PACIFIC JOURNAL OF MATHEMATICS
Vol 68, No. 2, 1977

ON A REPRESENTATION THEORY FOR IDEAL SYSTEMS

PAUL EZUST

In widely divergent branches of mathematics, objects emerge which bear sufficient formal resemblance to the ideals of rings for them to be called "ideals". In a series of papers, Karl E. Aubert developed an axiomatic theory of ideal systems which subsumes most of the existing "ideal" theories. The goal of this paper is a representation theory for ideal systems in commutative monoids which will allow the formation of a cohomology theory for these systems. One of the results is a theorem which gives at once a monadic (co)homology for each ideal system. The base category in the monad includes PTOP, the category of pointed topological spaces and basepoint-preserving continuous maps, as a full subcategory and, for each ideal system, the category of algebras associated with the monad consists of the module systems over the ideal system. It is the module systems which are the principal objects of this study.

Described below are some of the basic notions of Aubert's theory of ideal systems. For simplicity in connection with our own work we assume that S is a commutative monoid (written multiplicatively) with an annihilating zero element (denoted 0).

DEFINITION. A *closure operation* x on a set W is a function which assigns to each subset $A \subseteq W$ a unique subset $A_x \subseteq W$ subject to the following conditions:

(i) $A \subseteq A_x$ for all $A \subseteq W$

(ii) $A \subseteq B_x \Rightarrow A_x \subseteq B_x$ for all $A, B \subseteq W$

NOTE. We do *not* assume that a closure operation x satisfies the (topological) condition: $(A \cup B)_x = A_x \cup B_x$. In general this condition will not be satisfied.

DEFINITION. A pair (S, x) is an *ideal system* if S is a commutative monoid with zero and x is a closure operation on S which satisfies the following axioms.

x.1 $\{0\}_x = \{0\}$

x.2 $AB_x \subseteq B_x$ for all $A, B \subseteq S$ ["multiplicative ideal property"]

x.3 $AB_x \subseteq (AB)_x$ for all $A, B \subseteq S$ ["continuity axiom"].

TERMINOLOGY. The sets $A_x \subseteq S$ are called the *x-ideals* of S.

NOTATION. $A + B = (A \cup B)_x$

$A : B = \{s \in S \mid sb \in A \ \forall b \in B\}$

$a \equiv b(A_x)$ iff $A_x + \{a\} = A_x + \{b\}$.

Several examples of particular ideal systems are discussed in Aubert's extensive survey paper **[2]** and the reader is referred to that paper for definitions, etc.

In a brief note **[8]**, Aubert and Hansen introduced the notion of "module system" over an ideal system as an ancillary device to the theory of ideal systems. Despite the pessimism expressed in that paper, it is our purpose to show that the theory of module systems over ideal systems yields a representation theory analogous to the theory of modules over rings.

Throughout this paper the terminology and notations of category theory have been used as are found in such standard texts as Herrlich and Strecker **[14]** and Mitchell **[22]**. The author originally became interested in the problems discussed herein during a course given by Professor Karl E. Aubert at Tufts University during the academic year, 1969–70.

2. Axioms for module-systems.

DEFINITION. Let (S, x) be a fixed ideal system. A left *S-set* is a set M together with a map $S \times M \to M$, denoted by $(s, m) \to sm$, satisfying

(i) $s(tm) = (st)m \ \forall s, t \in S, \ \forall m \in M$

(ii) $1m = m \ \forall m \in M$ (where 1 denotes the identity element of the monoid S).

DEFINITION. A pair (M, y), where M is a (left) S-set and y is a closure operation on M, is a *module-system* over (S, x) if the following are satisfied:

y.1 $\exists \theta \in M$ such that $0m = \theta \ \forall m \in M$, and $\{\theta\}_y = \{\theta\}$. We shall denote $\theta = 0$.

y.2 $AU_y \subseteq U_y \ \forall A \subseteq S, \ \forall U \subseteq M$

y.3 $AU_y \subseteq (AU)_y \ \forall A \subseteq S, \ \forall U \subseteq M$

y.4 $A_xU \subseteq (AU)_y \ \forall A \subseteq S, \ \forall U \subseteq M$.

NOTATION. Let (M, y) be a module-system, let $U, V \subseteq M$, $A \subseteq S$, $u, v, w \in M$, and $s \in S$. Then,

$$U + V = (U \cup V)_y$$

$$U : V = \{s \in S \mid sv \in U \ \forall v \in V\}$$

$$U: A = \{m \in M \mid am \in U \ \forall a \in A\}$$
$$\text{Ann}(u) = 0: u \ (= \{0\}: \{u\}) \text{ in } S$$
$$\text{Ann}(a) = 0: a \ (= \{0\}: \{a\}) \text{ in } M$$
$$w \equiv v(U_y) \text{ iff } U_y + \{w\} = U_y + \{v\}.$$

OBSERVATION. Frequent use shall be made of the following two equivalences which were established by Aubert and Hansen **[8]**.

1. Axiom y.3 is equivalent to the following statement:

$$(U_y : s)_y = U_y : s \ \ \forall U \subseteq M, \ \forall s \in S.$$

2. Axiom y.4 is equivalent to the following statement:

$$(U_y : v)_x = U_y : v \ \ \forall U \subseteq M, \ \forall v \in M.$$

DEFINITION. Let (S, x) be a fixed ideal system. The category MS consists of objects which are module systems (M, y) over (S, x) and morphisms $\xi : (M_1, y_1) \to (M_2, y_2)$ which are set functions that satisfy the following conditions:

(i) $\xi(su) = s\xi(u) \ \forall s \in S, \ \forall u \in M_1$

(ii) $\xi(U_{y_1}) \subseteq (\xi(U))_{y_2} \ \forall U \subseteq M_1$.

REMARK. Morphism condition (ii), above, is equivalent to:

$$(\xi^{-1}(V_{y_2}))_{y_1} = \xi^{-1}(V_{y_2}) \forall V \subseteq M_1.$$

EXAMPLES.

1. For any fixed ideal system (S, x), let $M = A_x$ for some $A \subseteq S$, and $y = x$. Thus, for $B \subseteq M$, $B_y = B_x$, and (M, y) is an object of MS.

2. Let S be the multiplicative semigroup of a commutative ring with identity, and let x be the classical ideal closure, $A_x = A_d = \langle A \rangle$ $\forall A \subseteq S$. Then any module M over the ring, with the classical submodule closure, $U_y = \langle U \rangle$, is an object of MS.

3. Let S be a commutative monoid with 0 and for each $A \subseteq S$, let $A_x = SA$ [this closure is called the *s-closure*]. For any S-*set*, M, and any $U \subseteq M$, define $U_y = SU$ [this closure will be referred to as the s-closure also]. Then (M, y) is an object of MS.

4. Let (S, x) be an ideal system and let M be an S-set. For any $U \subseteq M$, define $U_y = U \cup \{0\}$ [this closure will be referred to as the *discrete* closure on M]. Then (M, y) is an object of MS.

5. Let $S = \{1/n \mid n \in Z, \ n > 0\} \cup \{0\}$. For each $A \subseteq S$, define $A_x = \{s \in S \mid s \leqq \sup A\}$; i.e., $A_x = [0, \bar{a}]$, where $\bar{a} = \sup A$. Then

(S, x) is an example of an ideal system for which the inclusion x.3 is proper.

3. The morphisms of MS.

DEFINITION. An S-set M with 0 is called an *(S, x)-set* provided $(0: u)_x = 0: u$ for all $u \in M$. A map $\varphi: M_1 \to M_2$ from one S-set to another is called an S-map if it satisfies (i) above.

PROPOSITION 1. *Let M be an (S, x)-set and $\{f_j: M \to M_j \mid j \in J\}$ a family of S-maps, where $\{(M_j, y_j) \mid j \in J\}$ is a family of objects of S_I. Then there exists a closure operator y such that (M, y) is an object and $f_j: M \to M_j$ is a morphism for all $j \in J$. The coarsest such system y is said to be induced in M by the family $\{f_j: M \to M_j \mid j \in J\}$.*

Proof. Let M be an (S, x)-set and $F = \{f_j: M \to M_j \mid j \in J\}$ be a family of S-maps into objects (M_j, y_j), for $j \in J$. Let $Q = \{f_j^{-1}(U^j_{y_j}) \mid U^j \subseteq M_j,\ j \in J\} \cup \{0\}$. For any $V \subseteq M$ define $V_y = \cap \{W \in Q \mid V \subseteq W\}$.

DEFINITION. Let M be an (S, x)-set and $G = \{g_j: M_j \to M \mid j \in J\}$ be a family of S-maps from objects (M_j, y_j) to M. The finest closure system, y, on M (if one exists) such that (M, y) is an object of MS and such that each g_j is a morphism, will be called the closure system which is *coinduced* in M by the family G. Let $P = \{U \subseteq M \mid (g_j^{-1}(U))_{y_j} = g_j^{-1}(U)$ for all $j \in J\}$. G is called a *covering family* of S-maps into M if (1) for each $U \in P$, $\exists j \in J$ such that $g_j(g_j^{-1}(U)) = U$; and (2) $0 \in P$.

PROPOSITION 2. *Let M be an (S, x)-set and let $G = \{g_j: M_j \to M \mid j \in J\}$ be a covering family of S-maps from objects (M_j, y_j) to M. Then there exists a coinduced closure system y for M* (with respect to G).

Proof. Let M and G be as described above and let P be as defined above. Let $Q = \{U \in P \mid (U: m)_x = U: m \ \ \forall m \in M\}$ and, for each $V \subseteq M$, define $V_y = \cap \{U \in Q \mid V \subseteq U\}$.

DEFINITION. An equivalence relation $\sim$ on an object (M, y) is a *congruence* if $u \sim v \Rightarrow su \sim sv \ \forall s \in S$. Let $[v] = \{u \in M \mid u \sim v\}$. A congruence $\sim$ is *admissible* if $[0]_y = [0]$.

PROPOSITION 3. *Let (M, y) be an object of MS and $\sim$ an admissible congruence on M. Then $(M/\sim, \tilde{y})$ is an object of MS, where $M/\sim$ is* *the set of* $\sim$ *classes in* M *and* $\tilde{y}$ *is coinduced by the map* $\pi: M \to M/\sim$ *defined by* $\pi(u) = [u]$ $\forall u \in M$.

Proof. By Proposition 2, one need only show that for any admissible congruence $\sim$ on an object (M, y), the set $M/\sim$ is an (S, x)-set and the map $\pi: M \to M/\sim$ is a covering S-map.

PROPOSITION 4. *Let* (M, y) *be an object and* $U_y \subseteq M$. *Then*

(a) U_y *determines an admissible congruence on* M *given by the rule*: $u \equiv v\ (U_y)$ *iff* $U_y + \{u\} = U_y + \{v\}$. *Denote the set of congruence classes "modulo* U_y*" by* M/U_y.

(b) *The inclusion map* $i: U_y \to M$ *induces a system* y' *on* U_y *given by the rule*: $V_{y'} = V_y \cap U_y = V_y$ $\forall V \subseteq U_y$. *Thus,* U_y *is a subobject of* M. *The prime is generally omitted.*

PROPOSITION 5. *The Zero object,* $M = \{0\}$, *is both initial and terminal in* MS *and will be denoted, simply,* 0.

THEOREM 6. *Let* $\varphi: (M_1, y_1) \to (M_2, y_2)$ *be a morphism. Then*

(a) φ *is a monomorphism iff* φ *is injective.*

(b) φ *is an epimorphism iff* φ *is surjective.*

(c) *If* φ *is monic then* $(\varphi^{-1}(U))_{y_1} \subseteq \varphi^{-1}(U_{y_2})$ $\forall U \subseteq M_2$.

Proof. (a) Suppose φ is a monomorphism such that $\varphi(u) = \varphi(v)$ for some $u, v \in M_1$. Define (M_3, y_3) by: $M_3 = S \vee S$, the disjoint union of two copies of S (labeled with u and v, respectively) with the zero elements identified, and $U_{y_3} = (U \cap S_u)_x \cup (U \cap S_v)_x$ $\forall U \subseteq M_3$. In fact, this construction is a special case of the more general construction of the coproduct of S with itself, which is discussed in §4. Let $\psi_1: M_3 \to M_1$ be defined by the rule: $\psi_1(s_u) = su$ and $\psi_1(s_v) = sv$ $\forall s \in S$. Define $\psi_2: M_3 \to M_1$ by the rule: $\psi_2(s_u) = sv$ and $\psi_2(s_v) = su$ $\forall s \in S$. ψ_1 and ψ_2 are morphisms such that $\varphi\psi_1 = \varphi\psi_2$. Since φ is monic, it follows that $\psi_1 = \psi_2$; i.e., $u = v$.

(b) Suppose φ is an epimorphism. Then $\varphi(M_1)$ is an S-set.

Claim. $\varphi(M_1) = M_2$. Let $M_3 = M_2/\varphi(M_1)$ be the S-set of congruence classes in M_2 modulo the S-set $\varphi(M_1)$; i.e., for $u, v \in M_2$, $u \equiv v\ (\varphi(M_1))$ means $Su \cup \varphi(M_1) = Sv \cup \varphi(M_1)$. For any $U \subseteq M_3$, define $U_{y_3} = SU$. Let $\pi: M_2 \to M_3$ be the S-map $\pi(u) = [u]$ and let $M = \{(u, [u]) \mid u \in M_2\} \cup \{(u, [0]) \mid u \in M_2\}$. For each $s \in S$, $s(u, [u]) = (su, s[u]) = (su, [su])$ and $s(u, [0]) = (su, [0])$. Also, for each $u \in M_2$, $s(u, [0]) = (0, [0])$ iff $su = 0$ and $s(u, [u]) = (0, [0])$ iff $su = 0$, so that $(0, [0]): (u, [0]) = 0: u = (0: u)_x$. Hence, M is an (S, x)-set. Define $\xi_1: M_2 \to M$ by the rule: $\xi_1(u) = (u, [u])$ $\forall u \in M_2$. Define $\xi_2: M_2 \to M$

by the rule: $\xi_2(u) = (u, [0])\ \forall u \in M_2$. Then $\{\xi_1, \xi_2\}$ is a covering family of S-maps into M. Let y be coinduced on M by $\{\xi_1, \xi_2\}$ and note that $\xi_1\varphi = \xi_2\varphi$; hence, $\xi_1 = \xi_2$. Thus, $\pi(u) = [0]\ \forall u \in M_2$; i.e., $\varphi(M_1) = M_2$.

(c) Suppose φ is a monomorphism. Then, by (a) above, φ is injective. Thus, $\varphi((\varphi^{-1}(U))_{y_1}) \subseteq U_{y_2}$; hence, $(\varphi^{-1}(U))_{y_1} \subseteq \varphi^{-1}(U_{y_2})$.

THEOREM 7. *MS has* (a) *Kernels,* (b) *Images,* (c) *Cokernels, and* (d) *Coimages.*

Proof. Let $\varphi: (M_1, y_1) \to (M_2, y_2)$ be a morphism. (a) $\operatorname{Ker}\varphi = \varphi^{-1}(0)$. (b) $\operatorname{Im}\varphi = (\varphi(M_1), y\varphi)$, where the closure operator $y\varphi$ is coinduced by the (surjective) map $\varphi': M_1 \to \varphi(M_1)$ defined by the rule: $\varphi'(u) = \varphi(u)\ \forall u \in M_1$. (c) Define the congruence $\sim$ by the rule: $u \sim u$ $\forall u \in M_2$ and, for $u \neq v$, $u \sim v$ iff $\{u, v\} \subseteq (\varphi(M_1))_{y_2}$. In forming $M_2/\sim$, the S-set of $\sim$ classes, $(\varphi(M_1))_{y_2}$ is compressed down to $[0]$ and the rest of M_2 remains unchanged. Let $\pi: M_2 \to M_2/\sim$ be the projection $u \to [u]$. Note that $[u] = [0]$ for $u \in (\varphi(M_1))_{y_2}$ and $[u] = u$ for $u \notin (\varphi(M_1))_{y_2}$. Also note that $M_2/\sim$ is an (S, x)-set and let $\tilde{y}$ be coinduced by $\{\pi\}$. Then $\operatorname{Coker}\varphi = (M_2/\sim, \tilde{y})$. (d) For each $u \in M_1$, let $\bar{u} = \varphi^{-1}(\varphi(u))$ and let $M_1/\varphi = \{\bar{u} \mid u \in M_1\}$. Let $\pi: M_1 \to M_1/\varphi$ be the projection, $u \to \bar{u}$. For each subset $\pi(U) \subseteq M_1/\varphi$, define $(\pi(U))_{\varphi y} = \hat{\varphi}^{-1}((\varphi(U))_{y_2})$, where $\hat{\varphi}: M_1/\varphi \to M_2$ is the map, $\hat{\varphi}(\bar{u}) = \varphi(u)$, for all $\bar{u} \in M_1/\varphi$. Then $\operatorname{Coim}\varphi = (M_1/\varphi, \varphi y)$.

REMARKS. (1) φ monic $\Rightarrow \operatorname{Im}\varphi \cong M_1$.

(2) In any exact category (e.g., the category of modules over a commutative ring with unity), for any morphism $\varphi: M_1 \to M_2$, $\operatorname{Im}\varphi \cong \operatorname{Coim}\varphi$. The following example shows that this is not generally true in *MS.*

EXAMPLE. Let $M = \{0, a, b, c\}$, $S = \{0, 1\}$, with the obvious multiplication. Let (M_1, y_1) and (M_2, y_2) be defined as follows. $M_1 = M_2 = M$. y_1 is the s-system on M_1, and y_2 is the *indiscrete* system on M_2: $\{0\}_{y_2} = \{0\}$, and $U \neq \{0\} \Rightarrow U_{y_2} = M_2$. Let $\varphi: M_1 \to M_2$ be the identity map. Then $(M_2, y\varphi) = \operatorname{Im}\varphi \not\cong \operatorname{Coim}\varphi = (M_1, \varphi y)$.

PROPOSITION 8. *Let* $\varphi: M_1 \to M_2$ *be a morphism. If* $\varphi(U_{y_1}) = (\varphi(U))_{y_2}$ *for all* $U \subseteq M_1$ *then* $\operatorname{Im}\varphi \cong \operatorname{Coim}\varphi$.

OBSERVATION. The example which precedes Proposition 8 also illustrates the fact that a morphism in *MS* might be both monic and epic and yet fail to be an isomorphism; i.e., *MS* is not balanced. Another way of characterizing this situation is to note that the forgetful functor $F: MS \to SET$ does not reflect isomorphisms. It follows (Proposition 32.5 **[14]**) that *MS* is not an algebraic category.

4. Categorical constructions in *MS*.

THEOREM 9. *MS has Products.*

Proof. Let $\{(M_j, y_j) \mid j \in J\}$ be a family of objects of *MS*. Let ΠM_j denote the cartesian product of the sets M_j $(j \in J)$. For each $(m_j) \in \Pi M_j$ and each $s \in S$, define $s(m_j) = (sm_j)$. Let 0 denote (0_j) and observe that, for all $(m_j) \in \Pi M_j$, $0\colon (m_j) = \cap \{0\colon m_j \mid j \in J\}$, the latter being an intersection of x-ideals in S. Thus, ΠM_j, is an (S, x)-set. For each $k \in J$, define $\pi_k\colon \Pi M_j \to M_k$ by the rule, $\pi_k((m_j)) = m_k$ (this is the canonical projection map from the cartesian product to its factors). Let Πy_j be the system induced in ΠM_j by the family of projections, $\{\pi_j \mid j \in J\}$. Then, for each $U \subseteq \Pi M_j$, $U_{\Pi y_j} = \cap \{\pi_j^{-1}((\pi_j(U))_{y_j}) \mid j \in J\} = \times \{(\pi_j(U))_{y_j} \mid j \in J\}$. It is easy to verify that $(\Pi M_j, \Pi y_j)$ is the product.

NOTATION. $M_1 \times M_2$ will frequently be used to denote the product, $\Pi\{M_j \mid j = 1, 2\}$, of two objects of *MS*. The corresponding closure system will be denoted, $y_1 \times y_2$.

THEOREM 10. *MS has Coproducts.*

Proof. Let $\{(M_j, y_j) \mid j \in J\}$ be a family of objects of *MS* and let ΣM_j denote the disjoint union, $\vee \{M_j \mid j \in J\}$ with all zeros identified. For each $k \in J$, let $\delta_k\colon M_k \to \Sigma M_j$ be the natural inclusion map. Let Σy_j be defined on ΣM_j as follows. For any $U \subseteq \Sigma M_j$, $U_{\Sigma y_j} = \vee \{(\delta_j^{-1}(U))_{y_k} \mid k \in J\}$. Note that $U_{\Sigma y_j} = \cup \{(U \cap M_k)_{y_k} \mid k \in J\}$ if we identify M_k with its set-theoretic image, $\delta_k(M_k)$ in ΣM_j. Clearly $(\Sigma M_j, \Sigma y_j)$ is an object of *MS* and each map δ_k is a morphism. Note that Σy_j is the closure system coinduced in ΣM_j by the family of inclusions, $\{\delta_j \mid j \in J\}$. It is not hard to verify that $(\Sigma M_j, \Sigma y_j)$ is the coproduct.

DEFINITION. An object of *MS* is *free* if it is of the form ΣM_j $(j \in J)$, where for each $j \in J$, $(M_j, y_j) \cong (S, x)$. We denote such an object $(F(J), y_*)$ and refer to the index set J as the *basis* for the free object $(F(J), y_*)$.

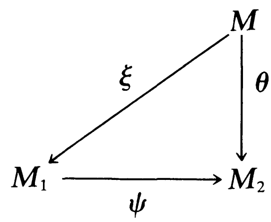
REMARK. In particular, (S, x) is free with basis $\{1\}$.

PROPOSITION 11. [Universal Mapping Property of Free Objects]. *$(F(U), y_*)$ is a free object with basis U iff for any object (M, y) and any set map $\sigma\colon U \to M$, there is a unique morphism $\varphi\colon F(U) \to M$ such that $\varphi\eta = \sigma$, where $\eta\colon U \to F(U)$ is the inclusion, $u \to 1_u$ for all $u \in U$.*

DEFINITION. The morphism φ described above is called the *lift* of σ.

PROPOSITION 12. *Let* (M, y) *be an object of* MS *and let* $\varphi: F(M) \to M$ *be the epimorphism that lifts the identity morphism* $1: M \to M$. *Then* $M \cong \mathrm{Coim}(\varphi)$.

DEFINITION. An object (M, y) is *projective* if for any morphism $\theta: M \to M_2$ and any epimorphism $\psi: M_1 \to M_2$ [where (M_1, y_1) and (M_2, y_2) are objects] there exists a morphism $\xi: M \to M_1$ such that $\psi\xi = \theta$.



REMARK. It follows immediately from the above definition that if (M, y) is projective, then (M, y') is projective for any closure system y' (on M) which is finer than y. Thus, since the s-system is the finest possible closure system for M, each projective object in the category ENS-S of all S-sets determines a family of projective objects of MS and, conversely, each projective object of MS determines a projective object of ENS-S.

PROPOSITION 13. *Let* (M, y) *be an object of* MS. *Then* M *is projective iff* M *is a retract of a free object of* MS [In particular, each free object of MS is projective.]

REMARK. In the category R-Mod, of left R-modules, an object is a retract of a free iff it is a direct summand of a free. The following example demonstrates that this is not the case in general in MS.

EXAMPLE. Let $S = \{0, 1, a, b\}$ with multiplication defined as follows: $aa = bb = ab = ba = a$. Let $M = \{0, a\}$ and let S and M each have the s-system closure. Then (M, y) is a projective object of MS and M is not a direct summand of S since $(S - M)_y \neq (S - M) \cup \{0\}$. Since a free object of MS must be a coproduct of copies of S it follows that M is not a direct summand of any free object.

PROPOSITION 14. *Let* $\{(M_j, y_j) \mid j \in J\}$ *be a family of objects of* MS. *Then* $(\Sigma M_j, \Sigma y_j)$ *is projective iff* (M_j, y_j) *is projective* $\forall j \in J$.

REMARK. In view of Theorems 9 and 10, it is clear that MS is not an additive category since finite products are not isomorphic to finite coproducts.

5. Completeness and cocompleteness of *MS*.

PROPOSITION 15. *MS is locally and colocally small.*

PROPOSITION 16. *MS has Intersections.*

Proof. Let $\{\alpha_j: (M_j, y_j) \to (M, y) \mid j \in J\}$ be a family of subobjects of (M, y). Since α_j monic $\Rightarrow$ $M_j \cong \operatorname{Im} \alpha_j$, for each $j \in J$ we take $M' = \bigcap \{\operatorname{Im} \alpha_j \mid j \in J\}$, a set-theoretic intersection of subsets of M. For each $j \in J$, let $\beta_j: M' \to \operatorname{Im} \alpha_j$ be the natural inclusion map. Then M' is an (S, x)-set and β_j is an S-map for each $j \in J$. Let y' be the system induced on M' by the family $\{\beta_j \mid j \in J\}$. Let $\alpha: M' \to M$ be the natural inclusion map. Then $\alpha: (M', y') \to (M, y)$ is the intersection of the subobjects $\{\alpha_j \mid j \in J\}$.

PROPOSITION 17. *MS has Equalizers.*

Proof. Let $\varphi, \theta: M_1 \to M_2$ be morphisms, and let $E = \{u \in M_1 \mid \varphi(u) = \theta(u)\}$. Then $\operatorname{Equ}(\varphi, \theta) = (E, y_e)$, where y_e is induced by the inclusion $\eta: E \to M_1$.

The following Theorem follows from Theorem 23.8 **[14]**.

THEOREM 18. *MS has the following properties*:
(a) *MS is complete* (in particular, *MS* has inverse limits).
(b) *MS has* (multiple) *pullbacks.*
(c) *MS has inverse images.*

From Theorems 10 and 18 and Proposition 15 we obtain the hypotheses of Theorem 23.12 **[14]**, and using the dual of Theorem 23.8 **[14]** we obtain the following

THEOREM 19. *MS has the following properties*:
(a) *MS is cocomplete* (in particular, *MS* has direct limits).
(b) *MS has* (multiple) *pushouts.*
(c) *MS has direct images.*
(d) *MS has coequalizers.*
(e) *MS has cointersections.*

6. Properties of the hom functor *MS* → *MS*.

THEOREM 20. *For each pair of objects* (M_1, y_1), (M_2, y_2) *of MS,* $(\hom_S(M_1, M_2), \hat{y})$ *is an object of MS, where, for* $\varphi \in \hom_S(M_1, M_2)$ *and* $s \in S$, $s\varphi$ *is defined by the rule*: $(s\varphi)(u) = s(\varphi(u))$ $\forall u \in M_1$, *and, for any*

$W \subseteq \hom_S(M_1, M_2)$, $W_{\hat{y}} = \cap\{[m, U_{y_2}] \mid W \subseteq [m, U_{y_2}]\}$, *where* $[m, U_{y_2}] = \{\xi \in \hom_S(M_1, M_2) \mid \xi(m) \in U_{y_2}\}$.

PROPOSITION 21. *For any object* (M, y) *of* MS, $(M, y) \cong (\hom_S(S, M), \hat{y})$, *where* (S, x) *is considered as an object of* MS.

THEOREM 22. *MS has an internal Hom functor,* Hom: $MS^{op} \times MS \to MS$.

Proof. By Theorem 20 it will suffice to verify that $\hom_S(\varphi, \theta)$: $\hom_S(M_1, M_2) \to \hom_S(M_1', M_2')$ is a morphism for all $\varphi \in \hom_S(M_1', M_1)$ and all $\theta \in \hom_S(M_2, M_2')$.

$$\begin{array}{ccc} \hom_S(M_1, M_2) & \qquad & M_1 \xrightarrow{\quad f \quad} M_2 \\ \Big\downarrow \hom_S(\varphi, \theta) & & \varphi \Big\uparrow \qquad\qquad \Big\downarrow \theta \\ \hom_S(M_1', M_2') & & M_1' \xrightarrow[\theta f \varphi]{\qquad} M_2' \\ & & = \hom_S(\varphi, \theta)(f) \end{array}$$

Indeed, it is true in any category that the corresponding construction yields a well defined set map. Thus, with $\hom_S(\varphi, \theta)(f) = \theta f \varphi$, we have the following equations:

$$\begin{aligned} \hom_S(\varphi, \theta)^{-1}([u, U_{y_2'}]) &= \{f \in \hom_S(M_1, M_2) \mid \theta f \varphi(u) \in U_{y_2'}\} \\ &= \{f \in \hom_S(M_1, M_2) \mid f(\varphi(u)) \in \theta^{-1}(U_{y_2'})\} = [\varphi(u), \theta^{-1}(U_{y_2'})]. \end{aligned}$$

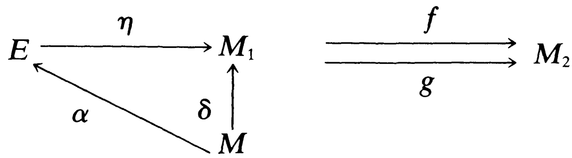
NOTATION. Since MS has an internal Hom functor, we will follow the practice of Herrlich and Strecker **[14]** and others and write it with a capital H. Also, we will suppress the subscript S when no confusion will result.

PROPOSITION 23. *For any family* $\{(M_j, y_j) \mid j \in J\}$ *of objects of* MS, $\mathrm{Hom}(\Sigma M_j, M) \cong \Pi \mathrm{Hom}(M_j, M)$ *for any object* (M, y).

PROPOSITION 24. *The functor* $\mathrm{Hom}(M, \_)$: $MS \to MS$ (for fixed object (M, y)) *preserves products; i.e., for any family* $\{(M_j, y_j) \mid j \in J\}$ *of objects,* $\mathrm{Hom}(M, \Pi M_j) \cong \Pi \mathrm{Hom}(M, M_j)$.

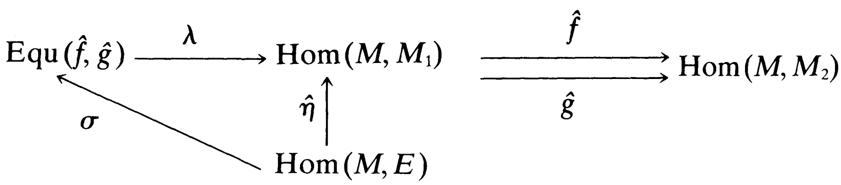
PROPOSITION 25. *The functor* $\mathrm{Hom}(M, \_)$: $MS \to MS$ *preserves equalizers.*

Proof. Let $f, g \in \mathrm{Hom}(M_1, M_2)$.



Then, by Prop. I.17, $\eta: E \to M_1$ is the equalizer of f and g, where $E = \{u \in M_1 \mid f(u) = g(u)\}$ and y_e, the closure on E, is induced by the canonical inclusion, $\eta: E \to M_1$.

To prove that $\mathrm{Hom}(M, \_\,)$ preserves equalizers we shall show that $\mathrm{Hom}(M, E) \cong \mathrm{Equ}(\hat{f}, \hat{g})$, where $\hat{f} = \mathrm{Hom}(M, f)$ and $\hat{g} = \mathrm{Hom}(M, g)$, and $\mathrm{Equ}(\hat{f}, \hat{g}) = \{\xi \in \mathrm{Hom}(M, M_1) \mid \hat{f}(\xi) = \hat{g}(\xi)\}$.



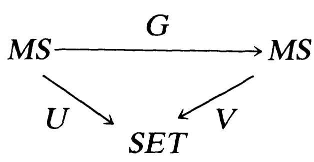
Let λ: $\mathrm{Equ}(f, g) \to \mathrm{Hom}(M, M_1)$ be the canonical inclusion and $\hat{\eta} = \mathrm{Hom}(M, \eta)$: $\mathrm{Hom}(M, E) \to \mathrm{Hom}(M, M_1)$; i.e., $\hat{\eta}(k) = \eta k$. Then $\hat{f}\hat{\eta} = \hat{g}\hat{\eta}$, hence there exists a morphism, σ: $\mathrm{Hom}(M, E) \to \mathrm{Equ}(\hat{f}, \hat{g})$ such that $\hat{\eta} = \lambda\sigma$. σ is the required isomorphism.

The next Proposition follows from Theorem 24.3 **[14]**.

PROPOSITION 26. $\mathrm{Hom}(M, \_\,)$: *MS* $\to$ *MS* *preserves pullbacks, multiple pullbacks, terminal objects, inverse images, finite intersections, and limits.*

THEOREM 27. $\mathrm{Hom}(M, \_\,)$: *MS* $\to$ *MS has a left adjoint.*

Proof. Consider the functor diagram, where $G = \mathrm{Hom}(M, \_\,)$, $U = \mathrm{hom}_S(M, \_\,)$, and V = Forgetful.



Clearly this diagram commutes. By Propositions 15, 18, and 19, *MS* is complete, cocomplete, locally small, and colocally small. By Proposition 26, G preserves limits. By Theorem 30.20 **[14]**, U has a left adjoint. Clearly V is faithful. The result follows from Theorem 28.12 **[14]**.

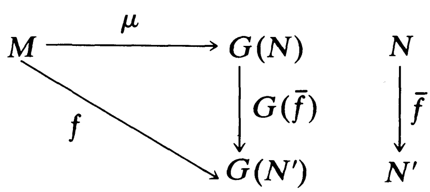
7. The tensor product in MS.

DEFINITION. We denote the left adjoint of $\mathrm{Hom}(M, \_): MS \to MS$ $\_ \otimes M$, and we refer to $M' \otimes M$ as the *tensor product* of M' and M. The closure system on $M' \otimes M$ is denoted $y' \otimes y$.

REMARKS. The adjoint situation, $(\eta, \delta): \_ \otimes M \dashv \mathrm{Hom}(M, \_)$, gives, for each object M_1 of MS a morphism $\eta_{M_1}: M_1 \to \mathrm{Hom}(M_2, M_1 \otimes M_2)$. Define $\psi: M_1 \times M_2 \to M_1 \otimes M_2$ by the rule, $\psi((u_1, u_2)) = (\eta_{M_1}(u_1))(u_2)$ and denote $\psi((u_1, u_2)) = u_1 \otimes u_2$. Note that $s(u_1 \otimes u_2) = su_1 \otimes u_2 = u_1 \otimes su_2$, for all $s \in S$. In fact, ψ is *bilinear*, in the sense that both $\psi(u_1, \_): M_2 \to M_1 \otimes M_2$ and $\psi(\_ , u_2)$: $M_1 \to M_1 \otimes M_2$ are morphisms (defined in the obvious ways). Indeed, $\psi(u_1, \_) = \eta_{M_1}(u_1) \in \mathrm{Hom}(M_2, M_1 \otimes M_2)$ by definition. To see that $\psi(\_ , u_2) \in \mathrm{Hom}(M_1, M_1 \otimes M_2)$, note that

$$\begin{aligned}\psi(\_ , u_2)^{-1}(U_{y_1 \otimes y_2}) &= \{u_1 \in M_1 \mid \psi(u_1, u_2) \in U_{y_1 \otimes y_2}\} \\ &= \{u_1 \in M_1 \mid (\eta_{M_1}(u_1))(u_2) \in U_{y_1 \otimes y_2}\} \\ &= \{u_1 \in M_1 \mid \eta_{M_1}(u_1) \in [u_2, U_{y_1 \otimes y_2}]\} \\ &= \eta^{-1}_{M_1}([u_2, U_{y_1 \otimes y_2}]).\end{aligned}$$

DEFINITION. Let $G: \underline{A} \to \underline{B}$ be a functor and let M be an object of $\underline{B}$ a pair (μ, N), where N is an object of $\underline{A}$ and $\mu : M \to G(N)$, is called a *universal map for M with respect to G* (or a G-universal map for M) provided that for each N' (object of $\underline{A}$) and each f: $M \to G(N')$, there is a unique $\underline{A}$-morphism $\bar{f}$: $N \to N'$ such that the triangle commutes.



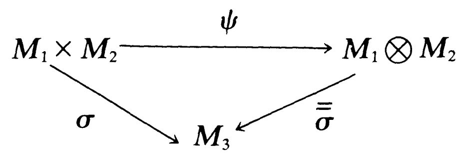
NOTATION. Given objects (M_j, y_j), for $j = 1,2,3$, let $\mathrm{Bihom}(M_1 \times M_2, M_3)$, denote the set of all bilinear maps $M_1 \times M_2 \to M_3$.

PROPOSITION 28. *The map* θ: $\mathrm{Bihom}(M_1 \times M_2, M_3) \to \mathrm{Hom}(M_1, \mathrm{Hom}(M_2, M_3))$ *given by,* $\theta(\sigma) = \bar{\sigma}$, *where* $(\bar{\sigma}(u_1))(u_2) = \sigma(u_1, u_2)$, *is a bijection.*

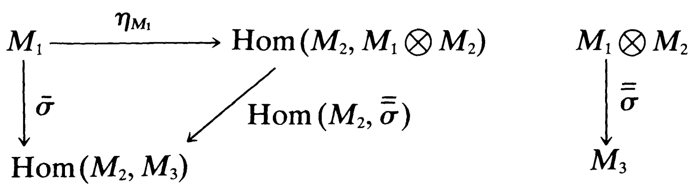
THEOREM 29. *Let* $a \in \mathrm{Bihom}(M_1 \times M_2, M_3)$. *Then there exists a unique* $\bar{\bar{\sigma}} \in \mathrm{Hom}(M_1 \otimes M_2, M_3)$ *such that* $\bar{\bar{\sigma}}\psi = \sigma$ [where

ψ: $M_1 \times M_2 \to M_1 \otimes M_2$ *is the canonical map,* $(m_1, m_2) \to m_1 \otimes m_2$; i.e., $\theta(\psi) = \eta_{M_1}$].

Proof.



To complete the first diagram with a morphism $\bar{\bar{\sigma}}$, we make use of the fact that, by Theorem 27.3 **[14]** $(\eta_{M_1}, M_1 \otimes M_2)$ is a universal map for M_1 with respect to $\mathrm{Hom}(M_2, \_\,)$.

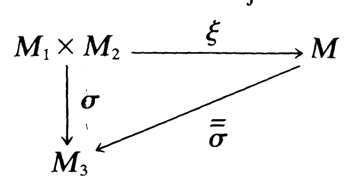


Thus, there exists a unique $\bar{\bar{\sigma}} \in \mathrm{Hom}(M_1 \otimes M_2, M_3)$ such that $\bar{\sigma} = \mathrm{Hom}(M_2, \bar{\bar{\sigma}})\eta_{M_1}$ i.e., $\bar{\sigma} = \bar{\bar{\sigma}}\eta_{M_1}$. Note that $\bar{\bar{\sigma}}$ makes the first diagram commute.

Proposition 30. *$M_1 \otimes M_2 = \{m_1 \otimes m_2 \mid m_1 \in M_1, \ m_2 \in M_2\}$ and $y_1 \otimes y_2$ is the closure operation coinduced on $M_1 \otimes M_2$ by the family,*

$$F = \{\eta_{M_1}(m_1) \mid m_1 \in M_1\} \cup \{\eta_{M_2}(m_2) \mid m_2 \in M_2\}.$$

Proof. Let $M = \{m_1 \otimes m_2 \mid m_1 \in M_1, \ m_2 \in M_2\}$. Then $M \subseteq M_1 \otimes M_2$. Although F is not a covering family, we can form the coinduced closure, y as follows: Let $Q_1 = \{U \subseteq M \mid (\eta_{M_1}(m_1)^{-1}(U))_{y_2} = \eta_{M_1}(m_1)^{-1}(U) \quad \forall m_1 \in M_1\} \cap \{U \subseteq M \mid (\eta_{M_2}(m_2)^{-1}(U))_{y_1} = \eta_{M_2}(m_2)^{-1}(U) \ \forall m_2 \in M_2\}$. Let $Q = \{U \in Q_1 \mid (U: (u_1 \otimes u_2))_x = U: (u_1 \otimes u_2), \ \forall u_1 \otimes u_2 \in M\}$ and note that $Q_1 = Q$. For each $V \subseteq M$, let $V_y = \cap \{U \in Q \mid V \subseteq U\}$. Then (M, y) is an object of MS and y is the finest closure system on M which permits all the S-maps in F to be morphisms into M. Define $\xi: M_1 \times M_2 \to M$ by the rule: $\xi(m_1, m_2) = m_1 \otimes m_2$. Then ξ is bilinear and surjective.



Let $\sigma: M_1 \times M_2 \to M_3$ be a bilinear map. Define $\overline{\overline{\sigma}}: M \to M_3$ by the rule: $\overline{\overline{\sigma}}(m_1 \otimes m_2) = \sigma(m_1, m_2)$. Then $\overline{\overline{\sigma}}$ is a morphism and the diagram commutes. In fact, $\overline{\overline{\sigma}}$ is the identity morphism, $m_1 \otimes m_2 \to m_1 \otimes m_2$ and, by Theorem 29, its inverse is also a morphism; hence, $M_1 \otimes M_2 = M$ and $y_1 \otimes y_2 = y$.

PROPOSITION 31. *For any objects (M_1, y_1), (M_2, y_2) in MS, $M_1 \otimes M_2 \cong M_2 \otimes M_1$.*

PROPOSITION 32. *For any object (M, y) of MS, $S \otimes M \cong M \cong M \otimes S$.*

Proof. Let $\mu: S \otimes M \to M$ be the map given by $\mu(s \otimes m) = sm$. Note that $\mu = \eta_S(1)^{-1}$. μ is the required isomorphism.

PROPOSITION 33. *$\otimes$ is associative.*

Proof. By Theorem 10 **[17]** it is enough to show that $\mathrm{Hom}(M_1 \otimes M_2, M_3) \cong \mathrm{Hom}(M_1, \mathrm{Hom}(M_2, M_3))$. By Theorem 27.9 **[14]**, the adjoint situation, $(\eta, \delta): \_ \otimes M \dashv \mathrm{Hom}(M, \_)$, gives a bijection $\alpha: \mathrm{Hom}(M_1 \otimes M_2, M_3) \to \mathrm{Hom}(M_1, \mathrm{Hom}(M_2, M_3))$ defined by the rule, $(\alpha(f)(m_1))(m_2) = f(m_1 \otimes m_2)$, for all $f \in \mathrm{Hom}(M_1 \otimes M_2, M_3)$. α is the required isomorphism since, for all $m_1 \otimes m_2 \in M_1 \otimes M_2$ and all $U_{y_3} \subseteq M_3$, $\alpha([m_1 \otimes m_2, U_{y_3}]) = [m_1, [m_2, U_{y_3}]]$.

PROPOSITION 34. *$\_ \otimes M$ preserves colimits. In particular, $\_ \otimes M$ preserves coproducts.*

PROPOSITION 35. *Let $\varphi \in \mathrm{Hom}(M_1, M_2)$. Then, for any object (M, y) in MS $\varphi \otimes M: M_1 \otimes M \to M_2 \otimes M$ is the map, $u_1 \otimes u \to \varphi(u_1) \otimes u$.*

Proof.

$$\begin{array}{ccc} M_1 & \xrightarrow{\quad \eta_{M_1} \quad} & \mathrm{Hom}(M, M_1 \otimes M) \\ \varphi \downarrow & & \downarrow \mathrm{Hom}(M, \varphi \otimes N) \\ M_2 & \xrightarrow{\quad \eta_{M_2} \quad} & \mathrm{Hom}(M, M_2 \otimes M) \end{array}$$

The adjoint situation $(\eta, \delta): \_ \otimes M \dashv \mathrm{Hom}(M, \_)$ makes the diagram commute for each object M. Thus, for each $m_1 \in M_1$, $\mathrm{Hom}(M, \varphi \otimes M)(\eta_{M_1}(m_1)) = \eta_{M_2}(\varphi(m_1))$; i.e., for all $m \in M$,

$$(\varphi \otimes M)(\eta_{M_1}(m_1))(m) = (\varphi \otimes M)(m_1 \otimes m)$$
$$= \eta_{M_2}(\varphi(m_1))(m) = \varphi(m_1) \otimes m.$$

NOTATION. $\varphi \otimes 1$ will sometimes be written instead of $\varphi \otimes M$ in cases where no confusion will result.

PROPOSITION 36. *For any object* (M, y), *the functor* $\_ \otimes M$ *preserves epimorphisms.*

DEFINITION. An object (m, y) is *Flat* if the functor $\_ \otimes M$ preserves monomorphisms.

PROPOSITION 37. *S is a flat object of MS.*

PROPOSITION 38. *Let* $\{(M_j, y_j) \mid j \in J\}$ *be a family of objects of* MS. *Then* $(\Sigma M_j, \Sigma y_j)$ *is flat iff* (M_j, y_j) *is flat for each* $j \in J$.

PROPOSITION 39. *Every projective object of MS is flat.*

8. Restriction and extension of scalars.

REMARKS. Let $\varphi: (S, x) \to (S', x')$ be a morphism of ideal systems; i.e., $\varphi(st) = \varphi(s)\varphi(t)$, for all $s, t \in S$, and $\varphi(A_x) \subseteq (\varphi(A))_{x'}$, for all $A \subseteq S$. Then any object (M', y') of MS' can be considered as an object of MS in the following manner: for each $s \in S$, $u' \in M'$, define $su' = \varphi(s)u'$. It is easy to verify that, with this S-set structure, (M', y') is an object of MS (the closure system y' does not change). This process is usually referred to as *restriction of scalars*. Let $\xi' \in \mathrm{Hom}_{S'}(M'_1, M'_2)$. If we restrict scalars as described above, we can consider both objects M'_1 and M'_2 as objects of MS and then ξ' becomes an S-morphism with its S-map structure defined by the rule, $\xi'(su') = \xi'(\varphi(s)u')$ for all $s \in S$.

PROPOSITION 40. *Let* $\varphi: (S, x) \to (S', x')$ *be a morphism of ideal systems. Then the process of restriction of scalars determines a faithful, covariant functor,* $R_\varphi: MS' \to MS$, *which preserves monomorphisms and epimorphisms.*

DEFINITION. A functor which preserves monomorphisms and epimorphisms shall be called *exact.*

PROPOSITION 41. *Let* $\varphi: (S, x) \to (S', x')$ *be a morphism of ideal systems. Then the functor* $R_\varphi: MS' \to MS$ *has a left adjoint* $E_\varphi: MS \to MS'$ *given by the rule,* $E_\varphi(M) = M \otimes R_\varphi S'$ *for all objects*

(M, y) of MS $[E_\varphi(M)$ is given S'-set structure by defining for each $s' \in S'$ and each $u \otimes t' \in E_\varphi(M)$, $s'(u \otimes t') = u \otimes s't']$ and $E_\varphi(\delta) = \delta \otimes R_\varphi S'$ for any morphism $\delta \in \mathrm{Hom}_S(M_1, M_2)$.

Proof. By Theorem 27.9 **[14]**, it is sufficient to show that the two set-valued bifunctors, $\mathrm{hom}_{S'}(E_\varphi \_, \_)$ and $\mathrm{hom}_S(\_, R_\varphi \_)$ are naturally isomorphic. Thus, let (M, y) be an object of MS and (M', y') be an object of MS', and define β: $\mathrm{hom}(E_\varphi M, M') \to \mathrm{hom}(M, R_\varphi M')$ by the rule: $\beta(f)(m) = f(m \otimes 1')$ $\forall m \in M$. Then β is a bijection.

REMARK. The functor E_φ: $MS \to MS'$ is usually referred to as *extension of scalars.*

PROPOSITION 42. *Let φ: $(S, x) \to (S', x')$ be a morphism of ideal systems. Then the functor R_φ: $MS' \to MS$ has a right adjoint H_φ: $MS \to MS'$ given by the rule: $H_\varphi(M) = \mathrm{Hom}_S(R_\varphi S', M)$ $\forall$ objects (M, y) of MS $[H_\varphi(M)$ becomes an object of MS' by defining for each $s' \in S'$ and each $\sigma \in H_\varphi(M)$, $(s'\sigma)(t') = \sigma(s't')$ $\forall t' \in R_\varphi S']$ and $H_\varphi(\lambda) =$* $\mathrm{Hom}_S(R_\varphi S', \lambda)$ $\forall \lambda \in \mathrm{Hom}_S(M_1, M_2)$.

Proof. By Theorem 27.9 **[14]**, it is sufficient to show that the two set-valued bifunctors, $\mathrm{hom}_S(R_\varphi \_, \_)$ and $\mathrm{hom}_{S'}(\_, H_\varphi \_)$ are naturally isomorphic. Thus, let (M, y) be an object of MS and (M', y') be an object of MS', and define γ: $\mathrm{hom}(R_\varphi M', M) \to \mathrm{hom}(M', H_\varphi M)$ by the rule: $(\gamma(g)(m'))(s') = g(s'm)$ $\forall s' \in S'$ and $\forall g \in \mathrm{hom}(R_\varphi M', M)$. Then γ is a bijection.

REMARK. Let φ: $(S, x) \to (S', x')$ be a morphism of ideal systems. Then for any object (M, y) of MS and any object (M', y') of MS', $M \otimes R_\varphi M'$ may be regarded as an object of MS' if it is given S'-set structure in the following manner: $s'(m \otimes m') = m \otimes s'm'$ $\forall s' \in S'$ and $\forall m \otimes m' \in M \otimes R_\varphi M'$.

PROPOSITION 43. *Let φ: $(S, x) \to (S', x')$ be a morphism of ideal systems. Let (M, y) be an object of MS and let (M', y') be an object of MS'. Then (in MS) $M \otimes R_\varphi M' \cong R_\varphi(E_\varphi M \otimes' M')$* [where $\otimes'$ indicates that the tensor product is formed in MS'].

Proof. Let α: $M \times R_\varphi M' \to R_\varphi(E_\varphi M \otimes' M')$ be defined by the rule: $\alpha(u, u') = (u \otimes 1') \otimes' u'$. Then α is S-bilinear; hence, there exists an S-morphism $\bar{\alpha}$: $M \otimes R_\varphi M' \to R_\varphi(E_\varphi M \otimes' M')$ such that the diagram commutes; i.e., $\bar{\alpha}(m \otimes m') = \alpha(m, m') = (m \otimes 1') \otimes' m'$ [ψ is the canonical bilinear map]. Note $\bar{\alpha}(m \otimes s'm') = (m \otimes s') \otimes' m'$.

$$\begin{array}{ccc} M\otimes R_\varphi M' & \xrightarrow{\bar{\alpha}} & R_\varphi(E_\varphi M\otimes' M') \\ \psi\Big\uparrow & \nearrow_{\alpha} & \\ M\times R_\varphi M' & & \end{array}$$

Let $\hat{\alpha}: M\otimes R_\varphi M'\to E_\varphi M\otimes' M'$ denote $\bar{\alpha}$ regarded as an S'-morphism. Now, for each $m'\in R_\varphi M'$, define $\underline{m}': M\times R_\varphi S'\to M\otimes R_\varphi M'$ by the rule: $\underline{m}'(m,s')=m\otimes s'm'$. Then $\underline{m}'$ is S-bilinear; hence, there exists an S-morphism $\hat{\underline{m}}': M\otimes R_\varphi S'\to M\otimes R_\varphi M'$ such that $\hat{\underline{m}}'(m\otimes s')=m\otimes s'm'$.

$$\begin{array}{ccc} M\otimes R_\varphi S' & \xrightarrow{\hat{\underline{m}}'} & M\otimes R_\varphi M' \\ \psi\Big\uparrow & \nearrow_{\underline{m}'} & \\ M\times R_\varphi S' & & \end{array}$$

Actually, $\hat{\underline{m}}'$ is an S'-morphism with domain $E_\varphi M$ and codomain $M\otimes R_\varphi M'$, the latter regarded as an object of MS'. Let $\beta: E_\varphi M\times M'\to M\otimes R_\varphi M'$ be defined by the rule: $\beta(m\otimes s', m')=m\otimes s'm'$. Note that β is well defined since, for each fixed $m'\in R_\varphi M'$, $\beta(\_\,, m')=\hat{\underline{m}}'$ and, hence, does not depend upon the choice of representative of $m\otimes s'$. Since β is S'-bilinear, it follows that there is an S'-morphism $\hat{\beta}: (M\otimes R_\varphi S')\otimes M'\to M\otimes R_\varphi M'$ such that $\hat{\beta}((m\otimes s')\otimes' m')=m\otimes s'm'$.

$$\begin{array}{ccc} (M\otimes R_\varphi S')\otimes' M' & \xrightarrow{\hat{\beta}} & M\otimes R_\varphi M' \\ \psi\Big\uparrow & \nearrow_{\beta} & \\ (M\otimes R_\varphi S')\times M' & & \end{array}$$

Thus, we have produced S'-morphisms, $\hat{\alpha}: M\otimes R_\varphi M'\to E_\varphi M\otimes' M'$ and $\hat{\beta}: E_\varphi M\otimes' M'\to M\otimes R_\varphi M'$ which are inverses of one another; i.e., $M\otimes R_\varphi M'\cong E_\varphi M\otimes M'$ in MS'. It follows that $M\otimes R_\varphi M'\cong R_\varphi(E_\varphi M\otimes' M')$ in MS.

PROPOSITION 44. *Let $\varphi: (S,x)\to(S',x')$ be a morphism of ideal systems. Then the two MS-valued bifunctors, $\_\otimes R_\varphi\,\_: MS\times MS'\to MS$ and $R_\varphi(E_\varphi\,\_\otimes'\,\_): MS\times MS'\to MS$ are naturally isomorphic.*

PROPOSITION 45. *Let $\varphi: (S,x)\to(S',x')$ be a morphism of ideal systems. Suppose that $R_\varphi S'$ is a flat object of MS. Then $R_\varphi M'$ is flat in MS for all flat objects (M',y') in MS'.*

Proof. The functor $\_\otimes R_\varphi M'\colon MS \to MS$ preserves monomorphisms whenever (M', y') is a flat object of MS'.

PROPOSITION 46. *Let $\varphi\colon (S,x)\to(S',x')$ be a morphism of ideal systems. Suppose that (M,y) is a flat object of MS. Then $E_\varphi M$ is flat in MS'.*

Proof. Let $\xi\colon M_1'\to M_2'$ be an S'-monomorphism. Then by Proposition 40, $R_\varphi\xi\colon R_\varphi M_1'\to R_\varphi M_2'$ is an S-monomorphism; hence, since M is flat in MS, $M\otimes R_\varphi\xi\colon M\otimes R_\varphi M_1'\to M\otimes R_\varphi M_2'$ is an S-monomorphism. By Proposition 44, we have that $R_\varphi(E_\varphi M\otimes'\xi)\colon R_\varphi(E_\varphi M\otimes' M_1')\to R_\varphi(E_\varphi M\otimes' M_2')$ is an S-monomorphism; hence, $E_\varphi M\otimes'\xi\colon E_\varphi M\otimes' M_1'\to E_\varphi M\otimes' M_2'$ is an S'-monomorphism (since R_φ is faithful).

9. Monads and algebras in $M\{0,1\}$.

NOTATION. We shall denote the category $M\{0,1\}$, simply, $\mathfrak{B}$. For any ideal system, (S,x), $\tau\colon\{0,1\}\to S$ will denote the map, $\tau(0)=0$, $\tau(1)=1$. Clearly τ is a morphism of ideal systems [$\{0,1\}$ is given the obvious (s-system) closure system]. In the sequel we will denote $R_\tau S$, simply, S.

THEOREM 47. *For any ideal system (S,x), $\bar{K}_S=(K_S,\eta,\mu)$ is a monad in $\mathfrak{B}$, where $K_S\colon\mathfrak{B}\to\mathfrak{B}$ is the functor, $\_\otimes S$, and $\eta\colon 1_{\mathfrak{B}}\to K_S$ is the natural transformation given by, $\eta_M(m)=m\otimes 1$, and $\mu\colon K_SK_S\to K_S$ is the natural transformation, $\mu_M\colon (M\otimes S)\otimes S\to M\otimes S$ given by $\mu_M((m\otimes s)\otimes t)=m\otimes st$.*

Proof. The "unit," η, and the "multiplication," μ, make the following diagrams commute:

$$\begin{array}{ccccc} K_S & \xrightarrow{\eta K_S} & K_SK_S & \xleftarrow{K_S\eta} & K_S \\ & \diagdown\!\!\diagdown & \Big\downarrow{\scriptstyle\mu} & \diagup\!\!\diagup & \\ & & K_S & & \end{array} \qquad \begin{array}{ccc} K_SK_SK_S & \xrightarrow{\mu K_S} & K_SK_S \\ {\scriptstyle K_S\mu}\Big\downarrow & & \Big\downarrow{\scriptstyle\mu} \\ K_SK_S & \xrightarrow[\mu]{} & K_S \end{array}$$

where

$$(\eta K_S)_M=\eta_{M\otimes S}\colon M\otimes S\to(M\otimes S)\otimes S$$

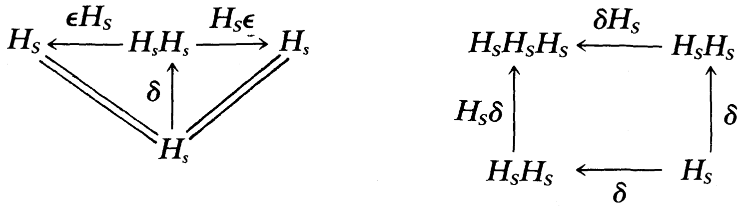
$$(K_S\eta)_M=\eta_M\otimes 1\colon M\otimes S\to(M\otimes S)\otimes S$$

$$(\mu K_S)_M=\mu_{M\otimes S}\colon ((M\otimes S)\otimes S)\otimes S\to(M\otimes S)\otimes S$$

$$(K_S\mu)_M=\mu_M\otimes 1\colon ((M\otimes S)\otimes S)\otimes S\to(M\otimes S)\otimes S.$$

THEOREM 48. *For any ideal system,* (S, x), $\bar{H}_S = (H_S, \epsilon, \delta)$ *is a comonad in* $\mathfrak{B}$, *where* $H_S\colon \mathfrak{B} \to \mathfrak{B}$ *is the functor,* $\mathrm{Hom}(S, \_)$, *and* $\epsilon\colon H_S \to 1_{\mathfrak{B}}$ *is the natural transformation,* $\epsilon_M\colon \mathrm{Hom}(S, M) \to M$, *given by,* $\epsilon_M(f) = f(1)$, *and* $\delta\colon H_S \to H_S H_S$ *is the natural transformation,* $\delta_M\colon \mathrm{Hom}(S, M) \to \mathrm{Hom}(S, \mathrm{Hom}(S, M))$, *given by,* $(\delta_M(f)(s))(t) = (sf)(t)$ *for all* $t \in S$.

Proof. One must verify here that the following diagrams commute:



where

$$(\epsilon H_S)_M = \epsilon_{\mathrm{Hom}(S,M)}\colon \mathrm{Hom}(S, \mathrm{Hom}(S, M)) \to \mathrm{Hom}(S, M)$$

$$(H_S\epsilon)_M = \mathrm{Hom}(S, \epsilon_M)\colon \mathrm{Hom}(S, \mathrm{Hom}(S, M)) \to \mathrm{Hom}(S, M)$$

$$(\delta H_S)_M = \delta_{\mathrm{Hom}(S,M)}\colon \mathrm{Hom}(S, \mathrm{Hom}(S, M)) \to \mathrm{Hom}(S, \mathrm{Hom}(S, \mathrm{Hom}(S, M)))$$

$$(H_S\delta)_M = \mathrm{Hom}(S, \delta_M)\colon \mathrm{Hom}(S, \mathrm{Hom}(S, M)) \to \mathrm{Hom}(S, \mathrm{Hom}(S, \mathrm{Hom}(S, M))).$$

REMARKS. Let (S, x) be an ideal system and let $\mathfrak{B}^S$ denote the category of $\bar{K}_S$-algebras. Let $G\colon MS \to \mathfrak{B}^S$ be defined as follows: For each object (M, y) of MS, $G(M) = \langle R_\tau M, h\rangle$, where $h\colon R_\tau M \otimes S \to R_\tau M$ is the $\mathfrak{B}$-morphism $m \otimes s \to sm$. For each S-morphism $f\colon M \to M'$, $G(f) = R_\tau f\colon R_\tau M \to R_\tau M'$. Then $G(f)$ is a $\mathfrak{B}^S$-morphism and, hence, G is a (covariant) functor.

Now let $F\colon \mathfrak{B}^S \to MS$ be defined as follows: For each object $\langle M, h\rangle$ of $\mathfrak{B}^S$ (where (M, y) is an object of $\mathfrak{B}$), $F(\langle M, h\rangle) = (\bar{M}, y)$, where $\bar{M} = M$, equipped with the S-multiplication, $sm = h(s \otimes m)$ for all $s \in S$, $m \in M$. For any $\mathfrak{B}^S$-morphism $g\colon \langle M, h\rangle \to \langle M', h'\rangle$, $F(g) = \bar{g}$, where $\bar{g} = g$, converted into an S-map by taking $\bar{g}(sm) = g(h(s \otimes m))$ for all $s \in S$, $m \in M$.

THEOREM 49. [Monadicity]. *For any ideal system,* (S, x), $\mathfrak{B}^S$ *is isomorphic to* MS.

Proof. With notation as in the remarks above we need only show that $FG = 1_{MS}$ and $GF = 1_{\mathfrak{B}^S}$. To show $FG = 1_{MS}$: since for any object (M, y) of MS, $FG(M) = F(\langle R_T M, h\rangle) = (\overline{R_T M}, y)$, where $\overline{R_T M} = R_T M$, endowed with an S-multiplication which is derived from the map h ; i.e., $(\overline{R_T M}, y) = (M, y)$. For each object $\langle M, h\rangle$ of $\mathfrak{B}^S$, $GF(\langle M, h\rangle) = G((\bar{M}, y)) = \langle R_T\bar{M}, \bar{h}\rangle$, where $\bar{h}: R_T\bar{M} \otimes S \to R_T\bar{M}$ is defined by the rule, $\bar{h}(m \otimes s) = sm = h(m \otimes s)$. Thus, $\bar{h} = h$, and, since it is clear that $R_T\bar{M} = M$, it follows that $GF(\langle M, h\rangle) = \langle M, h\rangle$.

Concluding remarks. The monads and comonads constructed above provide the tools with which resolutions and derived functors can be constructed which, in turn, lead to a (co)homology theory for $\mathfrak{B}$. The category of pointed topological spaces and basepoint preserving maps, PTOP, can be found in $\mathfrak{B}$. In fact, the inclusion functor PTOP$\to \mathfrak{B}$ is a full, faithful embedding.

References

1. M. F. Atiyah, and I. G. MacDonald, *Introduction to Commutative Algebra*, Addison-Wesley, (1969).
2. Karl E. Aubert, *Theory of X-ideals*, Acta Mathematica, **107** (1962), 1–52.
3. ———, *Ideal systems and lattice theory I*, Algebra Universalis, **1** (1971), No. 2, 204–213.
4. ———, *Ideal Systems and Lattice Theory II*, Tufts University (mimeographed notes) 1971.
5. Karl E. Aubert, and G. Gismarvik, *Chinese Remainder Theorems in Ideal Systems*, Tufts University (mimeographed notes) 1971.
6. Karl E. Aubert, *Un Nouvel Axiome dans la Théorie, des Systèmes d'idéaux*, C. R. Acad. Sci. Paris, t, **272** (1971), 422–425.
7. ———, *Localisation Dans les Systèmes D'Idéaux*, C. R. Acad. Sci. Paris, t, **272** (1971), 465–468.
8. Karl E. Aubert, and Erling R. Hansén, *Systèmes De Modules*, C. R. Acad. Sci. Paris, t, **272** (1971), 525–528.
9. ———, *Additive Ideal Systems*, J. Algebra, **18** (1971), 511–528.
10. M. Barr, and J. Beck, *Homology and Standard Constructions.* Seminar on Triples and Categorical Theory. Springer-Verlag, **80** (1969), 245–335.
11. Henri Cartan, and Samuel Eilenberg, *Homological Algebra*, Princeton University, (1956).
12. J. Duskin, *$K(\pi, n)$-torsors and the interpretation of "triple" cohomology*, Proc. Nat. Acad. Sci. USA, **71** (1974), 2554–2557.
13. Paul Ezust, *Systems of Modules over Systems of Ideals*, Thesis, Tuft University, 1975.
14. Horst Herrlich, and G. E. Strecker, *Category Theory*, Allyn and Bacon, (1973).
15. P. J. Hilton, and U. Stammbach, *A Course in Homological Algebra*, Springer-Verlag, (1970).
16. Peter Hilton, *Lectures in Homological Algebra*, Regional Conference Series in Mathematics #8. Am. Math. Soc., (1970).
17. G. M. Kelly, *Tensor products in categories*, J. Algebra, **2** (1965), 15–37.
18. F. E. J. Linton, *Applied Functorial Semantics*, Seminar on Triples and Categorical Theory, Springer-Verlag, **80** (1969), 53–74.
19. ———, *An outline on Functorial Semantics*, Seminar on Triples and Categorical Theory, Springer-Verlag, **80** (1969).
20. Saunders MacLane, *Homology*, Springer-Verlag, (1963).

21. ———, *Categories for the Working Mathematician*, Springer-Verlag, (1971).

22. Barry Mitchell, *Theory of Categories*, Academic Press, (1965).

23. Bodo Pareigis, *Categories and Functors*, Academic Press, (1970).

24. Bo Strenstrom, *Flatness and Localization over Monoids*, Math. Nachr., **48** (1971), 315–334.

25. F. Umer, *On Cotriples and Andre (co)Homology, Their Relationship with Classical Homological Algebra*, Seminar on Triples and Category Theory, Springer-Verlag, **80** (1969), 376–398.

Received June 30, 1975 and in revised form November 30, 1976. The results presented herein are drawn from the author's doctoral dissertation which was written at Tufts University under the direction of Professor George F. Leger.

SUFFOLK UNIVERSITY
BOSTON, MA 02114