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A CHARACTERIZATION OF THE GAUSSIAN DISTRIBUTION IN A HILBERT SPACE

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In this paper we consider the case in which random variables X_j take values in a real, separable Hilbert space $\mathscr{H}$. We look at a linear form $\Sigma A_j X_j$ where each A_j is a bounded linear operator in $\mathscr{H}$. We then assume that this linear form is identically distributed with a monomial and form conditions under which it is possible to deduce that the common distribution of the random variables is the Gaussian distribution.

The study of identically distributed linear forms of independent and identically distributed random variables has been undertaken by several authors. J. Marcinkiewicz studied linear forms in which all moments of the random variables are assumed to exist. He then proved that the common distribution of the random variables was the Normal distribution. R. G. Laha and E. Lukacs have considered the case where one of the linear forms is a monomial. They have obtained characterizations of the Normal distribution for both the case when the variance is assumed finite and when no assumption is made concerning the variance.

1. Statement of the main result. Suppose now that $X_1, X_2, \cdots$ is a sequence (possibly finite) of independent, identically distributed, nondegenerate $\mathscr{H}$-valued random variables, where X_1 has a finite variance (i.e. $\operatorname{Var} X_1 < +\infty$). Let $A_1, A_2, \cdots$ be a sequence of 1–1 bounded linear operators in $\mathscr{H}$, with the following two properties:

$$\sum_j \|A_j\|^2 < +\infty \quad \text{and} \quad \sum_j A_j^* A_j \geqq I \tag{1}$$

and

$$\sup_j \|A_j\| < 1. \tag{2}$$

We note that in the above A_j^* represents the adjoint of A_j and that the inequality $\Sigma_j A_j^* A_j \geqq I$ is true in the sense of positive-definiteness. (For example, see page 313 of [7].)

Our goal is to prove the following theorem.

THEOREM 1. *Suppose that* $\Sigma_j A_j X_j$ *converges with probability one. If* $\Sigma_j A_j X_j$ *has the same distribution as* X_1, *then* X_1 *has a Gaussian distribution.*

In §2 we will prove an important preliminary result (Theorem 2). Then in §3 we will present the proof of Theorem 1.

2. A preliminary result. In this section we will prove the following result.

THEOREM 2. *Let* $X_1, X_2, \cdots$ *be a sequence* (possibly finite) *of independent, identically distributed, nondegenerate,* $\mathscr{H}$*-valued random variables. Suppose that the sum* $\Sigma_j A_j X_j$ *exists with probability one, where* $A_1, A_2, \cdots$ *are bounded linear operators in* $\mathscr{H}$, *with* $\sup_j \|A_j\| < 1$.

If $\Sigma_j A_j X_j$ has the same distribution as X_1, then X_1 has an infinitely divisible distribution.

Note. The hypotheses of Theorem 2 are somewhat weaker than the hypotheses of Theorem 1.

Before beginning the proof of Theorem 2, let us fix some notation.

Let $\varphi(y)$ be the common characteristic functional of $X_1, X_2, \cdots$. Then $\varphi(y) = \mathscr{E} e^{i\langle X_1, y\rangle}$ for all $y \in \mathscr{H}$, where $\mathscr{E}$ denotes mathematical expectation.

The characteristic functional of $A_j X_j$ is then given by:

$$\mathscr{E} e^{i\langle A_j X_j, y\rangle} = \mathscr{E} e^{i\langle X_j, A_j^* y\rangle} = \varphi(A_j^* y) \tag{3}$$

where A_j^* denotes the adjoint operator of A_j.

Now, suppose that $\Sigma_j A_j X_j$ has the same distribution as X_1. Then equation (3) gives us:

$$\varphi(y) = \prod_j \varphi(A_j^* y), \qquad \text{for all} \qquad y \in \mathscr{H}, \tag{4}$$

where the product converges uniformly on bounded spheres. (See Theorem 4.4, pg. 171 of **[5]**.)

Since $\Sigma_j A_j X_j$ converges, then $\Sigma_{j=n}^{\infty} A_j X_j$ converges, with probability one, to the origin of $\mathscr{H}$ as $n \to \infty$. (Of course, if $X_1, X_2, \cdots$ is a finite sequence, the preceding statement is unnecessary.)

Thus, it is possible to choose N_0 for any $\epsilon > 0$, such that $P\{\|\Sigma_{j=N+1}^{\infty} A_j X_j\| > \epsilon\} < \epsilon$, whenever $N \geqq N_0$. Let $\varphi_N(y)$ denote the characteristic functional of $\Sigma_{j=N+1}^{\infty} A_j X_j$. Then using equation (4), we have:

$$\varphi(y) = \varphi(A_1^* y) \cdots \varphi(A_N^* y)\varphi_N(y). \tag{5}$$

Proof of Theorem 2. We assume that $\Sigma_j A_j X_j$ has the same distribution as X_1. Then equation (5) holds. If we replace y by $A_j^* y$ in equation (5), we obtain:

$$\varphi(A_j^* y) = \varphi(A_1^* A_j^* y) \cdots \varphi(A_N^* A_j^* y)\varphi_N(A_j^* y) \tag{6}$$

for each $j = 1, 2, \cdots, N$.

Combining equations (5) and (6) we have:

$$\varphi(y) = \prod_{j=1}^{N} \varphi((A_j^*)^2 y) \cdot \prod_{j \neq k} \varphi(A_j^* A_k^* y) \prod_{j=1}^{N} \varphi_N(A_j^* y)\varphi_N(y).$$

If we repeat the above process n times, we get the following result:

$$\varphi(y) = \Pi\varphi(A_{j_1}^* \cdots A_{j_n}^* y) \prod_{k=1}^{n-1} \Pi\varphi_N(A_{j_1}^* \cdots A_{j_{n-k}}^*)\varphi_N(y). \tag{7}$$

The product on the right hand side of equation (7) consists of $N^n + N^{n-1} + \cdots + N + 1$ factors, where each of the subscripts $j_1, \cdots, j_n$ can take any of the values $1, \cdots, N$ with repetitions allowed.

Thus, equation (7) says that X_1 is distributed as the sum of $k_n = \Sigma_{k=0}^{n} N^k$ independent, $\mathscr{H}$-valued random variables, $Y_{n,k}$ ($k = 1, 2, \cdots, k_n$), for any positive integer n.

We will now show that $Y_{n,k}$ is a *uniformly infinitessimal* collection of random variables. That is, we will show that for any $\epsilon > 0$, $\sup_{1 \leqq k \leqq k_n} P\{\| Y_{nk} \| > \epsilon\} \to 0$ as $n \to \infty$. Once this has been established, the infinite divisibility of X_1 will follow from Corollary 6.2, page 199 of **[5]**.

Consider the factors on the right hand side of equation (7). Let $\epsilon > 0$ be given. By definition $\varphi_N(y)$ is the characteristic functional of $\Sigma_{j=N+1}^{\infty} A_j X_j$ and $P\{\| \Sigma_{j=N+1}^{\infty} A_j X_j \| > \epsilon\} < \epsilon$, for all $N \geqq N_0$.

Consider now a factor of the form $\varphi_N(A_{j_1}^* A_{j_2}^* \cdots A_{j_{n-k}}^* y)$. This is the characteristic functional of

$$A_{j_{n-k}} \cdots A_{j_2} A_{j_1} \sum_{j=N+1}^{\infty} A_j X_j.$$

Also,

$$P\left\{\left\| A_{j_{n-k}} \cdots A_{j_2} A_{j_1} \sum_{j=N+1}^{\infty} A_j X_j \right\| > \epsilon\right\}$$

$$\leqq P\left\{\| A_{j_{n-k}} \| \cdots \| A_{j_2} \| \cdot \| A_{j_1} \| \cdot \left\| \sum_{j=N+1}^{\infty} A_j X_j \right\| > \epsilon\right\}$$

$$\leqq P\left\{\left\| \sum_{j=N+1}^{\infty} A_j X_j \right\| > \epsilon\right\} < \epsilon \quad \text{whenever} \quad N \geqq N_0,$$

since $\sup_j \| A_j \| < 1$.

Finally, we consider a factor of the form $\varphi(A^*_{j_1}\cdots A^*_{j_n}y)$, which is the characteristic functional of $A_{j_n}\cdots A_{j_1}X_1$. Set $\alpha = \sup_j \|A_j\|$. Then

$$\begin{aligned} P\{\|A_{j_n}\cdots A_{j_1}X_1\| > \epsilon\} \\ &\leqq P\{\|A_{j_n}\|\cdots\|A_{j_1}\|\cdot\|X_1\| > \epsilon\} \\ &\leqq P\left\{\|X_1\| > \frac{\epsilon}{\alpha^n}\right\}. \end{aligned}$$

Now choose an integer N' such that $P\{\|X_1\| > \epsilon/\alpha^n\} < \epsilon$, whenever $n \geqq N'$. (This is possible because $0 < \alpha < 1$). Set $n_0 = \max\{N_0, N'\}$.

Hence, we have shown that $P\{\|Y_{nk}\| > \epsilon\} < \epsilon$, for all $k = 1, 2, \cdots, k_n$, whenever $n \geqq n_0$. Therefore, the collection Y_{nk} is uniformly infinitesimal and X_1 is infinitely divisible. This completes the proof of the theorem.

3. Proof of the main result. For convenience, we now will make the assumption that $X_1, X_2, \cdots$ are symmetric random variables. Since the common distribution of these random variables is infinitely divisible, the common characteristic functional, $\varphi(y)$, has a unique Levy–Khintchine representation given by:

$$\ln \varphi(y) = -\tfrac{1}{2}\langle Sy, y\rangle + \int (\cos\langle x, y\rangle - 1)\,dL(x) \tag{8}$$

where S is an S-operator (a nonnegative, self-adjoint compact operator on $\mathscr{H}$, with a finite trace), and L is a σ-finite measure with finite mass outside every neighborhood of the origin and with the property that

$$\int_{\|x\|\leqq 1} \|x\|^2 dL(x) < +\infty.$$

(see **[5]**, page 181.)

Furthermore, since $X_1, X_2, \cdots$ have finite variance, $\varphi(y)$ has a unique Kolmogorov representation, given by:

$$\ln \varphi(y) = -\tfrac{1}{2}\langle Sy, y\rangle + \int_{\mathscr{H}\setminus\{0\}} \frac{\cos\langle x, y\rangle - 1}{\|x\|^2}\,dK(x) \tag{9}$$

where S is an S-operator and K is a finite measure on $\mathscr{H}$. (See **[6]**.)

By equations (4) and (8) we have:

$$\begin{aligned} -\frac{1}{2}\sum_j \langle SA^*_j y, A^*_j y\rangle + \sum_j \int (\cos\langle x, A^*_j y\rangle - 1)\,dL(x) \\ = -\tfrac{1}{2}\langle Sy, y\rangle + \int (\cos\langle x, y\rangle - 1)\,dL(x). \end{aligned}$$

Also,

$$
\begin{aligned}
&-\frac{1}{2}\sum_j \langle SA_j^* y, A_j^* y\rangle + \sum_j \int (\cos\langle x, A_j^* y\rangle - 1)\, dL(x) \\
(10)\qquad &= -\frac{1}{2}\Big\langle \sum_j A_j S A_j^* y, y\Big\rangle + \sum_j \int (\cos\langle A_j x, y\rangle - 1)\, dL(x) \\
&= -\frac{1}{2}\Big\langle \sum_j A_j S A_j^* y, y\Big\rangle + \sum_j \int (\cos\langle x, y\rangle - 1)\, dLA_j^{-1}(x).
\end{aligned}
$$

It is not difficult to show that $\Sigma_j A_j S A_j^*$ is an S-operator. Also, it is clear that LA_j^{-1} is the σ-finite measure which occurs in the Levy–Khintchine representation of $A_j X_j$, for each j.

We denote by $\mathcal{B}$, the class of Borel sets in $\mathcal{H}$. Then the measure K_j, defined by:

$$K_j(D) = \int_D \|x\|^2 dLA_j^{-1}(x), \quad \text{for all} \quad D \in \mathcal{B}$$

is the finite measure which occurs in the Kolmogorov representation of $A_j X_j$, for each j.

Since $X_1, X_2, \cdots$ have finite variance,

$$\int \|x\|^2 dL(x) < +\infty. \quad \text{(See [6].)} \tag{11}$$

By equation (10), $\ln \Pi_j \varphi(A_j^* y)$

$$= -\frac{1}{2}\Big\langle \sum_j A_j S A_j^* y, y\Big\rangle + \sum_j \int_{\{x \neq 0\}} \frac{(\cos\langle x, y\rangle - 1)}{\|x\|^2} dK_j(x). \tag{12}$$

We note that

$$
\begin{aligned}
\sum_j \int_{\{x\neq 0\}} \left| \frac{\cos\langle x, y\rangle - 1}{\|x\|^2} \right| dK_j(x) &\leqq \sum_j \int_{\{x\neq 0\}} \frac{\|x\|^2 \|y\|^2}{\|x\|^2} dK_j(x) \\
= \|y\|^2 \sum_j \int_{\{x\neq 0\}} dK_j(x) &= \|y\|^2 \sum_j \int \|x\|^2 dLA_j^{-1}(x) \\
= \|y\|^2 \sum_j \int \|A_j x\|^2 dL(x) &\leqq \Big(\sum_j \|A_j\|^2\Big) \|y\|^2 \int \|x\|^2 dL(x) < \infty
\end{aligned}
$$

because of relations (1) and (11).

Thus we may interchange the integral and summation signs in equation (12) to obtain:

$$\ln \prod_j \varphi(A_j^* y) = -\frac{1}{2}\left\langle \sum_j A_j S A_j^* y, y \right\rangle + \int_{\{x \neq 0\}} \frac{\cos\langle x, y\rangle - 1}{\|x\|^2} d\left(\sum_j K_j(x)\right).$$

Then, by the uniqueness of the Kolmogorov representation, we have:

$$\sum_j A_j S A_j^* = S \quad \text{and} \quad \sum_j K_j = K.$$

From the second of these relations, $\Sigma_j K_j(\mathscr{H}) = K(\mathscr{H})$, which leads to the following sequence of equations.

$$\sum_j \int \|x\|^2 dLA_j^{-1}(x) = \int \|x\|^2 dL(x)$$

$$\sum_j \int \|A_j x\|^2 dL(x) = \int \|x\|^2 dL(x)$$

$$\int \left[\sum_j \|A_j x\|^2 - \|x\|^2\right] dL(x) = 0$$

$$\int \left[\left\langle \sum_j A_j^* A_j x, x \right\rangle - \langle x, x\rangle\right] dL(x) = 0.$$

In view of relation (4), it must then be true that

$$L\left\{x : \sum_j \|A_j x\|^2 - \|x\|^2 > 0\right\} = 0. \tag{13}$$

We note that for n a positive integer, $\Sigma_{j=1}^n A_j X_j$ has characteristic functional $\Pi_{j=1}^n \varphi(A_j^* y)$, and

$$\ln \prod_{j=1}^n \varphi(A_j^* y) = -\frac{1}{2}\left\langle \sum_{j=1}^n A_j S A_j^* y, y \right\rangle + \int (\cos\langle x, y\rangle - 1) d\left(\sum_{j=1}^n LA_j^{-1}(x)\right). \tag{14}$$

Thus $\Sigma_{j=1}^n LA_j^{-1}$ converges weakly, outside closed neighborhoods of $0 \in \mathscr{H}$, to L, as $n \to \infty$. (See **[5]**, page 189).

It now becomes necessary to state and prove two technical lemmas.

LEMMA 1. *For any* $\epsilon > 0$,

$$\int_{\|x\|>\epsilon} \|x\|^2 dL(x) = \sum_j \int_{\|A_j x\|>\epsilon} \|A_j x\|^2 dL(x).$$

Proof. Let ϵ_1 and ϵ_2 be positive constants with $\epsilon_1 < \epsilon_2$. Define a function $f(x)$ by:

$$f(x) = \begin{cases} \|x\|^2, & \epsilon_1 < \|x\| \leqq \epsilon_2 \\ (\epsilon_2)^2, & \|x\| > \epsilon_2. \end{cases}$$

Then $f(x)$ is bounded and continuous. Thus by comment (14),

$$\int_{\|x\|>\epsilon_1} f(x)dL(x) = \lim_{n\to\infty} \sum_{j=1}^{n} \int_{\|x\|>\epsilon_1} f(x)dLA_j^{-1}(x)$$

which implies that

$$\sum_{j=1}^{n} \int_{\epsilon_1<\|x\|\leqq\epsilon_2} \|x\|^2\, dLA_j^{-1}(x) + (\epsilon_2)^2 \sum_{j=1}^{n} LA_j^{-1}\{\|x\| > \epsilon_2\}$$

converges to

$$\int_{\epsilon_1<\|x\|\leqq\epsilon_2} \|x\|^2 dL(x) + (\epsilon_2)^2 L\{\|x\| > \epsilon_2\}$$

as $n \to \infty$.

But, again because of comment (14), $\Sigma_{j=1}^{n} LA_j^{-1}\{\|x\| > \epsilon_2\}$ converges to $L\{\|x\| > \epsilon_2\}$ as $n \to \infty$.

Therefore, $\Sigma_{j=1}^{n} \int_{\epsilon_1<\|x\|\leqq\epsilon_2} \|x\|^2 dLA_j^{-1}(x)$ converges to

$$\int_{\epsilon_1<\|x\|\leqq\epsilon_2} \|x\|^2 dL(x),$$

whenever we choose $0 < \epsilon_1 < \epsilon_2$.

Let $\epsilon > 0$ be given. Let ϵ_n be a strictly increasing sequence of positive numbers, $\epsilon_n \uparrow +\infty$, with $\epsilon_1 > \epsilon$. For convenience we set $\epsilon = \epsilon_0$.

Then

$$\begin{aligned} \int_{\|x\|>\epsilon_0} \|x\|^2 dL(x) &= \sum_{k=0}^{\infty} \int_{\epsilon_k<\|x\|\leqq\epsilon_{k+1}} \|x\|^2 dL(x) \\ &= \sum_{k=0}^{\infty} \sum_{j} \int_{\epsilon_k<\|x\|<\epsilon_{k+1}} \|x\|^2 dLA_j^{-1}(x) \\ &= \sum_{j} \sum_{k=0}^{\infty} \int_{\epsilon_k<\|x\|\leqq\epsilon_{k+1}} \|x\|^2 dLA_j^{-1}(x) \\ &= \sum_{j} \int_{\|x\|>\epsilon} \|x\|^2 dLA_j^{-1}(x) = \sum_{j} \int_{\|A_jx\|>\epsilon} \|A_jx\|^2 dL(x). \end{aligned}$$

This completes the proof.

LEMMA 2. *$L(\{x: \|A_kx\|^2 \geqq \|x\|^2$, for some $k = 1, 2, \cdots\}) = L(\{0\})$.*

Proof. Let k be a fixed positive integer.

Set $E_k = \{x : \| A_k x \|^2 = \| x \|^2\}$. Then, using equation (13), $L(E_k) = L(E_k \cap \{x : \Sigma_j \| A_j x \|^2 = \| x \|^2\})$.

Thus, $L(E_k) = L\{x : \Sigma_{j \neq k} \| A_j x \|^2 = 0\} = L(\{0\})$, since each operator A_j is 1–1.

$$\text{Similarly,} \quad L\Big(\bigcup_k E_k\Big) = L(\{0\}). \tag{15}$$

Using the same type of argument, it is easy to show that for all $k = 1, 2, \cdots$

$$L\{x : \| A_k x \|^2 > \| x \|^2\} = 0. \tag{16}$$

Combining equations (15) and (16) we are done.

From relations (1) and (13), we see that $L\{x : \| x \|^2 \neq \Sigma_j \| A_j x \|^2\} = 0$. Hence, referring to Lemma 1, it is true that, for all $\epsilon > 0$,

$$\sum_j \int_{\|x\|>\epsilon} \| A_j x \|^2 dL(x) = \sum_j \int_{\|A_j x\|>\epsilon} \| A_j x \|^2 dL(x),$$

and this implies that

$$\sum_j \left[\int_{\|x\|>\epsilon} \| A_j x \|^2 \, dL(x) - \int_{\|A_j x\|>\epsilon} \| A_j x \|^2 \, dL(x) \right] = 0, \text{ for all } \epsilon > 0. \tag{17}$$

But

$$L\{x : x \neq 0 \quad \text{and} \quad \| A_j x \| \geqq \| x \|\} = 0, \quad \text{for all} \quad j. \tag{18}$$

Thus, each term in the sum of equation (17) must be nonnegative, which yields:

$$\int_{\|x\|>\epsilon} \| A_j x \|^2 dL(x) = \int_{\|A_j x\|>\epsilon} \| A_j x \|^2 dL(x), \quad \text{for all} \quad \epsilon > 0 \quad \text{and all} \quad j.$$

Or, using equation (18),

$$\int_{\{\|x\|>\epsilon\} \cap F_j} \| A_j x \|^2 dL(x) = \int_{\{\|A_j x\|>\epsilon\} \cap F_j} \| A_j x \|^2 dL(x),$$

for all $\epsilon > 0$ and all j, where $F_j = \{x : \| A_j x \| < \| x \|\}$, for each $j = 1, 2, \cdots$.

The above implies that

$$\int_{\{\|x\|>\epsilon\}\cap\{\|A_jx\|\leqq\epsilon\}\cap F_j} \|A_jx\|^2 dL(x) = 0, \quad \text{for all } \epsilon > 0 \text{ and all } j,$$

or,

$$\int_{\{\|x\|>\epsilon \text{ and } \|A_jx\|\leqq\epsilon\}} \|A_jx\|^2 dL(x) = 0, \quad \text{for all } \epsilon > 0 \text{ and all } j.$$

So, we must have that

$$(19) \quad L\{x: \|x\| > \epsilon \quad \text{and} \quad \|A_jx\| \leqq \epsilon\} = 0, \quad \text{for all } \epsilon > 0 \text{ and all } j.$$

Consider the set Q^+ of positive rational numbers. Let k be a fixed positive integer.

$$L\left[\bigcup_{r\in Q^+} \{x: \|x\| > r \quad \text{and} \quad \|A_kx\| \leqq r\}\right] = L[\{x: \|A_kx\| < \|x\|\} = L[\mathscr{H}\backslash\{0\}].$$

Therefore, $L[\mathscr{H}\backslash\{0\}] \leqq \Sigma_{r\in Q^+} L\{x: \|x\| > r \text{ and } \|A_kx\| \leqq r\} = 0$, by equation (19).

This last relation says that L is degenerate at $0 \in \mathscr{H}$, which means that the common characteristic functional of $X_1, X_2, \cdots$ is given by:

$$\ln \varphi(y) = -\tfrac{1}{2}\langle Sy, y\rangle \quad (\text{see Eq. (8)}).$$

Hence $X_1, X_2, \cdots$ have a common Gaussian distribution.

Recall that we have assumed $X_1, X_2, \cdots$ to be symmetric, but it is now easy to extend our result to the general case by using Cramer's Theorem (see page 141 of **[1]**).

The proof of Theorem 1 is now completed.

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