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A COMMUTATIVITY THEOREM FOR NON-ASSOCIATIVE ALGEBRAS OVER A PRINCIPAL IDEAL DOMAIN

JIANG LUH AND MOHAN S. PUTCHA

Let A be an algebra (not necessarily associative) over a principal ideal domain R such that for all a, $b \in A$, there exist α, $\beta \in R$ such that $(\alpha, \beta) = 1$ and $\alpha ab = \beta ba$. It is shown that A is commutative.

Throughout this paper N will denote the set of natural numbers and Z^+ the set of positive integers. A will denote an algebra with identity 1 over a Principal Ideal Domain R. If a, $b \in A$ then $[a, b] = ab - ba$. If $\alpha, \beta \in R$, then (α, β) denotes the greatest common divisor of α and β. If $a \in A$, then the *order* of a, $o(a)$ is the generator of the ideal $I = \{\alpha \mid a \in R,\ \alpha a = 0\}$ of R. $o(a)$ is unique up to associates. As a generalization of concepts in **[1]**, **[2]**, **[3]**, **[4]**, **[5]** we consider the following:

(*) For all a, $b \in A$, there exist α, $\beta \in R$ such that $(\alpha, \beta) = 1$ and $\alpha ab = \beta ba$.

We will show that if A satisfies (*), then A is commutative. This generalizes [**3**; Theorem 3.5].

LEMMA 1. *Let p be a prime in R, $m \in Z^+$ such that $p^m A = (0)$. If A satisfies (*), then A is commutative.*

Proof. Let C denote the center of A. Let $x \in A$, $o(x) = p^k$, $k \in N$. We prove by induction on k that $x \in C$. If $k = 0$, then $x = 0$. So let $k > 0$. Let $y \in A$. First we show

$$[x, y] \neq 0 \quad \text{implies} \quad [yx, y] = 0. \tag{1}$$

If $yx = 0$, this is trivial. So let $yx \neq 0$. Now for some $\alpha_1, \alpha_2 \in R$,

$$\begin{aligned} &\alpha_1 xy = \alpha_2 yx, (\alpha_1, \alpha_2) = 1 \\ &\beta_1(x+1)y = \beta_2 y(x+1), (\beta_1 \beta_2) = 1. \end{aligned} \tag{2}$$

So $\alpha_1\beta_1(x+1)y = \alpha_1\beta_2 y(x+1)$. Thus substituting the above, we get

$$(\alpha_2\beta_1 - \alpha_1\beta_2)yx = (\alpha_1\beta_2 - \alpha_1\beta_1)y. \tag{3}$$

We claim that $(\alpha_2\beta_1-\alpha_1\beta_2)yx\neq 0$. For otherwise $(\alpha_1\beta_2-\alpha_1\beta_1)y=0$. Since $y\neq 0$, we get $p\mid\alpha_1\beta_2-\alpha_1\beta_1$.

Also $(\alpha_1\beta_2-\alpha_1\beta_1)yx=0$. Since $(\alpha_2\beta_1-\alpha_1\beta_2)yx=0$, we get $(\alpha_2-\alpha_1)\beta_1yx=0$. Since $yx\neq 0$, $p\mid\beta_1(\alpha_2-\alpha_1)$. So

$$p\mid\alpha_1(\beta_2-\beta_1),\ p\mid\beta_1(\alpha_2-\alpha_1).$$

Case 1. $p\nmid\alpha_1$. Then since $\alpha_1(\beta_2-\beta_1)y=0$, we get $(\beta_2-\beta_1)y=0$. So by (2), $\beta_1[x,y]=0=\beta_2[x,y]$. Since $[x,y]\neq 0$, we get $p\mid\beta_1$, $p\mid\beta_2$, contradicting (2).

Case 2. $p\mid\alpha_1$. Then $p\nmid\alpha_2$ and so $p\nmid\alpha_2-\alpha_1$. Thus $p\mid\beta_1$. So $p\nmid\beta_2$, $p\nmid\beta_2-\beta_1$. Since $\alpha_1(\beta_2-\beta_1)y=0$ we get $\alpha_1y=0$. So $\alpha_1xy=0$. By (2), $\alpha_2yx=0$. Since $yx\neq 0$, we get $p\mid\alpha_2$, a contradiction.

Hence by (3)

$$(\alpha_2\beta_1-\alpha_1\beta_2)yx\neq 0.$$

In particular

$$\alpha_2\beta_1-\alpha_1\beta_2\neq 0.$$

So

$$\alpha_2\beta_1-\alpha_1\beta_2=p^t\delta,\ t\in N,\ \delta\in R,\ (\delta,p)=1.$$

If $t\geqq k$, then $(\alpha_2\beta_1-\alpha_1\beta_2)yx=0$, a contradiction. So $t<k$. Hence

$$p^{k-t}(\alpha_1\beta_2-\alpha_1\beta_1)y=p^{k-t}p^t\delta yx=0.$$

Let $o(y)=p^i$, $i\in N$. If $i<k$, then $y\in C$, a contradiction. So $i\geqq k$. Hence

$$p^k\mid p^i\mid p^{k-t}(\alpha_1\beta_2-\alpha_1\beta_1).$$

So $p^t\mid\alpha_2\beta_2-\alpha_1\beta_1$ and $\alpha_1\beta_2-\alpha_1\beta_1=p^t\gamma$, $\gamma\in R$. Then $p^t\delta yx=p^t\gamma y$. Hence $p^t(\delta yx-\gamma y)=0$. By induction hypothesis, $\delta yx-\gamma y\in C$. So $[\delta yx-\gamma y,y]=0$. Thus $\delta[yx,y]=0$. Since $(\delta,p)=1$, $[yx,y]=0$. This establishes (1).

Now let $u\in A$ and suppose $[x,u]\neq 0$. Then also $[x,u+1]\neq 0$. By (1), $[ux,u]=0=[(u+1)x,u]$. So $[x,u]=0$, a contradiction. So $x\in C$ and the lemma is proved.

LEMMA 2. *Suppose A satisfies* (*). *Let $a,b\in A,o(b)=0$. If $ba=0$, then $ab=0$.*

Proof. Suppose $ab \neq 0$. Then there exist $\beta_1, \beta_2, \gamma_1, \gamma_2 \in R$ such that

$$\begin{aligned} &\beta_1(a+1)b = \beta_2 b(a+1),\ (\beta_1, \beta_2) = 1, \\ &\gamma_1 a(b+1) = \gamma_2 b(a+1),\ (\gamma_1, \gamma_2) = 1. \end{aligned} \tag{4}$$

So

$$\beta_1 ab = (\beta_2 - \beta_1)b \quad \text{and} \quad (\gamma_2 - \gamma_1)a = \gamma_1 ab. \tag{5}$$

If $\beta_2 = \beta_1$, then β_1, β_2 are units and by (5) $ab = ba = 0$, a contradiction. So $\beta_2 - \beta_1 \neq 0$. Similarly $\gamma_2 - \gamma_1 \neq 0$. Since $o(b) = 0$, we get by (5) that $o(ab) = 0$. So $o(a) = 0$. Hence by (5), $\beta_1 \neq 0$, $\gamma_1 \neq 0$. Also by (5) $[\beta_1 ab, b] = 0$.

So

$$\begin{aligned} (\gamma_2 - \gamma_1)\beta_1 ab &= \gamma_1\beta_1(ab)b \\ &= \gamma_1\beta_1 b(ab) \\ &= \beta_1(\gamma_2 - \gamma_1)ba \\ &= 0. \end{aligned}$$

So $o(ab) \neq 0$, a contradiction. This proves the lemma.

LEMMA 3. *Suppose A satisfies $(*)$. Let $b \in A$, $o(b) = 0$. Then $b \in C$, the center of A.*

Proof. Let $a \in A$. There exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in R$ such that

$$\begin{aligned} &\alpha_1 ab = \alpha_2 ba,\ (\alpha_1, \alpha_2) = 1, \\ &\beta_1(a+1)b = \beta_2 b(a+1),\ (\beta_1, \beta_2) = 1. \end{aligned} \tag{6}$$

Multiplying the second equation by α_1 and substituting the first we obtain

$$b[(\alpha_2\beta_1 - \alpha_1\beta_2)a - (\alpha_1\beta_2 - \alpha_1\beta_1)\cdot 1] = 0.$$

By Lemma 2,

$$[(\alpha_2\beta_1 - \alpha_1\beta_2)a - (\alpha_1\beta_2 - \alpha_1\beta_1)\cdot 1]b = 0.$$

Let $\mu = \alpha_2\beta_1 - \alpha_1\beta_2$. Then $\alpha_1(\beta_2 - \beta_1)b = \mu ab = \mu ba$. By (6) $\alpha_1\mu ab = \alpha_2\mu ba = \alpha_2\mu ab$. So

$$(\alpha_2 - \alpha_1)\alpha_1(\beta_2 - \beta_1)b = 0.$$

Since $o(b) = 0$, we obtain by (6) that either $\alpha_1 = \alpha_2$ is a unit, $\beta_1 = \beta_2$ is a unit or else $\alpha_1 = 0$. The first two cases imply by (6) that $ab = ba$. So let $\alpha_1 = 0$. Then $\alpha_2 ba = 0$ and α_2 is a unit by (6). So $ba = 0$. By Lemma 2, $ab = 0$. Thus in any case $ab = ba$ and we are done.

THEOREM 4. *Suppose A satisfies* (*). *Then A is commutative.*

Proof. Suppose A is not commutative. We will obtain a contradiction. There exists $x \in A$ such that $x \notin C$, the center of A. So $x + 1 \notin C$. By Lemma 3 $o(x) \neq 0$ and $o(x+1) \neq 0$. Hence $o(1) \neq 0$. Let $o(1) = d \neq 0$. Then d is not a unit and hence $d = p_1^{\alpha_1} \cdots p_t^{\alpha_t}$ for some primes $p_1, \cdots, p_t \in A$ and some positive integers $\alpha_1, \cdots, \alpha_t$. Let $A_i = \{a \mid a \in A, p_i^{\alpha_i} a = 0\}$. Then each A_i is a nonzero subalgebra of A and $A = A_1 \oplus \cdots \oplus A_t$. Being subalgebras of A, the A_i's also satisfy (*). Being homomorphic images of A, all the A_i's have identity elements. By Lemma 1 each A_i and hence A is commutative, a contradiction. This proves the theorem.

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NORTH CAROLINA STATE UNIVERSITY
RALEIGH, NC 27607