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AN ANALOGUE OF OKA'S THEOREM FOR WEAKLY NORMAL COMPLEX SPACES

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Two well known results concerning normal complex spaces are the following. First, the singular set of a normal complex space has codimension at least two. Second, this property characterizes normality for complex spaces which are local complete intersections. This second result is a theorem of Abhyankar [1] which generalizes Oka's theorem. The purpose of this paper is to prove analogues of these facts for the class of weakly normal complex spaces, which were introduced by Andreotti–Norguet [3] in a study of the space of cycles on an algebraic variety. A weakly normal complex space can have singularities in codimension one, but it will be shown that an obvious class of such singularities is generic.

1. Preliminaries. All complex spaces are assumed to be reduced. If X is a complex space, there is the sheaf $\mathcal{O}_X$ of holomorphic functions on X, and the sheaf $\mathcal{O}_X^c$ of c-holomorphic functions on X. A section of $\mathcal{O}_X^c$ on an open subset U of X is a continuous function $f\colon U \to \mathbf{C}$ such that f is holomorphic on the regular points of U. The complex space X is said to be weakly normal if $\mathcal{O}_X = \mathcal{O}_X^c$. Examples of weakly normal spaces are normal spaces and unions of submanifolds of $\mathbf{C}^m$ in general position.

Let $V_j = \{(x_1, \cdots, x_m) \in \mathbf{C}^m : x_k = 0 \text{ for } n \leqq k < j \text{ and } j < k \leqq m\}$ where $n \leqq j \leqq m$. Then V_j is an n-dimensional linear subspace of $\mathbf{C}^m$. Let

$$V_{(n,m)} = \bigcup_{j=n}^{m} V_j = \{(x_1, \cdots, x_m) \in \mathbf{C}^m : x_i x_j = 0 \quad \text{for} \quad n \leqq i < j \leqq m\}$$

and let $S(V_{(n,m)})$ be the singular set of $V_{(n,m)}$.

LEMMA. *$V_{(n,m)}$ is a weakly normal complex space and* $\dim S(V_{(n,m)}) = n - 1$.

Proof. Since $S(V_{(n,m)}) = \{(x_1, \cdots, x_m) \in \mathbf{C}^m : x_n = \cdots = x_m = 0\}$, $\dim S(V_{(n,m)}) = n - 1$. Let $f\colon V_{(n,m)} \to \mathbf{C}$ be a continuous function which is holomorphic on the regular points of $V_{(n,m)}$. To prove weak normality of $V_{(n,m)}$, we need to show that f is holomorphic. Let $f_j = f|_{V_j}$. By the Riemann extension theorem, f_j is holomorphic on the n-plane V_j and thus $f_j = f_j(x', x_j)$ is a convergent power series, where $x' = (x_1, \cdots, x_{n-1})$ and x_j are coordinates on V_j. Since $f_j|_{x_j=0} = f_k|_{x_k=0}$ for $n \leqq j, k \leqq m$, we let $f_0(x') = f_j(x', 0)$ and set $g_j(x', x_j) = f_j(x', x_j) - f_0(x')$ for $n \leqq j \leqq m$. Then $f(x_1, \cdots, x_m) = f_0(x') + \sum_{j=n}^{m} g_j(x', x_j)$ and hence f is holomorphic on $V_{(n,m)}$.

If X is a complex space with $\dim X = n$, let $Sg(X) = S(X) \cup (\bigcup_{0 \leqq k < n} X^{(k)})$ where $S(X)$ is the singular set of X and $X^{(k)}$ is the analytic subset of X defined by $X^{(k)} = \{x \in X: X$ has a branch of dimension k at $x\}$. If $C_4(X, x)$ denotes the fourth Whitney tangent cone of X at x, then Stutz **[6]** has shown that $W_4 = Sg(X) \cap \{x \in X: \dim C_4(X, x) > n\}$ is an analytic subset of X of codimension at least two.

2. Codimension one singularities of weakly normal spaces. Let X be a complex space. A point $x \in X$ is said to be an elementary point of type (n, m), for $n \leqq m$, if the germ (X, x) is isomorphic to the germ $(V_{(n,m)}, 0)$. Note that if $x \in X$ is an elementary point of type (n, m), then the germ (X, x) is of pure dimension n and the imbedding dimension of (X, x) is m. The set of elementary points of X contains the set of regular points of X, i.e. the elementary points of type (n, n) for some n. In addition, it contains a particularly simple class of singular points of X. If x is an elementary point of type (n, m) with $n < m$, then x is singular and $\dim(S(X), x) = n - 1 = \dim(X, x) - 1$.

If $\dim X = n$, let $Y = \bigcup_{0 \leqq k < n} X^{(k)}$ and let $X_1 = \overline{X \setminus Y}$. By a theorem of Remmert, X_1 is an analytic set of pure dimension n. Let X_s denote the set of all elementary points of X of type (n, m) for some m with $m \geqq n = \dim X$. Hence $X_s \subseteq X_1$ and X_s contains the regular points of X of maximal dimension.

THEOREM 1. *Let X be a weakly normal complex space. Then $A = X_1 \setminus X_s$ is an analytic subset of X_1 of codimension at least 2.*

Proof. Let $n = \dim X$. If $\dim S(X) \leqq n - 2$ then $A = X_1 \cap S(X)$. Hence A is analytic and codimension $A \geqq 2$. Now suppose that $\dim S(X) = n - 1$. We will show that $A = X_1 \cap (Sg(Sg(X)) \cup W_4)$. Since $Sg(Sg(X)) \cup W_4$ is an analytic set of codimension at least 2 in X and since $\dim X = \dim X_1$, this will prove the theorem.

Let $x \in X_s$. If x is a regular point of X, then $x \notin Sg(Sg(X)) \cup W_4$. If x is an elemetary point of type (n, m) where $m > n$, then $\dim C_4(X, x) = n$. Hence $x \notin W_4$. Moreover, $S(X)$ is a manifold of dimension $n - 1$ in a neighborhood of x. Thus $x \notin Sg(Sg(X))$. Hence $X_s \subseteq X_1 \setminus (Sg(Sg(X)) \cup W_4)$ and $X_1 \cap (Sg(Sg(X)) \cup W_4) \subseteq A$.

Now suppose that $x_0 \in X_1 \cap S(X) \cap (X_1 \setminus (\mathrm{Sg}(\mathrm{Sg}(X)) \cup W_4))$. Thus

$x_0 \in \mathrm{Sg}(X) \backslash \mathrm{Sg}(\mathrm{Sg}(X))$ and $\dim C_4(X, x_0) = n$. Note also that the germ (X, x_0) is of pure dimension n. Since the result to be proved is local, we may assume that $X \subseteq \mathbf{C}^t$. By Proposition 4.2 of Stutz **[6]**, there is a neighborhood N of x_0 in X, a polydisc $D \subseteq \mathbf{C}^n$, and a choice of coordinates $x_1, \cdots, x_n$ in $\mathbf{C}^n$ and $y_1, \cdots, y_t$ in $\mathbf{C}^t$ centered at x_0 with the following properties.

If $B_0, \cdots, B_r$ are the global branches of $X \cap N$, then for each j $(0 \leqq j \leqq r)$ there is a holomorphic map $f_j \colon D \to B_j$ such that

(a) f_j is a homeomorphism;

(b) with respect to the coordinates $x_1, \cdots, x_n, y_1, \cdots, y_t$, $f_j(0) = 0$ and

$$f_j(x) = (x_1, \cdots, x_{n-1}, x_n^{p_j}, f_{n+1,j}(x), \cdots, f_{tj}(x))$$

where p_j is a positive integer for $0 \leqq j \leqq r$;

(c) $f_{ij}(x_1, \cdots, x_n) = \Sigma_{\nu=p_j}^{\infty} f_{ij}^{(\nu)}(x_1, \cdots, x_{n-1}) \cdot x_n^{\nu}$ for $n+1 \leqq i \leqq t$ and $0 \leqq j \leqq r$.

Let $g_j \colon B_j \to D$ be the continuous inverse of f_j and define a map $h \colon X \cap N \to \mathbf{C}^{n+r}$ by $\pi_j \circ h\,|_{B_j} = g_j$ where $\pi_j \colon \mathbf{C}^{n+r} \to \mathbf{C}_{x_1, \cdots, x_{n-1}, x_{n+j}}$ is the natural linear projection onto the n-plane with coordinates $x_1, \cdots, x_{n-1}, x_{n+j}$, for $0 \leqq j \leqq r$. To see that the map h is well defined, note first that $S(X)$ is an $n-1$ dimensional manifold in a neighborhood of x_0. Furthermore, $B_j \cap B_k \subseteq S(X) \cap N$ for all j, k. But $f_j(x', 0) = (x', 0, \cdots, 0) = f_k(x', 0)$ where $x' = (x_1, \cdots, x_{n-1})$. Therefore, if N is chosen small enough, then $B_j \cap B_k = S(X) \cap N = \{y_n = \cdots = y_t = 0\}$ for $0 \leqq j,\ k \leqq r$. For each $(y_1, \cdots, y_t) \in S(X) \cap N$, it follows that $g_j(y) = (y_1, \cdots, y_{n-1}, 0)$ for $0 \leqq j \leqq r$. Thus h is a well defined continuous map.

Since the jacobian matrix $\partial f_j / \partial x$ is given by

$$\frac{\partial f_j}{\partial x} = \begin{bmatrix} I_{n-1} & 0 \\ 0 & p_j x_n^{p_j - 1} \\ * & * \end{bmatrix}$$

h is holomorphic on the regular points of $X \cap N$. Since X is weakly normal and h is a homeomorphism onto its image, it follows that h is biholomorphic. Therefore x_0 is an elementary singularity of type $(n, n + r)$. Hence $A \subseteq X_1 \cap (\mathrm{Sg}(\mathrm{Sg}(X)) \cup W_4)$ and the theorem is proved.

REMARK. Let X be a weakly normal complex space and suppose that $\operatorname{codim} S(X) = 1$. Theorem 1 shows that there is an elementary singularity of type (n, m) where $m > n$. Since such a singular point is not normal, Theorem 1 implies the well-known theorem that $\operatorname{codim} S(X) \geqq 2$ for a normal complex space X.

THEOREM 2. *Let X be a pure dimensional local complete intersection. Then X is weakly normal if and only if* $\operatorname{codim} X \backslash X_s \geqq 2$.

Proof. Let $A = X \backslash X_s$. If X is weakly normal then $\operatorname{codim} A \geqq 2$ by Theorem 1.

Conversely, suppose $\operatorname{codim} A \geqq 2$. Since $X \backslash A = X_s$, the germ (X, x) is weakly normal for each $x \in X \backslash A$. Since X is a pure dimensional local complete intersection, $pf(\mathcal{O}_{X,x}) = \dim X$ for each $x \in X$, where pf = profondeur. From the Hartog theorem for weak normality **[2]**, we conclude that X is weakly normal.

REMARKS. (1) For the case of curves, the assumption of local complete intersection is not needed. A curve X is weakly normal if and only if $X \backslash X_s = \emptyset$. An algebraic proof of this fact was given by Bombieri **[5]**.

(2) If X is a pure dimensional hypersurface in $\mathbf{C}^{n+1}$, then Theorem 2 can be proved without the use of the Hartog theorem for weak normality. This case follows from the result of Becker in **[4]**.

(3) Let $X \subseteq \mathbf{C}^{n+1}$ be a pure dimensional hypersurface. If X is weakly normal, there is another characterization of $X \backslash X_s$ than that which is given by the proof of Theorem 1. This description is as follows. There is a holomorphic function $f \in \mathcal{O}(\mathbf{C}^{n+1})$ such that $X = V(f) = \{x \in \mathbf{C}^{n+1}: f(x) = 0\}$ and such that there is a sheaf equality $(f) \cdot \mathcal{O} = \mathcal{I}_X$ where $\mathcal{I}_X$ is the sheaf of ideals of X. Then

$$S(X) = \left\{ x \in X: \frac{\partial f}{\partial z_i}(x) = 0 \text{ for } 1 \leqq i \leqq n+1 \right\}.$$

At a point $x_0 \in S(X)$ the Hessian form is defined by

$$H(f)_{x_0}(u) = \sum_{i,j=1}^{n+1} \frac{\partial^2 f}{\partial z_i \partial z_j}(x_0) \cdot u_i u_j.$$

Let $\mu(x_0) = \operatorname{rank} H(f)_{x_0}$ and set $S_2(X) = \{x \in S(X): \mu(x) \leqq 1\}$.

Claim. If X is weakly normal and $\dim S(X) = n-1$, *then*

$$W_4 \cap (S(X) \backslash \operatorname{Sg}(S(X))) = S_2 \cap (S(X) \backslash \operatorname{Sg}(S(X))).$$

Proof. From the proof of Theorem 1, $X \backslash X_s = \operatorname{Sg}(S(X)) \cup W_4$. Suppose $x \in S(X) \backslash \operatorname{Sg}(S(X))$ but $x \notin W_4$. Then the proof of Theorem 1 shows that x is an elementary singular point of type $(n, n+1)$. A proper choice of local coordinates about x shows that (X, x) is isomorphic to $(V(z_1 z_2), 0)$. Hence $\mu(x) = 2$ and $x \notin S_2(X)$.

Now suppose that $x \in S(X) \setminus \mathrm{Sg}(S(X))$ but $x \notin S_2(X)$. Thus $\mu(x) \geqq 2$. If $\mu(x) > 2$ then the implicit function theorem shows that $\dim(S(X), x) \leqq n - 2$. Therefore $\mu(x) = 2$ and choosing convenient local coordinates centered at x gives $f(z) = az_1z_2 + 0(3)$ where $a \neq 0$. Hence x is an elementary singular point of type $(n, n+1)$. Therefore, $x \notin W_4$ and the claim is proved.

For weakly normal hypersurfaces this claim gives an easy differential criterion for computing the portion of the set W_4 which is contained in $S(X) \setminus \mathrm{Sg}(S(X))$. This claim is false for hypersurfaces which are not weakly normal.

EXAMPLE. Let $X = \{(x, y, z) \in \mathbf{C}^3 : x^2 - zy^2 = 0\}$ be the Cayley umbrella in $\mathbf{C}^3$. Then $X \setminus X_s = \{(0, 0, 0)\}$ so that X is weakly normal by Theorém 2. Remark (3) then shows that $W_4 = \{0, 0, 0)\}$.

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