta}, \psi_{u\eta})$ and $v = (x, i, j)$ then

$$((x(ja_u))\alpha, i\beta, j\phi_u\beta) = (v(u\theta_s))\eta = (v\eta)(u\eta\theta_T) = ((x\alpha)j\beta b_{u\eta}, i\beta, j\beta\psi_{u\eta}).$$

So $(x(ja_u))\alpha = (x(j\beta b_{u\eta})\alpha^{-1})\alpha$. Since $D(u)\beta = D(u\eta)$ then $a_u =$

$\beta b_{u\eta}\alpha^{-1}$. Likewise $\phi_u\beta = \beta\psi_{u\eta}$. Thus $m = m'$ and since $\bar{H} = \{(a_u, \phi_u); u \in S\backslash S_1\} \cup \{0\}$ then the structure data sets are equivalent.

Conversely, given that $\{n, r, m, G, \bar{H}\}$ and $\{n', r', m', G', \bar{H}'\}$ are equivalent structure data sets, let $\alpha: G^\circ \to (G')^\circ$ be an isomorphism, $\beta: Z_n \to Z_n$ be a bijection and $\gamma: \bar{H} \to \bar{H}'$ be the bijection so that $(a, \phi)\gamma = (\beta^{-1}a\alpha, \beta^{-1}\phi\beta)$. Define $\eta_1: S_1 \to T_1$ by $(x, i, j)\eta_1 = (x\alpha, i\beta, j\beta)$. Then η_1 is an isomorphism. As in the first part of the proof we get for $v \in S_1$ that $v\eta_1\theta_T = (\beta^{-1}a_v\alpha, \beta^{-1}\phi_v\beta)$. Hence there is a bijection $\gamma': (S)\theta_S \to (T)\theta_T$ given by

$$(a_u, \phi_u)\gamma' = (\beta^{-1}a_u\alpha, \beta^{-1}\phi_u\beta).$$

Define $\eta: S \to T$ by $u\eta\theta_T = u\theta_S\gamma'$. Then

$$\begin{aligned}(x, i, j)\eta_1(u\eta\theta_T) &= (x\alpha(j\beta\beta^{-1}a_u\alpha), i\beta, j\beta\beta^{-1}\phi_u\beta) = ((x(ja_u))\alpha, i\beta, j\phi_u\beta)\\ &= (x, i, j)u\theta_S\eta_1.\end{aligned}$$

So $u\eta\theta_T = \eta_1^{-1}(u\theta_S)\eta_1$ and clearly η is an isomorphism.

THEOREM 4.5. *Each finite inverse Δ-semigroup S with principal series $\{0\} \subset S_1 \subset S$ has a structure data set $\{n, r, m, G, \bar{H}\}$. A semigroup is isomorphic to S if and only if its structure data sets are equivalent to $\{n, r, m, G, \bar{H}\}$. Conversely, each structure data set $\{n, r, m, G, \bar{H}\}$ is a structure data set of some finite inverse Δ-semigroup T with principal series $\{0\} \subset T_1 \subset T$.*

Proof. The first two statements have been proved. Suppose $\{n, r, m, G, \bar{H}\}$ is a structure data set. Let $T_1 = (G \times Z_n \times Z_n) \cup \{0\}$ with binary operation as defined after Example 2. Then $T_1 \cong B(G, Z_n)$. Let $T = T_1 \cup \bar{H}\backslash(0, 0)$. For $(a, \phi), (b, \psi) \in \bar{H}\backslash(0, 0)$ and $(x, i, j), (y, h, k) \in T_1$ define a binary operation on T so that:

$$(a, \phi)(b, \psi) = \begin{cases} (a * b, \phi\psi) \in \bar{H} & \text{if } D(\phi\psi) = D(\phi) \\ (la(l\phi a), l, l\phi\psi) \in T_1 & \text{if } D(\phi\psi) = \{l\} \\ 0 & \text{if } D(\phi\psi) = \square, \end{cases}$$

$$(x, i, j)(a, \phi) = \begin{cases} (x(ja), i, j\phi) \in T_1 & \text{if } j \in D(\phi) \\ 0 & \text{if } j \notin D(\phi), \end{cases}$$

$$(a, \phi)(x, i, j) = \begin{cases} (i\phi^{-1}ax, i\phi^{-1}, j) \in T_1 & \text{if } i \in R(\phi) \\ 0 & \text{if } i \notin R(\phi), \end{cases}$$

$$(x, i, j)(y, h, k) = \begin{cases} (xy, i, k) \in T_1 & \text{if} \quad j = h \\ 0 & \text{if} \quad j \neq h. \end{cases}$$

Since $\bar{H} \cong B(H, Z_r)$ it can be routinely checked that T is an inverse semigroup. It can also be checked, using Lemmas 3.3 and 3.4 or 3.5, that S is a Δ-semigroup. Since $(x, i, j)(a, \phi) = (x(ja), i, j\phi)$ for $(x, i, j) \in T_1$, $(a, \phi) \in T/T_1$ and $j \in D(\phi)$ we see that $\{n, r, m, G, \bar{H}\}$ is a structure data set of T.

Let S be a finite inverse Δ-semigroup with principal series $S_0 \subset S_1 \subset \cdots \subset S_q = S$ where $q > 1$. We can uniquely determine, up to equivalence, the structure data sets of the semigroups S_i/S_{i-2} for $i = 2, \cdots, q$. Conversely, let $\{\{n_i, r_i, m_i, G_i, \bar{H}_i\}; i = 2, \cdots, q\}$ be a family of structure data sets so that $n_j = r_{j-1}$ and $G_j \cong H_{j-1}$ for $j = 3, \cdots, q$, where H_{j-1} is the basic group of $\bar{H}_{j-1}$. Then, by Theorem 3.6 and the proof of Theorem 4.5, we can construct a finite inverse Δ-semigroup T with principal series $T_0 \subset T_1 \subset \cdots \subset T_q = T$ so that $\{n_i, r_i, m_i, G_i, \bar{H}_i\}$ is a structure data set of T_i/T_{i-2}. Any finite inverse Δ-semigroup that is not a group or a Brandt semigroup can be so constructed.

The authors are grateful for the referee's kind suggestions on this paper.

REFERENCES

1. A. H. Clifford, and G. B. Preston, *The algebraic theory of semigroups*, Math. Surveys No. 7, Amer. Math. Soc., Providence, Vol. I (1961), Vol. II (1967).

2. J. M. Howie, and G. Lallement, *Certain fundamental congruences on a regular semigroup*, Proc. Glasgow Math. Assoc., **7** (1966), 145–156.

3. M. Hall Jr., *The Theory of Groups*, Macmillan, New York (1959).

4. W. D. Munn, *Fundamental inverse semigroups*, Quart. J. Math. Oxford (2), **21** (1970), 157–170.

5. M. Petrich, *The maximal semilattice decomposition of a sèmigroup*, Math. Zeitschr., **85** (1964), 68–82.

6. ———, *Introduction to semigroups*, Merrill, Columbus (1973).

7. G. B. Preston, *Congruences on Brandt semigroups*, Math. Ann., **139** (1959), 91–94.

8. B. M. Schein, *Commutative semigroups whose congruences form a chain*, Polonaise des Sciences, **9** (1969), 523–527.

9. T. Tamura, *The theory of construction of finite semigroups I*, Osaka Math. J., **8** (1956), 243–261.

10. ———, *Decomposition of a completely simple semigroup*, Osaka Math. J., **12** (1960), 269–275.

11. ———, *Commutative semigroups whose lattice of congruences form a chain*, Bull. Soc. Math. France, **97** (1969), 369–380.

12. M. Teissier, *Sur les équivalences regulières dans les demi-groupes*, C. R. Acad. Sci., Paris, **232** (1951), 1987–1989.

Received November 7, 1975 and in revised form September 13, 1976.

THE UNIVERSITY OF TASMANIA, HOBART, AUSTRALIA
AND
THE UNIVERSITY OF CALIFORNIA,
DAVIS, CA 95616