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ON THE DISTRIBUTION OF a-POINTS OF A STRONGLY ANNULAR FUNCTION

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This paper gives an example of a strongly annular function which omits 0 near an arc I on the unit circle C and which omits 1 near the complementary arc $C-I$. This example affirmatively answers the following question of Bonar: Does there exist any annular function for which we can find two or more complex numbers w such that the limiting set of its w-points does not cover C?

1. Introduction. The purpose of this paper is to study the distribution of a-points of annular functions. We recall that a holomorphic function in the open unit disk $D:|z|<1$ is said to be annular [1] if there is a sequence $\{J_n\}$ of closed Jordan curves about the origin in D, converging out to the unit circle $C:|z|=1$, such that the minimum modulus of $f(z)$ on J_n increases to infinity as n increases. When the J_n can be taken as circles concentric with C, $f(z)$ will be called strongly annular. Given a finite complex number a, the minimum modulus principle guarantees that every annular function f has infinitely many a-points in D and hence their limit points form a nonempty closed subset, say $Z'(f,a)$, of C. On the other hand, by virtue of the Koebe–Gross theorem concerning meromorphic functions omitting three points, it follows from the annularity of f that open sets $C-Z'(f,a)$ and $C-Z'(f,b)$ on the circle can not overlap if $a\neq b$ and consequently that the set of all values a for which $Z'(f,a)\neq C$ must be at most countable. Therefore we may well say such a to be singular for f.

For this reason we will be concerned with the set $S(f)=\{a:Z'(f,a)\neq C\}$ in this paper. We denote by $|S(f)|$ the cardinality of $S(f)$ and then, from the simple fact observed above, we have that $0\leqq|S(f)|\leqq\aleph_0$, which in turn conversely tempt us to raise the following question: Given a cardinality $N(0\leqq N\leqq\aleph_0)$, can we find any annular function f for which $|S(f)|=N$? ([1], [2]).

We know many examples of strongly annular functions such that $|S(f)|=0$ [4]. In particular if an annular function f belongs to the MacLane class, i.e., the family of all nonconstant holomorphic functions in D which have asymptotic values at each point of everywhere dense subsets of C, the set $S(f)$ becomes necessarily empty. As for $N=1$, Barth and Schneider [3] constructed an example of an annular function f for which $|S(f)|=1$. The example involved in their construction, however, did not appear to be strongly annular. An example of a strongly annular f with $|S(f)| = 1$ was constructed independently by Barth, Bonar and Carroll **[2]** and the author **[5]**. The aim of this paper is to give an example of a strongly annular function f for which $|S(f)| = 2$.

2. For this purpose we consider a class of functions holomorphic in D. Let I_0 and I_1 be a pair of complementary open arcs on the unit circle C and choose a Jordan arc J_j connecting the end points of I_j, which is contained, except for its end points, in the open sector

$$\{z : 0 < |z| < 1,\ z/|z| \in I_j\} \qquad (j = 0, 1).$$

Further denote by G_j the Jordan domain surrounded by I_j and J_j and consider

$$S(G_0, G_1) = \{g \in H(D) : g \text{ is bounded away from 0 (or 1) in } G_0 \text{ (or } G_1)\}$$

where $H(D)$ denotes the set of all functions holomorphic in D. In terms of this notation our purpose is in amount to find a strongly annular function which is locally a uniform limit of a sequence in $S(G_0, G_1)$. To construct such a function, we make essential use of the approximation theorem of Runge, which asserts that if K is a compact set with connected complement relative to the plane and a function g is holomorphic in an open set containing K, for any $\rho > 0$, there is a polynomial P such that

$$|P(z) - g(z)| < \rho \qquad (z \in K).$$

We call such P an approximating polynomial with respect to the triple (K, g, ρ). In our arguments to follow we may restrict ourselves to the special pair of G_0 and G_1 such that

$$G_0 = \{z = x + iy : |z| < 1,\ 2x + |y| > 1\} \quad \text{and} \quad G_1 = \{z : -z \in G_0\}$$

with no loss of generality, which serves to simplify the geometric formulation. Then the Runge theorem, in cooperation with our previous lemma, yields the following:

LEMMA. *Let there be given positive numbers ϵ and k, numbers a and b with $0 < a < b < 1$, and a function f in $S(G_0, G_1)$ (simply S), which is bounded in G_1. Then there exists a function g in S, which is also bounded in G_1, such that*

$$(1) \qquad |g(z)| > k \qquad (|z| = b)$$

and

(2) $$|g(z)-f(z)|<\epsilon \qquad (|z|\leqq a).$$

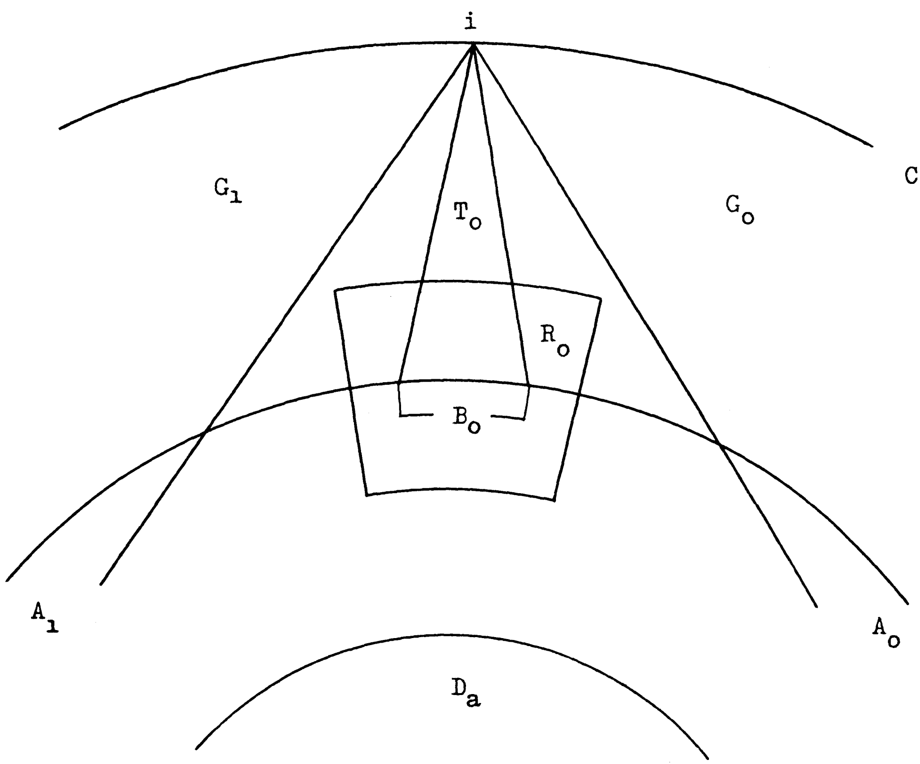
Proof. We first divide the circle $|z|=b$ into 4 closed arcs as follows:

$$A_0=[-bie^{it},\ bie^{-it}], \qquad A_1=\{z:-z\in A_0\}$$
$$B_0=[bie^{-it},\ bie^{it}], \qquad B_1=\{z:\bar{z}\in B_0\}.$$

Here $t(>0)$ should be chosen so small that we may apply our lemma **[5]** to an appropriately small open annular sector R_0, which is contained in

$$\{z=x+iy:y>0,\ |z|>a,\ 2|x|+|y|<1\}$$

and contains the arc B_0. Set $R_1=\{z:\bar{z}\in R_0\}$.



Next, to make use of the Runge theorem, we prepare two triples, which are defined, except for c_j and ρ_j, by the following:

$$(3)\quad \begin{cases} K_j = \bar{G}_j \cup A_j \cup A_{1-j} \cup \bar{D}_a,\ \bar{D}_a = \{z : |z| \leqq a\} \\ g_j(z) = 0 \qquad (z \in \bar{G}_j \cup A_j \cup \bar{D}_a) \qquad (j = 0, 1). \\ g_j(z) = c_j (>0) \qquad (z \in A_{1-j}) \end{cases}$$

As for c_j (or ρ_j) we shall later choose positive numbers large (or small) enough to satisfy our requirements. Obviously these definitions allow us to apply the Runge theorem to (K_j, g_j, ρ_j) $(j = 0, 1)$ and hence we can find an approximating polynomial P_j. On the other hand, if necessary, adding a small vector we may assume that $f(z) \neq 0, 1$ on the circle $|z| = b$. Combining these functions, define a function F holomorphic in D by

$$F(z) = \{(f(z) - 1)\exp(P_0(z)) + 1\}\exp(P_1(z)).$$

Then carefully observing (3) and suitably choosing values of c_j and ρ_j, we can conclude that the function F is a member of S, bounded in G_1 and has the following properties:

$$(4)\qquad |F(z)| > 2k \qquad (z \in \{z : |z| = b\} - B_0 - B_1)$$

$$(5)\qquad |F(z) - f(z)| < \epsilon/2 \qquad (z \in \bar{D}_a).$$

In addition it may be supposed that F does not vanish on $B_0 \cup B_1$.

Thus the last step in our construction of g is to make $|F(z)|$ large on the remaining arcs B_0 and B_1 without losing the properties described above of F. Given $c_2 > 0$ and $\rho_2 > 0$, applying our lemma **[5]** to the annular sectors R_0 and R_1 previously chosen, and successively using the standard "pole sweeping" method for the resulting rational functions, we can find a holomorphic function H_j in D such that

$$(6)\qquad |H_j(z)| > c_2 \qquad (z \in B_j),$$

$$(7)\qquad \operatorname{Re} H_j(z) > -\rho_2 \qquad (z \in R_j \cap \{z : |z| = b\} - B_j)$$

and

$$(8)\qquad |H_j(z)| < 2\rho_2 \qquad (z \in D - T_j)$$

where T_0 (or T_1) denotes an appropriate "pole sweeping route" ending at $z = i$ (or $-i$) which is contained in

$$E_0 = \{z = x + iy : y > 0,\ |z| > b,\ 2|x| + |y| < 1\}$$

(or $E_1 = \{z : \bar{z} \in E_0\}$) (see Figure 1). Using these functions and F defined above, set

$$F(z)\{1 + H_0(z)\}\{1 + H_1(z)\} = g(z).$$

Since F does not vanish on $B_0 \cup B_1$, if we appropriately choose a large (or small) positive number as a value of c_2 (or ρ_2), by virtue of (4) and (5) together with (6), (7) and (8), we can show that the function g belongs to the class S, is bounded in G_1 and further satisfies (1) and (2). This proves Lemma.

3. The following result is immediate from Lemma in 2.

THEOREM. *Let $\{r_n\}$ and $\{k_n\}$ be two sequences of positive numbers with $r_n \uparrow 1$ and $1 < k_n \uparrow +\infty$. Then there exists a function f, which is locally a uniform limit of a sequence in S and which furthermore satisfies that $|f(z)| \geqq k_n$ on the circle $|z| = r_n$.*

Proof. It is sufficient to construct a sequence $\{f_n(z)\}$ in S such that

(9) $$|f_n(z)| > k_j \quad \text{if} \quad 1 \leqq j \leqq n \qquad (z \in C_j = \{z : |z| = r_j\}),$$

(10) $$|f_n(z) - f_{n-1}(z)| < \epsilon_{n-1} \qquad (|z| \leqq r_{n-1},\ n \geqq 2)$$

and

(11) f_n is bounded in G_1

where $\{\epsilon_n\}$ is a preassigned sequence of positive numbers with $\Sigma\epsilon_n < +\infty$. In order to construct $\{f_n\}$ inductively, let $f_1(z) = 2k_1$ and suppose that $f_1, \cdots, f_{n-1}$ have already been defined. In Lemma in 2, on setting $f = f_{n-1}$, $a = r_{n-1}$, $b = r_n$, $k = k_n$ and $\epsilon = \min\{\epsilon_{n-1}, m_1, \cdots, m_{n-1}\}$ where $m_j = \min\{|f_{n-1}(z)| - k_j : z \in C_j\}$, we can find a function f_n in S satisfying (9), (10) and (11). Thus we obtain a sequence $\{f_n\}$ in S, which, by virtue of (10), converges uniformly on any compact subset of D. Obviously its limit f is a desired function in Theorem. Hence our proof is complete.

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