PACIFIC JOURNAL OF MATHEMATICS
Vol 68 No 2, 1977

THE ESSENTIAL UNIQUENESS OF BOUNDED NONOSCILLATORY SOLUTIONS OF CERTAIN EVEN ORDER DIFFERENTIAL EQUATIONS

G. J. ETGEN AND W. E. TAYLOR, JR.

Let n be a positive integer, let p be a positive continuous function on $[0, \infty)$, and consider the $2n$th order linear differential equation

$$(1) \qquad u^{(2n)} - p(x)u = 0.$$

It is well known that this equation has a solution $w = w(x)$ satisfying

$$(2) \qquad (-1)^k w^{(k)}(x) > 0, \qquad k = 0, 1, \cdots, 2n-1,$$

on $[0, \infty)$, and it is clear that w is positive and bounded. The purpose of this paper is to investigate the essential uniqueness of the solution w, where the statement "w is essentially unique" means that if y is any other solution of (1) which satisfies (2), then $y = kw$ for some nonzero constant k.

In addition to having solutions which satisfy (2), it is easy to show that equation (1) has solutions $z = z(x)$ satisfying

$$(3) \qquad z^{(k)}(x) > 0, \qquad k = 0, 1, \cdots, 2n-1,$$

on $[a, \infty)$ for some $a \geqq 0$. For some recent results concerning the behavior of solutions of (1) satisfying either (2) or (3), the reader is referred to the work of D. L. Lovelady **[6]**, and T. T. Read **[7]**.

A solution of (1) which satisfies (2) is said to be *strongly decreasing*, and a solution satisfying (3) is said to be *strongly increasing*. If y is a nontrivial solution of (1), then y is *oscillatory* if it has infinitely many zeros on $[0, \infty)$. Equivalently, y is oscillatory if the set of zeros of y is not bounded above. The differential equation (1) is oscillatory if it has at least one nontrivial oscillatory solution. Hereafter, the term "solution of (1)" shall be interpreted to mean "nontrivial solution." A solution of (1) which is not oscillatory is called *nonoscillatory*. Clearly, any solution satisfying either (2) or (3) is nonoscillatory. We shall say that equation (1) has *property* (H) if every nonoscillatory, eventually positive solution satisfies either (2) or (3).

S. Ahmad **[1]** has studied (1) in the case $n = 2$, and he has shown that (1) is oscillatory if and only if it has property (H). While this result is not known in general, Lovelady [**6**, Theorem 2] has shown that property (H) implies the oscillation of (1). Read [**7**] and G. W. Johnson [**4**] have obtained some results on the asymptotic properties of solutions of (1). In particular, they have obtained criteria which imply that any solution w satisfying (2) has the property $\lim_{x\to\infty} w(x)=0$. Finally, we refer to the work of G. D. Jones and S. M. Rankin [**5**] where the problem of the essential uniqueness of a solution w satisfying (2) was considered for the case $n=2$.

2. Preliminary results.

Let $\mathcal{S}$ denote the $2n$-dimensional vector space of solutions of equation (1). Our first result is essential in the work which follows. Since the proof is straightforward, using well known techniques, it will be omitted.

LEMMA 2.1. *If $y\in\mathcal{S}$ and $y^{(k)}(a)\geqq 0$, $k=0,1,\cdots,2n-1$, for some $a\geqq 0$, with at least one inequality being strict, then $y^{(k)}(x)>0$, $k=0,1,\cdots,2n-1$, on (a,∞) and*

$$\lim_{x\to\infty} y^{(k)}(x)=\infty,\qquad k=0,1,\cdots,2n-2.$$

If $z\in\mathcal{S}$ and $(-1)^k z^{(k)}(b)\geqq 0$, $k=0,1,\cdots,2n-1$, for some $b>0$, with at least one inequality being strict, then $(-1)^k z^{(k)}(x)>0$ on $[0,b)$.

Let J be the function defined on $\mathcal{S}\times\mathcal{S}$ by

$$J(u,v)(x)=\sum_{k=0}^{2n-1}(-1)^k v^{(k)}(x)u^{(2n-k-1)}(x) \tag{4}$$

For any pair of functions $u,v\in\mathcal{S}$, it is easy to verify by differentiating $J(u,v)$ that $J'(u,v)(x)=0$ for all $x\in[0,\infty)$. Thus $J(u,v)\equiv c$, a constant on $[0,\infty)$. The case where $J(u,v)\equiv 0$ shall be denoted by $u\perp v$. Fix $y\in\mathcal{S}$. Following the ideas introduced by J. M. Dolan in [**2**], we define the subset $\mathcal{S}(y)$ of $\mathcal{S}$ by

$$\mathcal{S}(y)=\{z\in\mathcal{S}\,|\,y\perp z\}$$

Let $u_1,u_2,\cdots,u_{2n-1}$ be $2n-1$ solutions of equation (1), and let $W(u_1,u_2,\cdots,u_{2n-1})$ denote their Wronskian. It is well known that W is a solution of (1), and that W is nontrivial if and only if the solutions are linearly independent. Let $y\in\mathcal{S}$ and let $T[y,u_1,u_2,\cdots,u_{2n-1}]$ denote the Wronskian of the $2n$ solutions. Then, by expanding T along its first column, we get the following relationship between T, W and the function J

$$T[y,u_1,u_2,\cdots,u_{2n-1}]=J[y,W(u_1,u_2,\cdots,u_{2n-1})] \tag{5}$$

THEOREM 2.2. *Let $y \in \mathcal{S}$. Then the following hold.*

(i) *$\mathcal{S}(y)$ is a $(2n-1)$-dimensional subspace of $\mathcal{S}$ and $y \in \mathcal{S}(y)$.*

(ii) *If $z \in \mathcal{S}(y)$, and y and z are linearly independent, then there exists a solution $u \in \mathcal{S}(y)$ such that $J(u,z) \neq 0$.*

(iii) *If $\{u_1, u_2, \cdots, u_{2n-1}\}$ is a basis for $\mathcal{S}(y)$, then $W(u_1, u_2, \cdots, u_{2n-1}) = ky$ for some nonzero constant k.*

(iv) *If $v \in \mathcal{S}$, then $\mathcal{S}(y) \cap \mathcal{S}(v)$ has dimension $2n-1$ if and only if y and v are linearly dependent; otherwise $\mathcal{S}(y) \cap \mathcal{S}(v)$ has dimension $2n-2$.*

Proof. Part (i) is easy to verify using (4) and the definition of $\mathcal{S}(y)$.

(ii) Let $z \in \mathcal{S}(y)$ be independent of y. Suppose z has a zero of multiplicity k, $1 \leqq k \leqq 2n-1$, at some point $c \geqq 0$. Since $\mathcal{S}(y)$ has dimension $2n-1$ we can construct a solution $u \in \mathcal{S}(y)$ such that

$$u(c) = u'(c) = \cdots = u^{(2n-k-2)}(c) = 0 = u^{(2n-k+1)}(c) = \cdots = u^{(2n-1)}(c) = 0,$$

$$u^{(2n-k-1)}(c) = 1, \quad u^{(2n-k)}(c) = \gamma,$$

where γ is some constant. Then, from (4), $J(u,z) = z^{(k)}(c) \neq 0$. If $z \neq 0$ on $[0,\infty)$, then choose a point c such that $y(c) \neq 0$, and choose $m \neq 0$ such that $y(c) - mz(c) = 0$. Let $v = y - mz$. Then $v \in \mathcal{S}(y)$ and $v \not\equiv 0$ since y and z are independent. Now, we can repeat the argument above to determine a solution $u \in \mathcal{S}(y)$ such that $J(u,v) \neq 0$. Since $J(u,v) = J(u, y - mz) = -mJ(u,z)$, we conclude that $J(u,z) \neq 0$.

(iii) Let $\{u_1, u_2, \cdots, u_{2n-1}\}$ be a basis for $\mathcal{S}(y)$. Since $y \in \mathcal{S}(y)$, $y = \sum_{i=1}^{2n-1} c_i u_i$ and thus

$$0 = T[y, u_1, u_2, \cdots, u_{2n-1}] = J[y, W(u_1, u_2, \cdots, u_{2n-1})]$$

Hence the solution $W(u_1, u_2, \cdots, u_{2n-1})$ is an element of $\mathcal{S}(y)$. The same reasoning shows that

$$J[z, W(u_1, u_2, \cdots, u_{2n-1})] = 0$$

for all $z \in \mathcal{S}(y)$, and we can conclude, from (ii), that $W(u_1, u_2, \cdots, u_{2n-1}) = ky$.

Part (iv) is an immediate consequence of either (ii) or (iii). This completes the proof of the theorem.

We now consider the properties of the subspace $\mathcal{S}(w)$ in the case where w satisfies (2).

THEOREM 2.3. *Assume that equation* (1) *has property* (H), *and suppose $w \in \mathcal{S}$ satisfies* (2). *Then*:

(i) *If* $y \in \mathcal{S}(w)$, *then either* y *satisfies* (2), *or* y *is oscillatory*

(ii) *If* $y \in \mathcal{S}(w)$ *and* $y^{(k)}(a) = 0$ *for some* $a \geqq 0$ *and some nonnegative integer* k, $0 \leqq k \leqq 2n-1$, *then* y *is oscillatory*

(iii) *If* $z \in \mathcal{S}$ *and* $z \notin \mathcal{S}(w)$, *then* z *is unbounded.*

Proof. (i) Let $y \in \mathcal{S}(w)$ and assume that y is nonoscillatory with $y > 0$ on $[a, \infty)$, $a \geqq 0$. Suppose y does not satisfy (2). Then y satisfies (3) and there is a number $b \geqq a$ such that $y^{(k)}(x) > 0$, $k = 0, 1, \cdots, 2n-1$, on $[b, \infty)$. By evaluating $J(w, y)$ at any $x \geqq b$, we have that $J(w, y) \neq 0$, contradicting the fact that $y \in \mathcal{S}(w)$.

Part (ii) follows immediately from (i).

(iii) Let $z \in \mathcal{S}$ and suppose $z \notin \mathcal{S}(w)$. Fix any point $a \geqq 0$. Since $\mathcal{S}(w)$ has dimension $2n-1$ we can construct a basis for $\mathcal{S}(w)$ consisting of w and $2n-2$ solutions $u_1, u_2, \cdots, u_{2n-2}$ such that u_k has a zero of multiplicity k at $x = a$, $k = 1, 2, \cdots, 2n-2$. By (ii) every linear combination of the solutions $u_1, u_2, \cdots, u_{2n-2}$ is oscillatory. Let y be the solution of (1) determined by the initial conditions $y(a) = y'(a) = \cdots = y^{(2n-2)}(a) = 0$, $y^{(2n-1)}(a) = 1$. Then y satisfies (3) on $[b, \infty)$ for every $b \geqq a$. Thus $y \notin \mathcal{S}(w)$ and the set $\{y, w, u_1, u_2, \cdots, u_{2n-2}\}$ is a basis for $\mathcal{S}$. Now

$$z = cy + dw + \sum_{i=1}^{2n-2} c_i u_i,$$

where $c \neq 0$. Since w is bounded, and $\Sigma_{i=1}^{2n-2} c_i u_i$ is oscillatory, we can conclude that z is unbounded.

Our next result has appeared in [**5**, Lemma 4] for the case $n = 2$. The proof is straightforward and, consequently, it will be omitted.

LEMMA 2.4. *Let* $\{u_1, u_2, \cdots, u_{2n}\}$ *be a basis for* $\mathcal{S}$. *Then there exists a basis* $\{z_1, z_2, \cdots, z_{2n}\}$ *for* $\mathcal{S}$ *and* $2n$ *nonzero constants* $k_1, k_2, \cdots, k_{2n}$, *such that*

$$u_i \equiv k_i W(z_1, z_2, \cdots, z_{i-1}, z_{i+1}, \cdots, z_{2n}), \qquad i = 1, 2, \cdots, 2n.$$

3. Main results.

It is easy to see that equation (1) has no oscillatory solutions when $n = 1$. Also, it is easy to show that the nonoscillatory solution w satisfying (2) is essentially unique in this case. Our first result shows that this situation holds in general.

THEOREM 3.1. *If equation* (1) *has no oscillatory solutions, then the nonoscillatory solution* w *satisfying* (2) *is essentially unique.*

Proof. Suppose that (1) has two linearly independent solutions w and v satisfying (2). Fix any $a \geqq 0$ and choose k such that

$w(a)-kv(a)=0$. Let y be the solution given by $y(x)=w(x)-kv(x)$. Since y is nonoscillatory, we shall assume that $y>0$, and that $\Pi_{k=1}^{2n-1} y^{(k)} \neq 0$ on $[b,\infty)$, $b>a$. Then $y^{(2n)}=py>0$. Since each of w and v is bounded on $[0,\infty)$, y is bounded and we can conclude that no two consecutive derivatives $y^{(k)}$, $y^{(k+1)}$, $1 \leqq k \leqq 2n-2$, can have the same sign on $[b,\infty)$. But this implies

$$sgn y = sgn y'' = \cdots = sgn y^{(2n)} \neq sgn y' = sgn y''' = \cdots = sgn y^{(2n-1)}$$

on $[b,\infty)$ and, with Lemma 2.1, contradicts the fact that $y(a)=0$.

We now consider the case where equation (1) is oscillatory. The next result gives a connection between the essential uniqueness of the solution w satisfying (2) and the maximum number of linearly independent oscillatory solutions in $\mathcal{S}$.

THEOREM 3.2. *Assume that equation* (1) *has property* (*H*). *The following two statements are equivalent*:

(a) *The solution* w *of* (1) *satisfying* (2) *is essentially unique.*

(b) *Equation* (1) *has at most* $2n-1$ *linearly independent oscillatory solutions.*

Proof. To show that (a) implies (b) we use a simple extension of the proof of the corresponding result for the case $n=2$ in [**5**, Theorem 4]. In particular, assume that w is essentially unique, and suppose $\mathcal{S}$ has a basis consisiting of $2n$ oscillatory solutions $u_1, u_2, \cdots, u_{2n}$. Using Lemma 2.4, let $\{z_1, z_2, \cdots, z_{2n}\}$ be a basis for $\mathcal{S}$ such that for each i, $1 \leqq i \leqq 2n$,

$$W(z_1, \cdots, z_{i-1}, z_{i+1}, \cdots, z_{2n}) = k_i u_i.$$

Consider the solution $u_1 = k_1 W(z_2, z_3, \cdots, z_{2n})$. Since u_1 is oscillatory, there is an increasing sequence $\{x_i\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty} x_i = \infty$ and $u_1(x_i)=0$ for all i. Therefore, for each positive integer i there are $2n-1$ constants $c_{2i}, c_{3i}, \cdots, c_{2n,i}$ such that $\Sigma_{j=2}^{2n} c_{ji}^2 = 1$ and the solution v_{1i},

$$v_{1i} = \sum_{j=2}^{2n} c_{ji} z_j,$$

has a zero of order $2n-1$ at $x=x_i$. Because the sequences $\{c_{ji}\}$, $j=2,3,\cdots,2n$, are bounded, we can assume, without loss of generality, that $\lim_{i\to\infty} c_{ji} = c_j$, $j=2,3,\cdots,2n$, and $\Sigma_{j=2}^{2n} c_j^2 = 1$. By using an argument similar to the one used in [**1**, Theorem 1],

$$\lim_{i\to\infty} v_{1i} = v_1 = c_2 z_2 + c_3 z_3 + \cdots + c_{2n} z_{2n}$$

is a bounded nonoscillatory solution of (1) satisfying (2). Repeating this process $2n-1$ more times with the solutions $u_2, u_3, \cdots, u_{2n}$, we obtain the bounded nonoscillatory solutions

$$v_2 = d_{21}z_1 + d_{23}z_3 + \cdots + d_{2,2n}z_{2n}, \quad \sum_{\substack{j=1 \\ j\neq 2}}^{2n} d_{2j}^2 = 1,$$

$$v_3 = d_{31}z_1 + d_{32}z_2 + d_{34}z_4 + \cdots + d_{3,2n}z_{2n}, \quad \sum_{\substack{j=1 \\ j\neq 3}}^{2n} d_{3j}^2 = 1,$$

$$\vdots$$

$$v_{2n} = d_{2n,1}z_1 + d_{2n,2}z_2 + \cdots + d_{2n,2n-1}z_{2n-1}, \quad \sum_{j=1}^{2n-1} d_{2n,j}^2 = 1.$$

The solution v_1 must be independent of at least one of the other v_i's, because, if not, then it is easy to show that $c_2 = c_3 = \cdots = c_{2n} = 0$ which contradicts $\Sigma_{j=2}^{2n} c_j^2 = 1$. Thus $\mathcal{S}$ cannot have more than $2n-1$ linearly independent oscillatory solutions.

Now assume that $\mathcal{S}$ contains at most $2n-1$ linearly independent oscillatory solutions. Let $w \in \mathcal{S}$ satisfy (2). As seen in the proof of Theorem 2.3 (iii), we can construct a solution basis for $\mathcal{S}(w)$ consisting of w and $2n-2$ oscillatory solutions $u_1, u_2, \cdots, u_{2n-2}$ such that u_k has a zero of multiplicity k at $x = a$, $k = 1, 2, \cdots, 2n-2$, $a \geqq 0$ fixed. Choose a point $b > a$ such that $u_1(b) \neq 0$ and let m be chosen such that $u_1(b) - mw(b) = 0$. Then $y = u_1 - mw \in \mathcal{S}(w)$, y is oscillatory, and $y, u_1, u_2, \cdots, u_{2n-2}$ are linearly independent. Suppose there exists a solution v satisfying (2) such that w and v are linearly independent. Then, from Theorem 2.2 (iv) $\mathcal{S}(w) \neq \mathcal{S}(v)$ and there exists a solution $z \in \mathcal{S}(v)$ such that $z \notin \mathcal{S}(w)$. Since $z \in \mathcal{S}(v)$ and v satisfies (2), z cannot satisfy (3). Since $z \notin \mathcal{S}(w)$, z must be unbounded. Therefore z is an unbounded oscillatory solution and it, together with the $2n-1$ independent oscillatory solutions in $\mathcal{S}(w)$ found above, constitute a solution basis for $\mathcal{S}$. This contradicts the hypothesis that $\mathcal{S}$ has at most $2n-1$ linearly independent oscillatory solutions, and completes the proof of the theorem.

COROLLARY 3.3. *Assume that equation* (1) *has property* (H). *If all the oscillatory solutions of* (1) *are bounded, then the solution* w *of* (1) *satisfying* (2) *is essentially unique.*

Proof. As seen in the proof of the theorem, if w is not essentially unique, then there exists an unbounded oscillatory solution $z \notin \mathcal{S}(w)$.

Our final result requires the concept introduced by Dolan and Klaasen in **[3]**. In particular, if $\mathcal{R}$ and $\mathcal{Q}$ are subsets of $\mathcal{S}$, then $\mathcal{R}$ is said to dominate $\mathcal{Q}$, denoted $\mathcal{R} > \mathcal{Q}$, if for each $y \in \mathcal{R}$ and $z \in Q$, $y + \lambda z \in \mathcal{R}$ for all real numbers λ.

Let $\mathcal{U}$ denote the unbounded nonoscillatory solutions of equation (1), $\mathcal{B}$ the set of bounded nonoscillatory solutions, and $\mathcal{O}$ the set of oscillatory solutions. When equation (1) has property (H), the sets $\mathcal{U}$ and $\mathcal{B}$ are easy to describe since $z \in \mathcal{U}$ implies either z or $-z$ is strongly increasing and $w \in \mathcal{B}$ implies either w or $-w$ is strongly decreasing.

THEOREM 3.4. *Assume that equation* (1) *has property* (H). *The following statements are equivalent*

(a) $\mathcal{U} > \mathcal{O}$

(b) $\mathcal{O} > \mathcal{B}$

(c) *The solution* w *of* (1) *satisfying* (2) *is essentially unique.*

Proof. Suppose (a) holds and suppose there is a number $k \neq 0$ such that $y + kw$ is nonoscillatory where $y \in \mathcal{O}$ and $w \in \mathcal{B}$, i.e., w satisfies (2). It is clear that the solution $v = y + kw$ does not satisfy (3), and so, by property (H), v satisfies (2). Obviously w and v are linearly independent. Fix any $a \geqq 0$. Let $u_1, u_2, \cdots, u_{2n-2}$ be the $2n-2$ linearly independent oscillatory solutions in $\mathcal{S}(w)$ such that u_k has a zero of multiplicity k, $k = 1, 2, \cdots, 2n-2$, at $x = a$. Let $z \in \mathcal{S}(v)$ such that $z \notin \mathcal{S}(w)$. We may assume that $z(a) = 0$ (which implies z oscillates), for if $z(a) \neq 0$, then choose $m \neq 0$ such that $z_1 = z - mw$ has a zero at a. Clearly $z_1 \in \mathcal{S}(v)$ and $z_1 \notin \mathcal{S}(w)$. Let y be the solution of (1) determined by the initial conditions $y(a) = y'(a) = \cdots = y^{(2n-2)}(a) = 0$, $y^{(2n-1)}(a) = 1$. From Lemma 2.1, $y \in \mathcal{U}$. The set $\{u_1, u_2, \cdots, u_{2n-2}, y\}$ forms a basis for the set of solutions of (1) having a zero at a. Therefore

$$z = \sum_{i=1}^{2n-2} c_i u_i + cy = u + cy.$$

Since $u(a) = 0$ and $u \in \mathcal{S}(w)$, u is oscillatory. Also, since $z \notin \mathcal{S}(w)$, $c \neq 0$. Thus $\bar{z} = (1/c)z = y + (1/c)u$ is oscillatory and contradicts the fact that $\mathcal{U} > \mathcal{O}$.

Suppose (b) holds and w is not essentially unique. Then there exists a solution v of (1) satisfying (2) which is independent of w. Let $u_1, u_2, \cdots, u_{2n-2}$ be the $2n-2$ linearly independent oscillatory solutions in $\mathcal{S}(w)$ such that u_k has a zero of multiplicity k, $k = 1, 2, \cdots, 2n-2$, at $x = a$, $a \geqq 0$ fixed. Then $\{w, u_1, u_2, \cdots, u_{2n-2}\}$ is a basis for $\mathcal{S}(w)$, and every linear combination of $u_1, u_2, \cdots, u_{2n-2}$ is oscillatory. Since v is bounded, we must have $v \in \mathcal{S}(w)$ by Theorem 2.3 (iii). Thus

$$v = \sum_{i=1}^{2n-2} c_i u_i + cw$$

where not all the c_i's are zero, that is, $v = u + cw$ is nonoscillatory where $u \in \mathcal{O}$ and $w \in \mathcal{B}$. This contradicts (b).

Finally, assume that (c) holds and suppose that $\mathcal{U}$ does not dominate $\mathcal{O}$. Then there exists $z \in \mathcal{U}$, $y \in \mathcal{O}$ and a nonzero number k such that $z + ky$ is oscillatory. It follows from Theorem 3.2 that $\mathcal{S}$ contains at most $2n - 1$ linearly independent oscillatory solutions. Since $\mathcal{S}(w)$ has a basis consisting of $2n - 1$ oscillatory solutions (see the proof of Theorem 3.2), we can conclude that both y and $z + ky$ are in $\mathcal{S}(w)$. But this implies $z \in \mathcal{S}(w)$ which is impossible since either z or $-z$ is strongly increasing. This completes the proof of the theorem.

References

1. S. Ahmad, *On the oscillation of solutions of a class of linear fourth order differential equations*, Pacific J. Math., **34** (1970), 289–299.

2. J. M. Dolan, *On the relationship between the oscillatory behavior of a linear third-order differential equation and its adjoint*, J. Differential Equations, **7** (1970), 367–388.

3. J. M. Dolan and G. A. Klaasen, *Dominance of* n-th *order linear equations*, Rocky Mountain J. Math. **5** (1975), 263–270.

4. G. W. Johnson, *A bounded nonoscillatory solution of an even order linear differential equation*, J. Differential Equations, **15** (1974), 172–177.

5. G. D. Jones and S. M. Rankin, *Oscillation properties of certain selfadjoint differential equations of the fourth order*, Pacific J. Math., (to appear).

6. D. L. Lovelady, *An asymptotic analysis of an even order differential equation*, submitted for publication.

7. T. T. Read, *Growth and decay of solutions of* $y^{(2n)} - py = 0$, Proc. Amer. Math. Soc., **43** (1974), 127–132.

Received September 17, 1976. The second author was partially supported by NSF grant No. HES 75-03730.

University of Houston
Houston, TX 77004

and

Prairie view A & M University
Houston, TX 77004