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ON NONSINGULARLY k-PRIMITIVE RINGS

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A ring R is called k-primitive if it has a faithful cyclic critical right module C with $|C| = k$. We first show that k-primitive rings with Krull dimension have many properties in common with prime rings. For the case where R is a PWD with a faithful critical right ideal, we obtain an internal characterization.

1. Introduction. Let R be a ring with Krull dimension. Then R is prime if and only if R has a faithful compressible right R-module. In this paper we consider a broader class of rings, those which have a faithful cyclic critical right R-module. From **[2]** such a ring is called a *k-primitive* ring where k denotes the Krull dimension of the faithful critical.

In the case where the faithful critical is nonsingular, these rings exhibit many of the properties of prime rings. Not all k-primitive rings have this additional property as an example in §4 shows. We call a k-primitive ring whose faithful critical is nonsingular, a *nonsingularly k-primitive* ring. Section 2 is devoted to showing some of the similarities with prime rings.

In §3 we consider piecewise domains (PWD) which are k-primitive rings. An internal characterization of PWD's with faithful critical right ideal is obtained, which is our main result.

All rings will have identity, and the modules are right unital. The singular submodule of a module M_R is denoted $Z(M)$. If X is a subset of R, then ann X'or X^r denotes the right annihilator of X in R. The Krull dimension of a module M_R is denoted by $|M|$. A certain familiarity with the definitions and basic results concerning Krull dimension is assumed. See **[5]** for reference.

2. Properties of k-primitive rings. If R is a prime ring with Krull dimension then R is nonsingular and has a faithful critical C such that $|C| = |R|$. These conditions are also true for nonsingularly k-primitive rings.

PROPOSITION 2.1. *Let R be a k-primitive ring with faithful cyclic critical C. Then $Z(R) = 0$ and $|C| = |R|$ if and only if R is nonsingularly k-primitive.*

Proof. Suppose $Z(C)=0$. This immediately implies $Z(R)=0$. Let X be the collection of annihilators of finite subsets of C. By [**4**, Theorem 1.24], X satisfies the descending chain condition. Since C is faithful $\bigcap_{x\in C}(x^r)=0$ and there exists a finite subset such that $\bigcap_{i=1}^{n}(x_i^r)=0$. This implies the existence of an R-monomorphism $R\to\Sigma_{i=1}^{n} R/x_i^r\to C^{(n)}$. Thus $|R|\leqq|C^{(n)}|=|C|\leqq|R|$.

Conversely if $Z(R)=0$, then $Z(C)=C$ or $Z(C)=0$. Suppose $Z(C)=C$. Then $C\cong R/K$ where K is a large right ideal. Let $L=\{D\mid D \text{ is a critical right ideal}\}$. Then $S=\Sigma D$, $D\in L$ is a two sided ideal of R. Since R/K is faithful, $S\not\subset K$. Hence there exists $D\in L$ such that $D\not\subset K$. Since K is large $D\cap K\neq 0$ and $|R/K|=|D+K/K|=|D/D\cap K|<|D|\leqq|R|$. This contradicts the fact that $|C|=|R|$, and therefore $Z(C)=0$.

Let C be the faithful k-critical of a nonsingularly k-primitive ring R. Then $P=\operatorname{ass} C$ is a prime ideal. In the remainder of this paper C and P will be used in this way.

LEMMA 2.2. *If R is a nonsingularly k-primitive ring with faithful critical C, then $P=\operatorname{ass} C$ is a nonessential minimal prime and $|R|=|R/P|$.*

Proof. That P is a nonessential minimal prime is straightforward.

The module C contains a nonzero submodule C^* where ann $C^*=P$. Since C^* is a nonsingular, faithful R/P-module, then by 2.1 $|R|=|C^*|=|R/P|$.

PROPOSITION 2.3. *Let R be a nonsingularly k-primitive ring with faithful, critical C and let $P=\operatorname{ass} C$. Then*

(1) *P contains all nonessential two sided ideals.*
(2) *R has exactly one nonessential prime ideal, namely P.*
(3) *If R is semiprime, then R is prime.*
(4) *Every uniform right ideal which misses P is compressible.*
(5) *Every uniform right ideal is critical and subisomorphic to C.*

Proof. (1) If H is not essential, there exists a right ideal I such that $I\cap H=0$. Then $IH=0$ which implies $H\subset\operatorname{ass} C=P$ by [**2**, Proposition 3.2].

(2) This follows from (1).

(3) If $0=P_1\cap\cdots\cap P_n$ is an irredundant intersection of minimal primes, then P_i is not large and hence $P_i=P$ by (2) for each i. Thus $P=0$.

(4) If U is uniform and $U\cap P=0$, then $U\cong U+P/P\subseteq R/P$. Since uniform right ideals of a prime ring are compressible, the result follows.

(5) Let U be a uniform right ideal. Then $CU \neq 0$. Thus there exists $x \in C$ such that $xU \neq 0$. Since $Z(C) = 0$, $U \cong xU \subset C$ and U is critical.

A ring R with Krull dimension is called *very smooth* if every right ideal of R has the same Krull dimension. In a prime ring any two critical right ideals are subisomorphic and thus the ring is very smooth.

PROPOSITION 2.4. *Let R be a nonsingularly k-primitive ring with faithful critical C.*

(1) *If D is a critical right ideal, then* ass $D = P$.
(2) *R is a very smooth ring.*
(3) *If D is a critical right ideal then $D \cap P = 0$ or $D \subseteq P$.*
(4) *If R is not prime and if $C \subset R$, then $C \subset P$.*

Proof. (1) Let $P_0 = ass\ D$. Then D has a submodule D^* such that ann $D^* = P_0$. Since $Z(D^*) = 0$, P_0 is not essential and hence by 2.3(2), $P_0 = P = ass\ C$.

(2) It suffices to show that if D is critical, $|D| = |R|$. By (1) D has a submodule D^* which is nonsingular and faithful as an R/P-module and hence $|D| = |D^*| = |R/P| = |R|$.

(3) If $D \cap P \neq 0$ then $D/D \cap P \cong D + P/P \subset R/P$. Since R/P is very smooth and $|R/P| = |D|$ then $|D/D \cap P| < |D|$ implies $D = D \cap P$.

(4) If $C \subset R$ and $C \cap P = 0$, then $CP = 0$ contradicting the faithfulness of C. Thus $C \cap P \neq 0$. Therefore by (3) $C \subset P$.

PROPOSITION 2.5. *Let R be a nonsingularly k-primitive ring. The compressible right ideals of R are subisomorphic.*

Proof. Since P = ass C is not large, there exists a critical right ideal $D \neq 0$ such that $D \cap P = 0$. By 2.3 (4) D is compressible. Let $K \neq 0$ be any compressible right ideal. Then $KD \neq 0$ since $D \not\subset P$ = ass C. Thus there exists $a \in K$ and a monomorphism $D \to aD \subseteq K$. Since K is compressible, K is subisomorphic to aD and hence to D. Since being subisomorphic is a transitive property, any two compressible right ideals are subisomorphic.

COROLLARY 2.6. *Let R be a nonsingularly k-primitive ring with the nonessential prime $P \neq 0$. Then P contains an isomorphic copy of all uniform right ideals.*

3. k-primitive piecewise domains. Let R be a ring with Krull dimension and suppose that R is a piecewise domain with a faithful critical right ideal. Then R is nonsingularly k-primitive and is, in general, not prime. We assume these rings have a faithful critical right ideal. In § 4 we provide an example to show that this need not always be the case. From [**7**] we have

DEFINITION 3.1. A ring R is a *piecewise domain* (PWD) with respect to a complete set of orthogonal idempotents $e_1, \cdots, e_n$ if $x \in e_i \mathrm{Re}_k$, $y \in e_k \mathrm{Re}_l$ then $xy = 0$ implies $x = 0$ or $y = 0$.

In [**7**, p. 554] the following criterion is given for R to be a PWD. R is a PWD with respect to the complete set of orthogonal idempotents $\{e_i\}$ if and only if every nonzero element of $\mathrm{Hom}_R(e_iR, R)$ is a monomorphism.

PROPOSITION 3.2. *Let R be a ring with Krull dimension. Then R is a PWD with faithful critical right ideal if and only if $R = \Sigma \oplus e_iR$ where e_iR is critical for every i and e_jR is faithful and nonsingular for some j. In this case R is nonsingularly k-primitive.*

Proof. Suppose R is a PWD with faithful critical right ideal. Then $R = \Sigma \oplus e_iR$ where $e_1, \cdots, e_n$ is a complete set of orthogonal idempotents. Since C is faithful, $Ce_iR \neq 0$ for all i. Hence for any given i there exists $c \in C$ such that $ce_iR \neq 0$. We can therefore define a homomorphism θ of e_iR into C using c and by [**7**, p. 554] the mapping θ is a monomorphism. Thus e_iR is critical.

Now $RC \neq 0$ which implies $e_jRC \neq 0$ for some j. By [**8**, Lemma 1], $Z(R) = 0$ and thus the mapping determined by the relation $e_jrC \neq 0$ is a monomorphism. So C is subisomorphic to at least one e_jR and $Z(e_jR) = 0$.

Conversely suppose $R = \Sigma \oplus e_iR$ where e_iR is critical. Since e_jR is faithful and nonsingular by 2.4 we know that R is very smooth. So consider a mapping $f\colon e_iR \to R$. If $\mathrm{Ker}\ f \neq 0$ then $|f(e_iR)| = |e_iR/\mathrm{Ker}\, f| < |e_iR| = |R|$. Thus necessarily $f = 0$.

The same technique employed in the above proof shows that any two faithful critical right ideals of a PWD are subisomorphic. Furthermore the faithful critical right ideal contains an isomorphic copy of every critical right ideal.

PROPOSITION 3.3. *Let R be a ring with Krull dimension. If R is a PWD with faithful critical right ideal, say $R = \Sigma \oplus e_iR$ where $C = e_1R$ is faithful, then*

(1) *$P = \mathrm{ass}\ C = \Sigma_i\, e_iR$, where e_iR is not compressible.*
(2) *If Q is a prime ideal not equal to P then $|R/Q| < |R|$.*
(3) *If R is not a prime ring, then not all the e_iR are compressible.*

Proof. (1) and (3). If R is prime, then the e_iR are all compressible and ass $C = P = 0$. If R is not prime, then C cannot be compressible because it is faithful.

Now suppose e_iR are compressible for $i = m+1, \cdots, n$ and e_jR are not compressible for $j = 1, \cdots, m$. Then

$$P = RP = \left(\sum_{i=1}^{n} e_iR\right)P \subseteq \sum_{i=1}^{m} e_iRP \subseteq \sum_{i=1}^{m} e_iR.$$

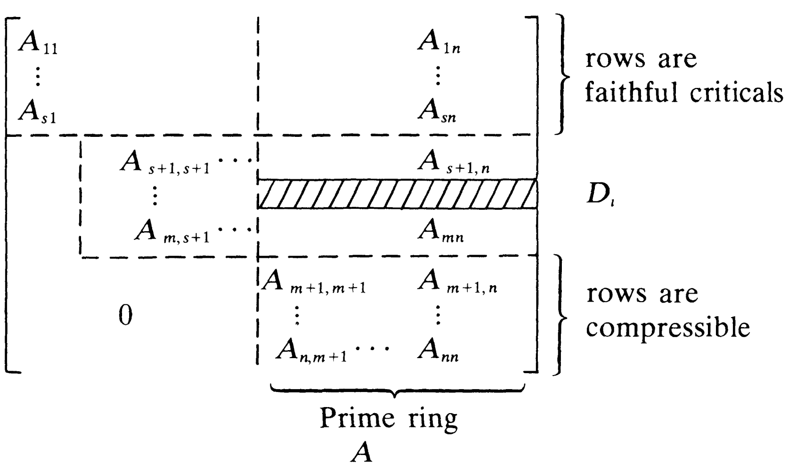
Conversely if D is critical either $D \cap P = 0$ or $D \subset P$ by 2.4(3). Since $e_iR \cap P = 0$ implies by 2.3(4) that e_iR is compressible, necessarily $e_iR \subseteq P$ $1 \leqq i \leqq m$ and hence $P = \Sigma_{i=1}^{m} e_iR$.

(2) If Q is a prime ideal not equal to P, then Q is large. Suppose $|R/Q| = |R|$. As in the proof of 2.4(3) if D is critical $D \subseteq Q$. Thus Q contains all critical right ideals and hence $R \subseteq Q$ which is impossible.

In [**8**, Theorem 2] Gordon obtains an internal characterization of prime right Goldie rings which are PWD's. In the following theorem we obtain an internal characterization for a nonsingularly k-primitive ring with a faithful critical right ideal which is a PWD.

THEOREM 3.4. *Let R be a PWD with Krull dimension, $|R| = k$. Then R is nonsingularly k-primitive with faithful critical right ideal if and only if $R \cong (A_{ij})_{n\times n}$ where for some s, m where $1 \leqq s < m < n$,*

(1) *$A_{ij} = 0$ for $i > s$ and $j \leqq s$ or $i > m$ and $j \leqq m$.*
(2) *$A_{ij} \neq 0$ for $i \leqq s$ or $j > m$.*
(3) *A_{ii} is a domain for $1 \leqq i \leqq n$.*
(4) *If $j \leqq m$ then $|(A_{ij})_{A_{jj}}| < k$.*
(5) *$A = \{(a_{ij}) \in (A_{ij}) \mid a_{ij} = 0\ 1 \leqq i \leqq m$ or $1 \leqq j \leqq m\}$ is a prime ring of Krull dimension k and $D_i = \{(a_{kj}) \mid a_{ij} = 0\ 1 \leqq j \leqq m,\ a_{kj} = 0\ k \neq i\}$ as a right A-module is k-critical for all i.*



Proof. By 3.2 and 3.3 $R = \Sigma_{i=1}^{n} \oplus C_i$ with C_i critical for all i where C_i is faithful for $1 \leqq i \leqq s$ and C_j is compressible for $m+1 \leqq j \leqq n$. Then R is isomorphic to the matrix ring with (i,j) entries in $A_{ij} = \mathrm{Hom}_R(C_j, C_i)$, which are monomorphisms or zero. Since $Z(R) = 0$ by **[7]**, then $Z(C_i) = 0$.

(1) If $i > s$, and $j \leqq s$, then $\mathrm{Hom}(C_j, C_i) = 0$ since these C_i are not faithful. Similarly, $\mathrm{Hom}(C_j, C_i) = 0$ $i > m$, $j \leqq m$, since these C_j are not compressible, and the C_i are compressible (using the fact that submodules of compressible modules are compressible). This proves (1).

(2) If C_i is faithful, then $C_iC_j \neq 0$ for all j and $\mathrm{Hom}(C_j, C_i) \neq 0$ for all j. If C_j is compressible, then $C_iC_j \neq 0$ for any i. Hence $\mathrm{Hom}(C_j, C_i) \neq 0$. This proves (2).

(3) A_{ii} is a domain, since R is *PWD* ring.

(4) For $j \leqq m$, let K_{ij} be an A_{jj} submodule of A_{ij}. Let $K^*_{ij} = \{(a_{rt}) \mid a_{ij} \in K_{ij}$, and $a_{rt} = 0$ otherwise$\}$. Let D_i be as in the theorem, then $D_i \neq 0$ by (2) and D_i is a right ideal of R because of (1). Let $S_i = K^*_{ij}R + D_i/D_i$. Then $S_i \subset C_i/D_i$. Then the mapping $K_{ij} \rightarrow S_i$ is a lattice isomorphism of the lattice of submodules of A_{ij} over A_{jj} into the lattice of submodules of C_i/D_i over R. Since C_i is critical and since $D_i \neq 0$, then the Krull dimension of A_{ij} over A_{jj} must be less than k.

(5) The first part follows since ass $C = P$ is equal to $C_1 + \cdots + C_m$ by 3.3 and $A \cong R/P$. Now each D_i is a submodule of a critical module, and hence is critical over R. But the lattice of modules of D_i over R is the same as the lattice of D_i over A, and D_i is critical over A, and since the Krull dimension of D_i over R is k, then $|D_i|$ over A is also k.

Conversely, let R be a *PWD* satisfying these conditions and let C_i denote the ith row of the matrix. We will show that C_i is critical, and that C_i, $i = 1, 2, \cdots, s$ is faithful. Since R is a *PWD*, and using (2), we have that C_i, for $i = 1, 2, \cdots, s$ is faithful. We now show C_1 is k-critical. Let M_1 be a *nonzero* submodule of C_1 over R. Let $M_{1j} = \{a \in A_{1j} \mid a$ is the $(1,j)$ entry of some member of $M_1\}$. Then $M_{1j} \neq 0$ for $j > m$. Now for $j \leqq m$, M_{1j} is an A_{jj} module. Thus by (4), the Krull dimension of A_{1j}/M_{1j} over A_{jj} is less than k. If $N = \Sigma_{i=m+1}^{n} M^*_{1i}$ where $M^*_{1i} = \{(a_{rt}) \mid a_{1i} \in M_{1i}$ and $a_{rt} = 0$ otherwise$\}$, then N is not zero, and by (4), the Krull dimension of D_1/N over A is less than k. Thus with each submodule of C_1/M_1, we can associate an A_{jj} submodule of A_{1j}/M_{1j}, one for each $j \leqq m$, and a factor module D_1/N of D_1 as an A-module. We construct a lattice isomorphism from the submodules of C_1/M_1 into the lattice of submodules of $T = A_{11}/M_{11} \oplus \cdots \oplus A_{1m}/M_{1m} \oplus D_1/N$ over $S = A_{11} \oplus \cdots \oplus A_{mm} \oplus A$, where in T_S we have scalar multiplication defined as coordinate multiplication by elements of the ring S. Now the Krull dimension of A_{1i}/M_{1i} is less than k over A_{ii} for $1 \leqq i \leqq m$, and similarly for D_1/N over A. Hence the Krull dimension of T over S is less than k. Thus $|C_1/M_1| < k$. Since D_1 over A has Krull dimension k, one can show that C_1 has Krull dimension k over R in a similar fashion. Thus C_1 is k-critical. Since $R = C_1 \oplus \cdots \oplus C_n$ and C_1 is faithful, then each C_i is R isomorphic to a submodule of C_1, and hence is critical of Krull dimension k. Thus $|R| = k$.

4. Examples and questions. Using the results of §3, one can easily construct nonsingularly k-primitive rings. Let I a right Noetherian integral domain, where $|I| = k$, and $I[x]$ be the polynomial ring with commuting x. We construct three matrix rings of this type

$$R = \begin{bmatrix} I & I & I[x] & I[x] & I[x][y] & I[x][y] \\ I & I & I[x] & I[x] & I[x][y] & I[x][y] \\ 0 & 0 & I[x] & A & I[x][y] & I[x][y] \\ 0 & 0 & B & I[x] & I[x][y] & I[x][y] \\ 0 & 0 & 0 & 0 & I[x][y] & I[x][y] \\ 0 & 0 & 0 & 0 & I[x][y] & I[x][y] \end{bmatrix}$$

where (1) $A = B = 0$, (2) $A = I[x]$, $B = 0$, (3) $A = I[x]$, $B = I[x]$. These are nonsingularly k-primitive rings of Krull dimension $k+2$.

In each of these three rings, with the notation of Theorem 3.4, C_1 is the faithful critical right ideal and $s = 2$, $m = 4$. In the case where $A = B = I$, the two nonfaithful, noncompressible modules C_3 and C_4 are (sub)isomorphic. Clearly, in all cases, $P = C_1 + C_2 + C_3 + C_4$.

One can show, in general, that if R is nonsingularly k-primitive, then the complete ring of quotients Q is a simple Artinian ring. If R is *PWD*, then by **[9]**, R has an Artinian Classical quotient ring Q_{cl}. However, Q_{cl} is never k-primitive unless R is prime. To illustrate this consider for a field F the ring, $R = \begin{bmatrix} F & F[x] \\ 0 & F[x] \end{bmatrix}$. Then $Q(R) = \begin{bmatrix} F(x) & F(x) \\ F(x) & F(x) \end{bmatrix}$ and $Q_{cl}(R) = \begin{bmatrix} F & F(x) \\ 0 & F(x) \end{bmatrix}$.

Question 1. In general does R have a classical quotient ring Q_{cl}? Does Q_{cl} contain a maximal k-primitive subring which contains R?

Question 2. Does $N(R)$ being prime imply R is prime? (True for *PWD*'s.)

Question 3. If R has left Krull dimension, is R a prime ring?

In regard to questions 1 and 2, if our ring R has the regularity condition (See [**6**]), and if N is prime, then R is a prime ring.

To give an example of a nonsingularly k-primitive PWD where the faithful cyclic is not imbeddable in R consider a field F with a derivation $(')$. Let $M = F[x] \oplus F[x]$. Then M is a right $F[x]$ module under $(f, g)h = (fh, gh)$, and M is a left F module under $a(f, g) = (af + a'g, ag)$, $a \in F$. Thus M is a bimodule, and the matrices

$$R = \begin{bmatrix} F & M \\ 0 & F[x] \end{bmatrix} \text{form a ring.}$$

Let $I = \begin{bmatrix} 0 & N \\ 0 & F[x] \end{bmatrix}$, where $N = \{(0, f(x) \mid f(x) \in F[x]\}$. Then I is not large, and contains no two-sided ideals. In addition $C = R/I$ is critical. In fact, if $C_0 \neq 0$ is a submodule of C, then C/C_0 is Artinian. One can show that C cannot be embedded in R, and $Z(C) = 0$. One can also show that no right ideal of R is faithful and critical. Hence R is a nonsingularly 1-primitive ring without a faithful critical right ideal.

In the case where R is k-primitive but the faithful critical is not nonsingular, then R may not have the properties established in §2. Consider the following example.

Let $A = F[x, (')][z]$ where F is a field with derivative $(')$ as in [**3**, p. 55], and z commutes with x. Let

$$R = \begin{bmatrix} F & A/xA \\ 0 & A \end{bmatrix}.$$

The first row of R is a faithful cyclic critical C which is not nonsingular. Now $|C| = 1$ and $|R| = 2$. Thus R is not very smooth. In addition, R does not satisfy the regularity condition. Hence R does not have an Àrtinian classical right quotient ring.

References

1. A. K. Boyle and E. H. Feller, *Smooth Noetherian modules*, Comm. Algebras, **4** (1976), 617–637.
2. M. G. Deshpande and E. H. Feller, *The Krull radical*, Comm. Algebras, **3** (1975), 185–193.
3. N. J. Divinsky, *Ring and Radicals*, University of Toronto Press, Mathematical Expositions No. 14, 1965.
4. K. R. Goodearl, *Singular torsion and the splitting properties*, Memoirs of Amer. Math. Soc. No. 124 (1972).
5. R. Gordon and J. C. Robson, *Krull dimension*, Memoirs of Amer. Math. Soc. No. 133 (1973).
6. R. Gordon, *Gabriel and Krull dimension*, Ring Theory, Lecture notes in Pure and Applied Math. Marcel Dekker, New York, (1974), 241–295.

7. R. Gordon and L. W. Small, *Piecewise Domains*, J. Algebra, **23**, No. 3 (1972), 553–564.

8. R. Gordon, *Semiprime right Goldie rings which are a direct sum of uniform right ideals*, Bull. London Math. Soc., **3** (1971), 277–282.

9. ———, *Classical quotient rings of PWD's*, Proc. Amer. Math. Soc., **36** (1972), 39–46.

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